Exotic Characteristic Classes of Lie Algebroids

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Summary. We show how the (intrinsic) exotic or secondary characteristic classes of Lie algebroids can be seen as characteristic classes of representations. We present two alternative ways: The first one consists of thinking of the adjoint representation as a connection up to homotopy. The second one is by viewing the adjoint representation as a honest representation on the first jet bundle of a Lie algebroid.

1 Introduction

Noncommutative geometry is usually characterized as the study of noncommutative algebras *as if* they were algebras of functions of spaces. This is motivated, of course, by the fact that a classical space (a topological space, a manifold, a variety, etc.) can be characterized by an appropriate commutative algebra of functions (continuous functions, smooth functions, rational functions, etc.). In this work, we concentrate on Lie (or differentiable) groupoids, whose role in noncommutative geometry has been stressed by Alan Connes.

In fact, the kind of noncommutative algebras we will be interested in are *Lie algebroids*, which are geometric versions of vector bundles, and are the infinitesimal approximations to Lie groupoids. These objects are extremely useful to describe various geometric setups. As important classes of examples one can mention foliated geometry, equivariant geometry, or Poisson geometry. The example of Poisson geometry is particularly relevant, since it lies in-between classical (or commutative) geometry and quantum (or noncommutative) geometry. We refer the reader to [4] for an introduction to the subject and its relation to the noncommutative world.

A fundamental problem, both in geometry and physics, is the study of global invariants. A Lie algebroid combines an algebraic with a geometric flavor, which makes the study of invariants extremely rich. For example, the

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classical theory of characteristic classes, such as Pontriagin classes or Chern classes, extends to Lie algebroids. The reason is that the usual Chern-Weil construction can be defined in the general context of Lie algebroids, as was explained in [8]. However, the invariants one obtains in this way, are not so interesting: all these classes are the image (by the anchor map) of the usual characteristic classes. Much more interesting are the exotic or secondary characteristic classes of Lie algebroids, introduced in [5, 8], and which generalize the secondary characteristic classes of foliations, introduced in the 70's by Bott *et al.*

The secondary classes for Lie algebroids, take two distinct disguises. On one hand, Lie algebroids have representations, which are flat Lie algebroid connections generalizing the flat vector bundles of ordinary geometry. It was shown in [5] that one can define secondary characteristic classes of representations of Lie algebroids, much like the characteristic classes of ordinary flat bundles. On the other hand, every Lie algebroid has an underlying characteristic foliation, which will be singular in general. Again, similar to the theory of foliations (see e.g. [3, 11]), it was shown in [8] that one can define intrinsic characteristic classes of the Lie algebroid.

Recall that in the theory of foliations one can describe the intrinsic secondary characteristic classes as the characteristic classes of a special representation. Namely, the normal bundle to the foliation carries a canonical flat connection, the Bott connection ([3]), which plays the role of the "adjoint representation" of the foliation. The purpose of this work is to give a similar relation between the intrinsic secondary characteristic classes of a Lie algebroid and the characteristic classes of a representation. The additional complication in this case is that, in general, a Lie algebroid does not carry an adjoint representation. We will give two distinct, alternative, solutions to this problem. Both solutions consist in giving an appropriate meaning to the notion of an "adjoint representation" of a Lie algebroid.

In the first solution to our problem, one enlarges the notion of representation, allowing *representations up to homotopy*. This was first proposed by Evans, Lu and Weinstein in [7], where they view the adjoint representation as a representation up to homotopy, and they use it to construct the most simple example of a secondary characteristic class, namely, the modular class. Here, we will see that one can define characteristic classes of representations up to homotopy, and that for the adjoint representation (up to homotopy) one obtains the intrinsic secondary characteristic classes of the Lie algebroid.

For the second solution to our problem, we observe that the first jet bundle of a Lie algebroid has a natural prolonged Lie algebroid structure. Moreover, this jet Lie algebroid carries a natural, honest, representation, which one can also view as the "adjoint representation" of the original Lie algebroid. By a straightforward application of the theory of characteristic classes of representations, we obtain classes in the Lie algebroid cohomology of the jet bundle. We then check that these classes are the pull-back of the intrinsic characteristic classes of the original Lie algebroid.

The remainder of the paper is organized into three sections. In Section 2, we recall the constructions of the intrinsic characteristic classes and of the characteristic classes of representations. In Section 3, we clarify the relevance of connections up to homotopy to the theory of characteristic classes, and we recover the intrinsic characteristic classes from the adjoint representation up to homotopy. In Section 4, we discuss the prolonged Lie algebroid structure on the jet bundle of a Lie algebroid, and we construct the intrinsic characteristic classes via the jet adjoint representation.

2 Secondary characteristic classes of Lie algebroids

In this work we will denote by A a **Lie algebroid** $\pi : A \to M$, with **anchor** $\# : A \to TM$, and **Lie bracket** $[\ ,\ ] : \Gamma(A) \times \Gamma(A) \to \Gamma(A)$. Underlying the Lie algebroid we have a (singular) foliation $\mathcal{F}$, which integrates the (singular) involutive distribution $\operatorname{Im}\#$. We recall that the space of A**-forms** $\Omega^\bullet(A)$ is formed by the sections of the exterior bundles $\Gamma(\wedge^\bullet(A^*))$, and that the A**-differential**

$$d : \Omega^\bullet(A) \to \Omega^{\bullet+1}(A)$$

is given by the usual Cartan formula:

$$\begin{aligned} \mathrm{d}\omega(\alpha_0, \dots, \alpha_k) = &\sum_{i=0}^{k+1} (-1)^i \#\alpha_i(\omega(\alpha_0, \dots, \widehat{\alpha}_i, \dots, \alpha_k)) \\ &+ \sum_{i<j} (-1)^{i+j} \omega([\alpha_i, \alpha_j], \alpha_0, \dots, \widehat{\alpha}_i, \dots, \widehat{\alpha}_j, \dots, \alpha_k). \quad (1) \end{aligned}$$

The cohomology of the complex $(\Omega^\bullet(A), d)$ is the **Lie algebroid cohomology** of A (with trivial coefficients), and is denoted $H^\bullet(A)$.

2.1 The Chern-Weil construction

Let us recall briefly the Chern-Weil construction for a Lie algebroid A (see [5, 8]). In the case $A = TM$ we recover the usual construction.

Given a vector bundle $E \to M$ we will consider the E-valued A-forms:

$$\Omega^\bullet(A; E) = \Omega^\bullet(A) \otimes_{C^\infty(M)} \Gamma(E).$$

An A**-connection** on E is a linear operator $\nabla : \Omega^0(A; E) \to \Omega^1(A; E)$, satisfying the Leibniz identity

$$\nabla_\alpha(fs) = f\nabla_\alpha s + \#\alpha(f)s.$$

It has a unique extension to an operator

$$\mathrm{d}_\nabla : \Omega^\bullet(A; E) \to \Omega^{\bullet+1}(A; E),$$

also satisfying the Leibniz identity. Explicitly

$$\mathrm{d}_\nabla \omega(\alpha_0, \ldots, \alpha_k) = \sum_{i=0}^{k+1} (-1)^i \nabla_{\alpha_i}(\omega(\alpha_0, \ldots, \widehat{\alpha}_i, \ldots, \alpha_k)) + \sum_{i<j} (-1)^{i+j} \omega([\alpha_i, \alpha_j], \alpha_0, \ldots, \widehat{\alpha}_i, \ldots, \widehat{\alpha}_j, \ldots, \alpha_k). \quad (2)$$

This will be a differential provided the curvature

$$R_\nabla(\alpha, \beta) = \nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha - \nabla_{[\alpha,\beta]},$$

vanishes. If this is the case, we obtain the **Lie algebroid cohomology with coefficients** in E, denoted $H^\bullet(A; E)$. In general, the curvature will not vanish, but it will satisfy the Bianchi identity

$$\mathrm{d}_\nabla R_\nabla = 0, \quad (3)$$

where on $\mathrm{End}(E)$ we take the induced connection from E.

Now the usual trace on $\mathrm{End}(E)$ induces a **trace**

$$\mathrm{Tr} : (\Omega^\bullet(A; \mathrm{End}(E)), \mathrm{d}_\nabla) \to (\Omega^\bullet(A), \mathrm{d}),$$

which satisfies $\mathrm{d}\,\mathrm{Tr} = \mathrm{Tr}\,\mathrm{d}_\nabla$. Hence, we can define the **Chern characters** by setting

$$\mathrm{ch}_k(\nabla) = \mathrm{Tr}(R_\nabla^k) \in \Omega^{2k}(A). \quad (4)$$

and we have:

Lemma 1. *The Chern characters* $\mathrm{ch}_k(\nabla)$ *are closed* A*-forms.*

A basic fact is that the cohomology class $[\mathrm{ch}_k(\nabla)] \in H^{2k}(A)$ does not depend on the connection. This can be seen through the Chern-Simons construction, which we also recall briefly in the context of Lie algebroids.

Let $\nabla_0, \ldots, \nabla_l$ be A-connections on E. Also let

$$\Delta^l = \{(t_0, \ldots, t_l) : t_i \geq 0, \sum_{i=0}^{l} t_i = 1\},$$

be the standard l-simplex, and denote by $p : M \times \Delta^l \to M$ the projection on the first factor. Then both E and A can be pull-backed to $M \times \Delta^l$ and p^*A had a natural Lie algebroid structure. We can define a p^*A-connection on p^*E by forming the affine combination:

$$\nabla^{\mathrm{aff}} = \sum_{i=0}^{l} t_i \nabla_i.$$

The classical integration along the fibers has also an analogue:

$$\int_{\Delta^l} : \Omega^\bullet(p^*A) \to \Omega^{\bullet-l}(A),$$

which is given explicitly by the formula:

$$(\int_{\Delta^l} \omega)(\alpha_1, \dots, \alpha_{n-l}) = \int_{\Delta^l} \omega(\frac{\partial}{\partial t_1}, \dots, \frac{\partial}{\partial t_l}, \alpha_1, \dots, \alpha_{n-l})\mathrm{d}t_1 \dots \mathrm{d}t_l.$$

We can now define the **Chern-Simons** transgression by

$$\mathrm{cs}_k(\nabla_0, \dots, \nabla_l) = \int_{\Delta^l} \mathrm{ch}_k(\nabla^{\mathrm{aff}}). \tag{5}$$

With the convention that for $l = 0$ we set $\mathrm{cs}_k(\nabla) = \mathrm{ch}_k(\nabla)$, we obtain the following lemma:

Lemma 2. *The Chern-Simons transgressions satisfy:*

$$\mathrm{d}\,\mathrm{cs}_k(\nabla_0, \dots, \nabla_l) = \sum_{i=0}^{l} (-1)^i \,\mathrm{cs}_k(\nabla_0, \dots, \widehat{\nabla}_i, \dots, \nabla_l). \tag{6}$$

The proof is a simple application of integration by parts. We conclude that for any vector bundle E we have a well defined **Chern character** $\mathrm{ch}(A; E)$, with components the cohomology classes of $\mathrm{ch}_k(\nabla)$, for any choice of A-connection ∇ on E. Whenever there is no confusion, we shall abbreviate the Chern character to $\mathrm{ch}(E)$.

2.2 Characteristic classes of a representation

Let E be a representation of a Lie algebroid A. This just means that E is endowed with a flat A-connection ∇. From the previous paragraph we conclude immediately that:

Corollary 1. *For a representation E of A we have* $\mathrm{ch}(E) = 0$.

The vanishing of the Chern character of a representation is the origin of new *secondary characteristic classes*. These characteristic classes of a representation were first introduced in [5].

From now on, unless otherwise stated, we assume that E is a *complex* vector bundle. We choose a hermitian metric g on E. The connection ∇ induces an adjoint connection ∇^g on E, which is defined in the usual way:

$$\#\alpha(g(s_1, s_2)) = g(\nabla_\alpha s_1, s_2) + g(s_1, \nabla_\alpha s_2).$$

We leave it to reader the easy check that:

Lemma 3. *Let ∇, ∇_0, ∇_1 be connections on E. For any metric g:*

$$\mathrm{ch}_k(\nabla^g) = (-1)^k \overline{\mathrm{ch}_k(\nabla)},$$
$$\mathrm{cs}_k(\nabla_0^g, \nabla_1^g) = (-1)^k \overline{\mathrm{cs}_k(\nabla_0, \nabla_1)}.$$

For any representation E we fix a metric g on E and we define elements

$$u_{2k-1}(E,\nabla) = \mathrm{i}^{k+1}\,\mathrm{cs}_k(\nabla,\nabla^g) \in \Omega^{2k-1}(A),$$

where $\mathrm{i} = \sqrt{-1}$.

Proposition 1. *The A-forms u_{2k-1} are real, closed, and their cohomology class is independent of the metric.*

Proof. By (6) and the previous lemma, we find

$$\begin{aligned} \mathrm{d}\,\mathrm{cs}_k(\nabla,\nabla^g) &= \mathrm{ch}_k(\nabla) - \mathrm{ch}_k(\nabla^g) \\ &= \mathrm{ch}_k(\nabla) - (-1)^k\overline{\mathrm{ch}_k(\nabla)}. \end{aligned}$$

Sine ∇ is flat, both terms vanish. Therefore, the u_{2k-1}'s are closed, and they are also real by the previous lemma.

If g and g' are metrics on E, formula (6) gives:

$$\mathrm{cs}_k(\nabla,\nabla^g) - \mathrm{cs}_k(\nabla,\nabla^{g'}) = \mathrm{cs}_k(\nabla^g,\nabla^{g'}) - \mathrm{d}\,\mathrm{cs}_k(\nabla,\nabla^g,\nabla^{g'}).$$

Therefore, u_{2k-1} will be independent of the metric provided the A-form $\mathrm{cs}_k(\nabla^g,\nabla^{g'})$ is exact. For that we choose a family g_t of metrics joining $g = g_0$ to $g' = g_1$, and define $u_t \in \mathrm{End}(E)$ by $g_t(e_1,e_2) = g(u_t(e_1),e_2)$. Then

$$\mathrm{cs}_k(\nabla^g,\nabla^{g_t}) = \mathrm{Tr}(\omega_t^{2k-1}),$$

and all we need to show is that $\frac{\partial}{\partial t}\mathrm{Tr}(\omega_t^{2k-1})$ is an exact form. A simple computation shows that

$$\frac{\partial\omega_t}{\partial t} = \mathrm{d}_{\nabla^g}(v_t) + [\omega_t, v_t],$$

where $v_t = u_t^{-1}\frac{\partial u_t}{\partial t}$. Since $\mathrm{d}_{\nabla^g}(\omega_t^2) = 0$, this implies that:

$$\frac{\partial\omega_t}{\partial t}\omega_t^{2k-2} = \mathrm{d}_{\nabla^g}(v_t\omega_t^{2k-2}) + [\omega_t, v_t\omega_t^{2k-2}].$$

Now, by the properties of the trace, it follows that

$$\frac{\partial}{\partial t}\mathrm{Tr}(\omega_t^{2k-1}) = \mathrm{d}\,\mathrm{Tr}(v_t\omega_t^{2k-2}),$$

as desired. □

We can now introduce:

Definition 1. *The* ***characteristic classes of a representation*** *E are the cohomology classes*

$$u_{2k-1}(E) = [u_{2k-1}(E,\nabla)] \quad (k = 1,\dots,r),$$

where r is the rank of E.

Notice that, if E admits an invariant metric g, then these classes vanish, so they can be seen as obstructions to the existence of an invariant metric. The main properties of these classes are:

(i) $u_{2k-1}(E \oplus F) = u_{2k-1}(E) + u_{2k-1}(F)$;
(ii) $u_{2k-1}(E \otimes F) = \mathrm{rank}(E) u_{2k-1}(F) + \mathrm{rank}(F) u_{2k-1}(E)$;
(iii) $u_{2k-1}(E^*) = -u_{2k-1}(E)$.

We refer to [5] for a proof of these facts. They can also be summarized by saying that the map $\mathrm{Rep}(A) \to \mathbb{Z} \times H^{\mathrm{odd}}(A)$ defined by

$$E \mapsto (\mathrm{rank}(E), u_1(E), \dots, u_{2r-1}(E)),$$

is a morphism of *-semi-rings.

Let us consider now the case of a *real* vector bundle. For these we have:

Proposition 2. *Assume that (E, ∇) is a real representation of A. If k is even then $u_{2k-1}(E, \nabla) = 0$. If k is odd then, for any metric connection ∇_m, the differential form*

$$(-1)^{\frac{k+1}{2}} \mathrm{cs}_k(\nabla_0, \nabla_m) \in \Omega^{2k-1}(A)$$

is closed, and its cohomology class equals $\frac{1}{2} u_{2k-1}(E, \nabla)$.

Proof. Let ∇_m be a metric connection for some metric g, so that $\nabla_m^g = \nabla_m$. From Lemma 3, we find

$$\mathrm{cs}_k(\nabla_m, \nabla^g) = (-1)^k \mathrm{cs}_k(\nabla_m^g, \nabla) = (-1)^{k+1} \mathrm{cs}_k(\nabla, \nabla_m) \ .$$

This, combined with the transgression formula (6), implies

$$\begin{aligned} \mathrm{d}\, \mathrm{cs}_k(\nabla, \nabla_m, \nabla^g) &= \mathrm{cs}_k(\nabla_m, \nabla^g) - \mathrm{cs}_k(\nabla, \nabla^g) + \mathrm{cs}_k(\nabla, \nabla_m) \\ &= (1 + (-1)^{k+1}) \mathrm{cs}_k(\nabla, \nabla_m) - \mathrm{cs}_k(\nabla, \nabla^g), \end{aligned}$$

which proves the proposition. □

2.3 Intrinsic secondary characteristic classes

In order to motivate the introduction of these characteristic classes let us start by looking at the special case of regular Lie algebroids.

Let A be a regular Lie algebroid so that the characteristic foliation $\mathcal{F}$ integrating $\mathrm{Im}\,\#$ is non-singular. Denote the normal bundle by $\nu = TM/T\mathcal{F}$, and the kernel of the anchor by $K = \mathrm{Ker}\,\#$. These are both vector bundles, since A is regular, and they carry canonical flat connections, namely the *Bott connections*:

$$\nabla_\alpha \beta = [\alpha, \beta], \quad \beta \in \Gamma(K), \tag{7}$$

$$\nabla_\alpha \overline{X} = \overline{\mathcal{L}_{\#\alpha} X}, \quad \overline{X} \in \Gamma(\nu). \tag{8}$$

This means that we can define intrinsic secondary characteristic classes of A, by letting:

$$u_{2k-1}(A) = u_{2k-1}(K) - u_{2k-1}(\nu).$$

Notice that the origin of these secondary classes is the vanishing of the Chern character of the formal difference $K-\nu$. Now observe that we have the following short exact sequences of vector bundles:

$$0 \longrightarrow K \longrightarrow A \longrightarrow T\mathcal{F} \longrightarrow 0,$$

$$0 \longrightarrow T\mathcal{F} \longrightarrow TM \longrightarrow \nu \longrightarrow 0.$$

Hence, the difference $K-\nu$ equals the difference $A-TM$, and we have the following:

Corollary 2. *For any regular Lie algebroid,* $\mathrm{ch}(A-TM)=0$.

Let us now turn to the non-regular case. While the difference $K-\nu$ only makes sense for regular Lie algebroids, the difference $A-TM$ always makes sense. Also, we can introduce A-connections on A and TM, which are not flat, but which give rise to a flat connection on the formal difference. These naturally extend Bott's basic connections for foliations (see [3]).

Definition 2. *A connection* $(\hat{\nabla},\check{\nabla})$ *on* $A\oplus T^*M$ *is called a* basic connection *if* $\#\hat{\nabla}=\check{\nabla}\#$ *and if they restrict to the Bott connections on each leaf* L *of the characteristic foliation* $\mathcal{F}$.

Notice that what is left in the formal difference $A-TM$ is precisely the Bott part of the basic connection. The vanishing of the corollary above is now replaced by the following result (see [8]):

Lemma 4. *The curvature* R *of a basic connection vanishes along* $K\oplus(T\mathcal{F})^0$.

A simple procedure to obtain basic connections is as follows. One chooses an ordinary connection $\bar{\nabla}$ on A, and defines A-connections on A and on TM by the formulas:

$$\hat{\nabla}_\alpha\beta = \bar{\nabla}_{\#\beta}\alpha + [\alpha,\beta], \quad \check{\nabla}_\alpha X = \bar{\nabla}_X\alpha + [\#\alpha, X]. \tag{9}$$

One checks readily that the pair $\nabla=(\hat{\nabla},\check{\nabla})$ is a basic connection.

Now we can define our characteristic classes. We pick a basic connection ∇ and a metric connection ∇_{m} (i.e., ∇_{m} preserves some metric g on $A\oplus T^*M$).

Definition 3. *The* ***intrinsic characteristic classes*** *of* A *are the cohomology classes*

$$u_{2k-1}(\mathrm{Ad}\,A) = 2[(-1)^{\frac{k+1}{2}}\,\mathrm{cs}_k(\nabla,\nabla_m)] \in H^{2k-1}(A),$$

where $1\le 2k-1\le 2r-1$, *and* k *is an odd integer.*

The fact that these classes are well-defined and independent of any choices, is similar to the proof of the same fact for the characteristic classes of representations, given in the previous paragraph. We refer to [8] for details.

The notation $u_{2k-1}(\mathrm{Ad}\, A)$ suggests that these are the characteristic classes of the adjoint representation of A. To which extent this is true, is the main subject of this paper, and will be discussed in the next sections. Before we do that, we look at the intrinsic characteristic class of lowest degree.

Example 1. The modular class of a Lie algebroid was introduced in [14], and further discussed in [7, 9, 15]. We recall here the construction given in [7]. Consider the line bundle $Q_A = \wedge^r A \otimes \wedge^m T^*M$. On this line bundle we have a flat A-connection ∇ defined by:

$$\nabla_\alpha(\alpha^1 \wedge \cdots \wedge \alpha^r \otimes \mu) = \sum_{j=1}^{r} \alpha^1 \wedge \cdots \wedge [\alpha, \alpha^j] \wedge \cdots \wedge \alpha^r \otimes \mu + \alpha^1 \wedge \cdots \wedge \alpha^r \otimes \mathcal{L}_{\#\alpha}\mu, \quad (10)$$

whenever $\alpha, \alpha^1, \ldots, \alpha^r \in \Gamma(A)$ and $\mu \in \Gamma(\wedge^m T^*M)$.

Assume first that Q_A is trivial. Then we have a global section $s \in \Gamma(Q_A)$ so that

$$\nabla_\alpha s = \theta_s(\alpha) s, \qquad \forall \alpha \in \Gamma(A).$$

Since ∇ is flat, we see that θ_s defines a section of $\Gamma(A^*)$ which is closed: $\mathrm{d}\theta_s = 0$. If s' is another global section in $\Gamma(Q_A)$, we have $s' = fs$ for some non-vanishing smooth function f on M, and we find

$$\theta_{s'} = \theta_s + \mathrm{d} \log |f|.$$

Therefore, we have a well defined cohomology class

$$\mathrm{mod}(A) \equiv [\theta_s] \in H^1(A)$$

which is independent of the section s. If the line bundle Q_A is not trivial one considers the square $L = Q_A \otimes Q_A$, which is trivial, and defines

$$\mathrm{mod}(A) = \frac{1}{2}[\theta_s],$$

for some global section $s \in \Gamma(L)$. The class $\mathrm{mod}(A)$ is called the **modular class** of the Lie algebroid A.

Now we have proved in [5, 8] that

$$u_1(\mathrm{Ad}\, A) = \frac{1}{2\pi} \mathrm{mod}(A). \quad (11)$$

This gives a geometric interpretation of $u_1(\mathrm{Ad}\, A)$ as an obstruction class. In fact, as was argued in [7] one can think of global sections of Q_A (or $Q_A \otimes Q_A$) as “transverse measures” to A. The modular class is trivial iff there exists a transverse measure which is invariant under the flows of every section $\alpha \in \Gamma(A)$. Therefore, the modular class (i.e., the class $u_1(\mathrm{Ad}\, A)$) is an obstruction lying in the first Lie algebroid cohomology group $H^1(A)$ to the existence of a transverse invariant measure to A.

3 Characteristic classes and connections up to homotopy

3.1 Non-linear connections

As we have mentioned before, for a general Lie algebroid, there is no adjoint representation. One way around this difficulty is to relax the notion of connection.

Let $\pi : A \to M$ be a Lie algebroid and let $E = E^1 \oplus E^0$ be a super-vector bundle over M. We consider $\mathbb{R}$-linear operators

$$\Gamma(A) \otimes \Gamma(E) \to \Gamma(E), \quad (\alpha, s) \mapsto \nabla_\alpha s,$$

which satisfy the identity

$$\nabla_\alpha(f\mathbf{s}) = f\nabla_\alpha s + \#\alpha(f)s,$$

for all $f \in C^\infty(M)$, and preserve the grading. We will say that ∇ is a **non-linear connection** if ∇_α is local in α. This is a relaxation of the $C^\infty(M)$-linearity that one usually requires.

A **non-linear differential form** is an anti-symmetric, $\mathbb{R}$-multilinear map

$$\omega : \Gamma(A) \times \cdots \times \Gamma(A) \to C^\infty(M),$$

which is local. Many of the usual operations on forms don't use $C^\infty(M)$-linearity. For example, we have a de Rham operator $d : \Omega^\bullet_{\mathrm{nl}}(A) \to \Omega^{\bullet+1}_{\mathrm{nl}}(A)$. We can also consider E-values non-linear forms, which we denote by $\Omega^\bullet_{\mathrm{nl}}(A; E)$.

The Chern-Weil and all other constructions of Section 2.1 immediately generalize to non-linear connections provided we use non-linear forms. For example, the usual super-trace on $\mathrm{End}(E)$ induces a **super-trace**

$$\mathrm{Tr} : (\Omega^\bullet_{\mathrm{nl}}(A; \mathrm{End}(E)), \mathrm{d}_\nabla) \to (\Omega^\bullet_{\mathrm{nl}}(A), \mathrm{d}),$$

and we obtain the **Chern characters** of the non-linear connection

$$\mathrm{ch}_k(\nabla) = \mathrm{Tr}(R^k_\nabla) \in \Omega^{2k}_{\mathrm{nl}}(A).$$

As before, the Chern characters $\mathrm{ch}_k(\nabla)$ are closed, non-linear A-forms, and up to a boundary these classes do not depend on the connection. This, of course, is because the Chern-Simons construction also generalizes to this setting, giving a non-linear version of the **Chern-Simons transgressions** $\mathrm{cs}_k(\nabla^0, \dots, \nabla^l)$ which still satisfy equation (5), which is now a equality between non-linear forms.

From now on, we let (E, ∂) be a super-complex of vector bundles over the manifold M,

$$(E, \partial) : \; E^0 \underset{\partial}{\overset{\partial}{\rightleftarrows}} E^1 \; . \tag{12}$$

We consider also a non-linear connection ∇ on E such that $\nabla_\alpha \partial = \partial \nabla_\alpha$ for all $\alpha \in \Gamma(A)$. The notion of **connection up to homotopy** [7] on (E, ∂) is obtained by requiring linearity up to homotopy. In other words we require that

$$\nabla_{f\alpha} s = f \nabla_\alpha s + [H_\nabla(f, \alpha), \partial],$$

where $H_\nabla(f, \alpha) \in \Gamma(\mathrm{End}(E))$ are odd elements which are $\mathbb{R}$-linear and local in α and f.

We say that two non-linear connections ∇ and ∇' are **equivalent** (or homotopic) if, for all $\alpha \in \Gamma(A)$, we have

$$\nabla'_\alpha = \nabla_\alpha + [\theta(\alpha), \partial],$$

for some $\theta \in \Omega^1_{\mathrm{nl}}(A; \mathrm{End}(E))$ of odd degree. In this case, we write $\nabla \sim \nabla'$. There are two basic properties of this equivalence relation which we state as our next two lemmas.

Lemma 5. *A non-linear connection is a connection up to homotopy if and only if it is equivalent to a (linear) connection.*

Proof. Assume that ∇ is a connection up to homotopy. Let U_a be the domain of local coordinates x^k for M over which the bundle A trivializes, and denote by $\{e_1, \dots, e_r\}$ a basis of local sections. We define a local linear connection

$$\nabla^a_X = \nabla_X + [u^a(X), \partial],$$

where $u_a \in \Omega^1_{\mathrm{nl}}(A|_{U_a}; \mathrm{End}(E))$ is given by

$$u_a(\sum_k f_k e_k) = -\sum_k H_\nabla(f_k, e_k),$$

for all $f_k \in C^\infty(U_a)$. Next we take $\{\phi_a\}$ to be a partition of unity subordinate to an open cover $\{U_a\}$ by such coordinate domains and set

$$\nabla'_\alpha = \sum_a \phi_a \nabla^a_\alpha, \quad u(\alpha) = \sum_a \phi_a u^a(\alpha).$$

Then $\nabla' = \nabla + [u, \partial]$ is a connection equivalent to ∇. □

Lemma 6. *If* ∇^0 *and* ∇^1 *are equivalent connections, then* $\mathrm{ch}(\nabla^0) = \mathrm{ch}(\nabla^1)$.

Proof. Let ∇^0 and ∇^1 be connections such that $\nabla^1 = \nabla^0 + [\theta, \partial]$. A simple computation shows that

$$R_{\nabla^1} = R_{\nabla^0} + [d_{\nabla^0}\theta + Q, \partial] \;, \tag{13}$$

where $Q(\alpha, \beta) = [\theta(\alpha), [\theta(\beta), \partial]]$. Let us denote by $Z \subset \Omega^\bullet_{\mathrm{nl}}(A; \mathrm{End}(E))$ the space of non-linear forms ω with the property that $[\omega, \partial] = 0$, and by $B \subset Z$ the subspace consisting of element of the form $[\eta, \partial]$ for some non-linear form η. Since we have

$$[\partial, \omega\eta] = [\partial, \omega]\eta + (-1)^{|\omega|}\omega[\partial, \eta],$$

we see that $ZB \subset B$, hence (13) implies that $R^k_{\nabla^1} \equiv R^k_{\nabla^0}$ modulo B. The desired equality follows now from the fact that Tr vanishes on B. □

Observe now that if ∇ is a *linear* connections on (E, ∂), then the Chern characters $\mathrm{ch}_k(\nabla)$ are *linear* differential forms, whose cohomology classes are the components of the Chern character $\mathrm{ch}(E) = \mathrm{ch}(E^0) - \mathrm{ch}(E^1)$. Hence, an immediate consequence of the previous two lemmas is the following:

Proposition 3. *If ∇ is a connection up to homotopy on (E, ∂), then*

$$\mathrm{ch}_k(\nabla) = \mathrm{Tr}(R_\nabla^k),$$

are closed differential forms whose cohomology classes are the components of the Chern character $\mathrm{ch}(E)$.

The Chern-Simons transgressions forms of non-linear connections are non-linear forms, and they satisfy the obvious properties. We list here the relevant properties:

Lemma 7. *Let ∇, ∇_0, ∇_1 be non-linear connections. Then:*

(i) If ∇_0 and ∇_1 are connections up to homotopy then $\mathrm{cs}_k(\nabla_0, \nabla_1)$ are linear *differential forms;*
(ii) If $\nabla_0 \sim \nabla_1$, then $\mathrm{cs}_k(\nabla_0, \nabla_1) = 0$;
(iii) For any metric g:

$$\mathrm{ch}_k(\nabla^g) = (-1)^k \overline{\mathrm{ch}_k(\nabla)} \quad \text{and} \quad \mathrm{cs}_k(\nabla_0^g, \nabla_1^g) = (-1)^k \overline{\mathrm{cs}_k(\nabla_0, \nabla_1)}.$$

Proof. Part (i) follows from the fact that Chern characters of connections up to homotopy are differential forms.

For part (ii) we observe that the affine combination ∇^{aff} used in the definition of $\mathrm{cs}_k(\nabla_0, \nabla_1)$ is equivalent to the pull-back $\tilde{\nabla}_0$ of ∇_0 to $M \times \Delta^1$, since $\nabla^{\mathrm{aff}} = \tilde{\nabla}_0 + t[\theta, \partial]$. But $\mathrm{ch}_k(\tilde{\nabla}_0)$ vanishes so, by Lemma 6, we conclude that $\mathrm{cs}_k(\nabla_0, \nabla_1) = \mathrm{ch}_k(\nabla^{\mathrm{aff}}) = 0$.

Finally, if g is a metric on E, a simple computation shows that $R_{\nabla^g} = -R_\nabla^*$, where $*$ denotes the adjoint (with respect to g). Then (iii) follows from the equality $\mathrm{Tr}(C^*) = \overline{\mathrm{Tr}(C)}$, for any matrix C. □

3.2 Characteristic classes of representations up to homotopy

A **representation up to homotopy** is a super-vector bundle (E, ∂) with a connection up to homotopy which is flat. We are now ready to extend the construction of the characteristic classes of representations to representations up to homotopy. Again, the origin of these classes is the following vanishing result, which is an immediate consequence of Proposition 3.

Corollary 3. *If (E, ∂) is a representation up to homotopy, then* $\mathrm{ch}(E) = 0$.

For any representation up to homotopy (E, ∂, ∇), we choose a metric g on E and we consider the forms:

$$u_{2k-1}(E, \partial, \nabla) = i^{k+1}\, \mathrm{cs}(\nabla, \nabla^g) \in \Omega^{2k-1}(A).$$

Proposition 4. *Let (E, ∂, ∇) be a flat representation up to homotopy. Then:*

(i) The differential forms $u_{2k-1}(E, \partial, \nabla)$ are real and closed, and the induced cohomology classes do not depend on the choice of the metric.
(ii) If $\nabla \sim \nabla'$, then $u_{2k-1}(E, \partial, \nabla) = u_{2k-1}(E, \partial, \nabla')$.
(iii) If ∇ is equivalent to a metric connection (i.e., a connection which is compatible with a metric), then all the classes $u_{2k-1}(E, \partial, \nabla)$ vanish.

Proof. If we use Proposition 3 and Lemma 7, the proof of (i) is analogous to the proof of Proposition 1, and so we omit it. To prove (ii), we observe that the non-linear version of the transgression formula (5) gives, for any connection ∇_0,

$$\begin{aligned} \mathrm{d}\,\mathrm{cs}_k(\nabla, \nabla_0, \nabla^g) &= \mathrm{cs}_k(\nabla_0, \nabla^g) - \mathrm{cs}_k(\nabla, \nabla^g) + \mathrm{cs}_k(\nabla, \nabla_0) \\ \mathrm{d}\,\mathrm{cs}_k(\nabla^g, \nabla_0, \nabla_0^g) &= \mathrm{cs}_k(\nabla_0, \nabla_0^g) - \mathrm{cs}_k(\nabla^g, \nabla_0^g) + \mathrm{cs}_k(\nabla^g, \nabla_0). \end{aligned}$$

Adding up these two relations, we conclude that the class $u_{2k-1}(E, \partial, \nabla)$ equals the cohomology class of

$$\mathrm{i}^k \left(\mathrm{cs}_k(\nabla, \nabla_0) + \mathrm{cs}_k(\nabla_0, \nabla_0^g) + \mathrm{cs}_k(\nabla_0^g, \nabla)\right),$$

for any connection ∇_0. On the other hand, if ∇' is equivalent to ∇, Lemma 7 (ii) gives:

$$\mathrm{cs}_k(\nabla, \nabla_0) - \mathrm{cs}_k(\nabla', \nabla_0) = \mathrm{d}\,\mathrm{cs}_k(\nabla, \nabla', \nabla_0).$$

Hence, we conclude that

$$\begin{aligned} u_{2k-1}(E, \partial, \nabla) &= \mathrm{i}^k \left[\mathrm{cs}_k(\nabla, \nabla_0) + \mathrm{cs}_k(\nabla_0, \nabla_0^g) + \mathrm{cs}_k(\nabla_0^g, \nabla)\right] \\ &= \mathrm{i}^k \left[\mathrm{cs}_k(\nabla', \nabla_0) + \mathrm{cs}_k(\nabla_0, \nabla_0^g) + \mathrm{cs}_k(\nabla_0^g, \nabla')\right] \\ &= u_{2k-1}(E, \partial, \nabla'), \end{aligned}$$

which proves (ii).

Finally, (iii) follows from (i) and (ii). □

In the real case, we obtain the analogue of Proposition 2. The proof is entirely similar.

Proposition 5. *Assume that E is a real vector bundle. If k is even then $u_{2k-1}(E, \partial, \nabla) = 0$. If k is odd, then for any connection ∇_0 equivalent to ∇, and any metric connection ∇_m,*

$$(-1)^{\frac{k+1}{2}}\, \mathrm{cs}_k(\nabla_0, \nabla_m) \in \Omega^{2k-1}(A)$$

is a closed differential form whose cohomology class equals $\frac{1}{2} u_{2k-1}(E, \partial, \nabla)$.

In this way we have extended the theory of secondary characteristic classes of representations to representations up to homotopy. Note that the construction presented here actually works for connections which are *flat up to homotopy*, i.e., whose curvature forms are of the type $[-, \partial]$. Moreover, this notion is stable under equivalence, and the characteristic classes will only depend on the equivalence class of ∇ (cf. Proposition 4 (ii)).

Note also that, as in [6] (and following [2]), there is a version of our discussion for super-connections ([13]) up to homotopy. Some of our constructions can then be interpreted in terms of the super-connection $\partial + \nabla$.

If E is regular in the sense that $\mathrm{Ker}\,\partial$ and $\mathrm{Im}\,\partial$ are vector bundles, then so is the cohomology $H^\bullet(E,\partial) = \mathrm{Ker}\,\partial/\,\mathrm{Im}\,\partial$, and any connection up to homotopy ∇ on (E,∂) defines a linear connection on $H^\bullet(E,\partial)$. Moreover, this connection is flat if ∇ is, and the characteristic classes $u_{2k-1}(E,\partial,\nabla)$ coincide with the classical characteristic classes of the flat vector bundle $H^\bullet(E,\partial)$ (see [2, 10]). In general, the classes $u_{2k-1}(E,\partial,\nabla)$ should be viewed as invariants of $H^\bullet(E,\partial)$ constructed in such a way that no regularity assumption is required.

3.3 Intrinsic classes via representations up to homotopy

Let us turn now to the adjoint representation of a Lie algebroid A. The case of a regular Lie algebroid, considered at the start of Section 2.3, suggests that one should look at the formal difference $A - TM$. This can be made precise, by working up to homotopy. We consider the super-vector bundle

$$\mathrm{Ad}(A):\ A \underset{\#}{\overset{0}{\rightleftarrows}} TM, \tag{14}$$

with the flat connection up to homotopy ∇^{ad} given by:

$$\nabla^{\mathrm{ad}}_\alpha \beta = [\alpha,\beta], \qquad \nabla^{\mathrm{ad}}_\alpha X = [\#\alpha, X],$$

for which the homotopies are $H(f,\alpha)(\beta) = 0$ and $H(f,\alpha)(X) = X(f)\alpha$, for all $\alpha,\beta \in \Gamma(A)$, $X \in \mathfrak{X}(M)$, $f \in C^\infty(M)$.

The following result shows that the characteristic classes of the adjoint representation up to homotopy, as in the previous paragraph, coincide with the intrinsic characteristic classes we have discussed in Section 2.3.

Theorem 1. *For any Lie algebroid A and any k:*

$$u_{2k-1}(\mathrm{Ad}\,A) = u_{2k-1}(A,\partial,\nabla^{\mathrm{ad}}).$$

Proof. The clue is the following proposition relating the basic connections we have discussed in Section 2.3, to the adjoint connection.

Proposition 6. *If a linear connection ∇ on A is equivalent to the adjoint connection ∇^{ad} then it is a basic connection.*

Assuming the proposition holds, by Lemma 5 there exists a basic connection ∇ equivalent to ∇^{ad}. Fixing also a metric connection ∇_m, we find:

$$u_{2k-1}(\operatorname{Ad} A) = 2[(-1)^{\frac{k+1}{2}} \operatorname{cs}_k(\nabla, \nabla_{\mathrm{m}})] \qquad \text{(by Definition 3)},$$
$$= u_{2k-1}(A, \partial, \nabla^{\mathrm{ad}}) \qquad \text{(by Proposition 5)},$$

which proves the theorem.

Proof of Proposition 6. The condition that ∇ is equivalent to ∇^{ad} means that there exists some $\theta \in \Omega^1_{\mathrm{nl}}(A; \operatorname{End}(A \oplus TM))$ of odd degree such that:

$$\nabla_\alpha = \nabla^{\mathrm{ad}}_\alpha + [\theta(\alpha), \partial]$$

for all $\alpha \in \Gamma(A)$. Writing $\nabla = (\hat{\nabla}, \check{\nabla})$, this condition translates into:

$$\hat{\nabla}_\alpha \beta = [\alpha, \beta] + \theta(\alpha)\#\beta,$$
$$\check{\nabla}_\alpha X = [\#\alpha, X] + \#\theta(\alpha)X.$$

If we restrict, over a leaf L of A, the first connection to $\operatorname{Ker} \#|_L$ and the second connection to $\nu(L)$, the terms involving θ vanish. Therefore, both $\hat{\nabla}$ and $\check{\nabla}$ restrict over a leaf to the Bott connections. On the other hand, we compute:

$$\#\hat{\nabla}_\alpha \beta = \#[\alpha, \beta] + \#\theta(\alpha)\#\beta$$
$$= [\#\alpha, \#\beta] + \#\theta(\alpha)\#\beta = \check{\nabla}_\alpha \#\beta.$$

Hence, $\nabla = (\hat{\nabla}, \check{\nabla})$ is a basic connection. □

Notice that Proposition 6 gives some further geometric insight to the notion of a basic connection. Moreover, in the regular case, it is easy to check that a linear connection is basic iff it is equivalent to the adjoint connection.

4 Jets and characteristic classes

In the previous section we saw that the adjoint representation is a representation only up to homotopy, and we used this fact to show how the intrinsic classed can be seen as classes of representations. In this section we consider a different interpretation of the adjoint representation, as a honest representation. The price to pay is that we have to work on the jet Lie algebroid.

4.1 The jet of a Lie algebroid

Let us explain that for any Lie algebroid A, each jet bundle $J^k A$ inherits a natural Lie algebroid structure. This construction of the *jet Lie algebroid* can be traced back to the works of Kumpera, Libermann and Spencer (see [1] and references thereof).

Let $\pi : E \to M$ be a vector bundle. For each non-negative integer k, we will denote by $\pi^k : J^k E \to M$ the vector bundle of k-order jets of sections of E. If $l < k$, we denote by $\pi^k_l : J^k E \to J^l E$ the canonical projection. Since

$J^0E = E$ we have $\pi_0^k = \pi^k$. If $\alpha \in \Gamma(E)$ is a section of E, we will denote by $j^k\alpha$ the induced section of J^kE. For basic facts on jet bundles we refer the reader to [12]. Although we will assume that $0 \leq k < \infty$, many constructions below hold if $k = \infty$.

Our first observation is the following:

Proposition 7. *If A is a Lie algebroid, there exists a unique Lie algebroid structure on J^kA such that:*

(i) For any section $\alpha \in \Gamma(A)$ the anchors are related by:

$$\# j^k\alpha = \#\alpha.$$

(ii) For any sections $\alpha, \beta \in \Gamma(A)$ the Lie brackets are related by

$$[j^k\alpha, j^k\alpha] = j^k([\alpha, \beta]).$$

Proof. First we prove uniqueness. Property (i) clearly defines (uniquely) the anchor as the composition $\# \circ \pi_k$, where $\pi_k : J^kA \to A$ is the canonical projection. So let $[\ ,\ ]$ be a Lie algebroid bracket on J^kA with anchor $\# = \# \circ \pi_k$. The sections of J^kA are generated over $C^\infty(M)$ by sections of the form $j^k\alpha$, where $\alpha \in \Gamma(A)$. Using the Leibniz identity, we obtain:

$$\begin{aligned}[g_1 j^k\alpha_1, g_2 j^k\alpha_2] = g_1 g_2 [j^k\alpha_1, j^k\alpha_2] + \\ + g_1\#\alpha_1(g_2) j^k\alpha_2 - g_2\#\alpha_2(g_1) j^k\alpha_1. \quad (15)\end{aligned}$$

This shows that, if (ii) also holds, then $[\ ,\ ]$ is uniquely determined.

Since uniqueness holds, it remains to show every point $x \in M$ has a neighborhood U where such a Lie bracket exists. So let $(x^1, \ldots, x^m)$ be local coordinates on a open set $U \subset M$, over which the bundle A trivializes. Let $\{e_1, \ldots, e_r\}$ be a basis of sections of $A|_U$. For a multi-index $I = (i_1, \ldots, i_m)$ of non-negative integers, we set $|I| = i_1 + \cdots + i_m$ and denote by x^I the monomial $(x^1)^{i_1} \cdots (x^m)^{i_r}$. The sections defined by

$$e_a^I = \frac{1}{|I|!} j^k(x^I e_a), \quad 1 \leq a \leq m, |I| \leq k,$$

form a generating set of sections for $J^kA|_U$. A basis can be obtained by considering, for example, multi-indices I with $i_1 \leq i_2 \leq \ldots i_m$. Now we can define a Lie bracket satisfying (ii), by setting

$$[e_a^I, e_b^J] = \frac{1}{|I|!|J|!} j^k([x^I e_a, x^J e_b]),$$

and requiring the Leibniz identity to hold. □

For the jet adjoint representation, to be introduced in the next section, we will be interested in the case of J^1A. So let us give the local expression for the structure constants of the jet Lie algebroid J^1A. Let $(x^1, \ldots, x^m)$ be local coordinates on a open set $U \subset M$, over which the bundle A trivializes. Let $\{e_1, \ldots, e_r\}$ be a basis of sections of $A|_U$. The Lie algebroid A has structure functions B_a^i and C_{ab}^c defined by

$$\#e_a = B_a^i \frac{\partial}{\partial x^i}, \qquad [e_a, e_b] = C_{ab}^c e_c,$$

where we have used the convention of summing over repeated indices. Now, we have an induced basis $\{e_a, e_a^i\}$ of J^1A so that, for every local section $s \in \Gamma(A)$,

$$j^1 s(x) = s^a(x) e_a + \frac{\partial s^a}{\partial x^i}(x) e_a^i.$$

Explicitly, the section e_a^i is given by:

$$y \mapsto j^1((x^i - y^i)e_a)|_{x=y}.$$

A straightforward computation using properties (i) and (ii) of Proposition 7, gives the structure functions for J^1A:

$$[e_a, e_b] = C_{ab}^c e_c + \frac{\partial C_{ab}^c}{\partial x^i} e_a^i, \tag{16}$$

$$[e_a, e_b^i] = C_{ab}^c e_c^i + \frac{\partial B_a^i}{\partial x^j} e_b^j, \tag{17}$$

$$[e_a^i, e_b^j] = B_a^j e_b^i - B_b^i e_a^j. \tag{18}$$

There are similar formulas for the higher jet Lie algebroids J^kA.

Note that the characteristic foliations of A and J^kA coincide. Also, it is easy to check that the Lie algebroid structure on J^kA makes the projections $\pi_l^k : J^kA \to J^lA$ into Lie algebroid homomorphisms.

The operation of taking jets is functorial: if $\phi : A_1 \to A_2$ is a Lie algebroid homomorphism then $j^k\phi : J^kA_1 \to J^kA_2$ is also a homomorphism of Lie algebroids, and we have a commutative diagram:

$$\begin{array}{ccc} J^kA_1 & \xrightarrow{j^k\phi} & J^kA_2 \\ {\scriptstyle \pi_l^k}\downarrow & & \downarrow{\scriptstyle \pi_l^k} \\ J^lA_1 & \xrightarrow{j^l\phi} & J^lA_2 \end{array}$$

If a Lie algebroid $A \to M$ integrates to a Lie groupoid $\mathcal{G} \rightrightarrows M$, then the jet Lie algebroid J^kA integrates to the *jet Lie groupoid* $J^k\mathcal{G} \rightrightarrows M$: the arrows of $J^k\mathcal{G}$ are the k-order jets of bisections of $\mathcal{G}$, and the operations are the obvious ones. This groupoid structure, makes the natural projection

$\pi^k_l : J^k\mathcal{G} \to J^l\mathcal{G}$ into a Lie groupoid homomorphism, and the map of bisections $j^k : \mathcal{B}(\mathcal{G}) \to \mathcal{B}(J^k\mathcal{G})$ into a group homomorphism. Also, if $\Phi : \mathcal{G}_1 \to \mathcal{G}_2$ is a Lie groupoid homomorphism, then it induces a Lie groupoid homomorphism $j^k\Phi : J^k\mathcal{G}_1 \to J^k\mathcal{G}_2$ of the associated jet groupoids and, for each pair of indices, we have a commutative diagram:

$$\begin{array}{ccc} J^k\mathcal{G}_1 & \xrightarrow{j^k\Phi} & J^k\mathcal{G}_2 \\ {\scriptstyle \pi^k_l}\downarrow & & \downarrow{\scriptstyle \pi^k_l} \\ J^l\mathcal{G}_1 & \xrightarrow{j^l\Phi} & J^l\mathcal{G}_2 \end{array}$$

4.2 The jet adjoint representation

We saw above that the adjoint representation of a Lie algebroid A is a representation only up to homotopy. It turns out that we can also view the adjoint representation A as a honest representation of J^1A.

Proposition 8. *There is a unique representation ∇ of J^1A on the bundle A, such that for any sections $\alpha, \beta \in \Gamma(A)$:*

$$\nabla_{j^1\alpha}\beta = [\alpha, \beta]. \tag{19}$$

Proof. First we observe that there exists at most one connection satisfying (19). In fact, the sections of J^1A are generated over $C^\infty(M)$ by sections of the form $j^1\alpha$, where $\alpha \in \Gamma(A)$. Hence, any J^1A-connection ∇ is determined by its values on sections of this form. In particular, if ∇ satisfies (19), we find:

$$\nabla_{gj^1\alpha}\beta = g[\alpha, \beta],$$

so uniqueness holds.

Since uniqueness holds, existence will follow if we show that every point $x \in M$ has a neighborhood U where there exists a connection satisfying (19). Again, we let $(x^1, \dots, x^m)$ be local coordinates on a open set $U \subset M$, over which the bundle A trivializes, and we let $\{e_1, \dots, e_r\}$ be a basis of sections of $A|_U$. Using the notation above, we define a connection in $A|_U$ by

$$\nabla_{e_a} e_b = C^c_{ab} e_c, \quad \nabla_{e^i_a} e_b = -B^i_b e_a.$$

This connection clearly satisfies (19).

Finally, this connection is flat, since we have:

$$R(j^1\alpha, j^1\beta)\gamma = [\alpha, [\beta, \gamma]] - [\beta, [\alpha, \gamma]] - [[\alpha, \beta], \gamma] = 0,$$

which implies that $R \equiv 0$. □

4.3 Intrinsic classes via jet representations

For the jet adjoint representation ∇^{j^1}, which is a representation of the Lie algebroid J^1A on A, we can take its characteristic classes (see Section 2.2), which we denote by

$$u_{2k-1}(A, \nabla^{j^1}) \in \Omega^{2k-1}(J^1A).$$

Now, the Lie algebroid morphism $\pi^1 : J^1A \to A$ determines a pull-back map $(\pi^1)^* : \Omega^\bullet(A) \to \Omega^\bullet(J^1A)$, which preserves differentials. Hence, we also have a map at the level of cohomology:

$$(\pi^1)^* : H^\bullet(A) \to H^\bullet(J^1A).$$

We have:

Theorem 2. *The intrinsic characteristic classes of A pull-back to the characteristic classes of the jet adjoint representation:*

$$u_{2k-1}(A, \nabla^{j^1}) = (\pi^1)^* u_{2k-1}(\mathrm{Ad}\, A).$$

Proof. For a section α of J^1A we will denote by $\pi^1_*\alpha$ the induced section of A. If ∇ is a A-connection on a vector bundle E, then we have a pull-back J^1A-connection on E, denoted $(\pi^1)^*\nabla$, and which is defined by the formula

$$(\pi^1)^*\nabla_\alpha s = \nabla_{\pi^1_*\alpha} s,$$

for any sections $\alpha \in \Gamma(J^1A)$ and $s \in \Gamma(E)$. If we twist the connection ∇ by a metric g in E, then its pull-back is the twisted pull-back connection:

$$(\pi^1)^*\nabla^g = ((\pi^1)^*\nabla)^g.$$

Also, it follows from the definitions of the Chern-Simmons transgressions, that we have:

$$\mathrm{cs}_k((\pi^1)^*\nabla^0, \ldots, (\pi^1)^*\nabla^l) = (\pi^1)^* \,\mathrm{cs}_k(\nabla^0, \ldots, \nabla^l).$$

Now, all this is still true even for non-linear connections. For example, the adjoint connection up to homotopy ∇^{ad} pulls-back to the jet adjoint connection ∇^{j^1}:

$$\nabla^{j^1} = (\pi^1)^*\nabla^{\mathrm{ad}}.$$

Note that this example shows that a non-linear connection can pull-back to a linear connection.

These remarks immediately yield the theorem. In fact, we have:

$$\begin{aligned}
u_{2k-1}(A, \nabla^{j^1}) &= \mathrm{i}^{k+1}\,\mathrm{cs}_k(\nabla^{j^1}, (\nabla^{j^1})^g) \\
&= \mathrm{i}^{k+1}\,\mathrm{cs}_k((\pi^1)^*\nabla^{\mathrm{ad}}, ((\pi^1)^*\nabla^{\mathrm{ad}})^g) \\
&= \mathrm{i}^{k+1}\,\mathrm{cs}_k((\pi^1)^*\nabla^{\mathrm{ad}}, (\pi^1)^*(\nabla^{\mathrm{ad}})^g) \\
&= (\pi^1)^*(\mathrm{i}^{k+1}\,\mathrm{cs}_k(\nabla^{\mathrm{ad}}, (\nabla^{\mathrm{ad}})^g) \\
&= (\pi^1)^* u_{2k-1}(A, \partial, \nabla^{\mathrm{ad}}) = (\pi^1)^* u_{2k-1}(\mathrm{Ad}\, A),
\end{aligned}$$

where the last equality holds by Theorem 1. □

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