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Euler-like vector fields and normal forms

Master thesis

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Introduction

Throughout the field of Differential Geometry (and related areas), one can find so-called normal form theorems. One is given a submanifold $N \subseteq M$ and some geometric structure Γ on M, that behaves nicely around N. The theorem would then typically state that Γ has some "simple", standard form in a neighborhood of N, with respect to a certain choice of local coordinates. Examples are Darboux' theorem and Weinstein's Lagrangian neighborhood theorem in symplectic geometry, the Morse Lemma for smooth functions and Conn's linearization theorem in the case of Poisson structures. A common element in proofs for all such theorems is showing the existence of a tubular neighborhood around N that is somehow adapted to the structure Γ. In two papers [16][2], E. Meinrenken, along with Lima and Bursztyn, showed that such tubular neighborhoods correspond with particular vector fields around N. These are called Euler-like vector fields. As we will see, each Euler-like vector field X determines a unique maximal tubular neighborhood, in which it looks like the standard Euler vector field. This result allows one to prove a range of normal form theorems in a similar way. One finds an Euler-like vector field that is adapted to the structure, and then uses the unique tubular neighborhood it determines. Such vector fields often arise quite naturally from the structure. We will see this technique in action for many examples of normal forms theorems. Particularly we will prove (part of) Conn's linearization theorem using this method, which has not been done before.

We will start our treatment of the subject by looking at the normal bundle of a submanifold, and the concept of the *linear approximation* of maps around N. This is important for the definition of Euler-like vector fields, which we discuss afterwards. We will then look at various examples of normal form theorems and how they are proved using this method. When necessary, we will introduce first some basic notions of the field in question before looking at the theorem. For example, an introduction to Poisson structures and Poisson cohomology is given in Chapter 4.

In the final chapter we will describe a generalization to a weighted version of the theory. Essentially, the vector bundle structure of the normal bundle is replaced by that of a *graded bundle* (as in Grabowski, [1]). This leads to the notion of the *weighted normal bundle*, and corresponding weighted Euler-like vector fields. We will introduce the ideas based on the paper by Meinrenken [16] as well as a talk by Y. Loizides, in which some initial concepts are defined/discussed. This generalization of the theory can be used to prove a wider range of normal form theorems, notably the Isotropic embedding theorem.

Chapter 1

Normal bundle and linear approximation

A normal form theorem typically consists of two parts, a geometric structure and a submanifold around which it has a certain nice property. In this first section we will start by focusing on the latter, and study what surroundings of the submanifold are like. In other words, how it is embedded in the ambient manifold. This information can be obtained by considering the normal bundle of the submanifold, and so it is not surprising that this object plays large role in the theory we will build. In this section, we will look at its definition and the related notion of tubular neighborhoods. We will also see that there is a natural way to approximate all kinds of objects on the normal bundle, called the linear approximation.

Assumption. *In this thesis, we will always assume that $N \subseteq M$ is an* embedded submanifold*, without boundary.*

This assumption is required for the notion of a tubular neighborhood of N (see Definition 1.1.6), since N is always embedded as the zero section in its normal bundle.

1.1 The normal bundle

We recall the following definition.

Definition 1.1.1. The *normal bundle of a submanifold N in M* is defined as the quotient of vector bundles

$$\nu(M, N) := TM|_N/TN.$$

This is itself a vector bundle over N, with the projection given by restricting the natural projection map $\pi : TM \to M$ to $TM|_N$. We will also use the notation ν_N for this bundle.

Identifying N with the zero section of $\nu(M, N)$, we can view N as a submanifold of $\nu(M, N)$.

Remark. In a way, the normal bundle can be viewed as the 'linear approximation' of M around N (as in [5]). For example, consider the case that $\nu(M, N)$ is trivial. Then we can find a neighborhood U of N in M that is diffeomorphic to $N \times \mathbb{R}^{m-n}$ (where $\dim M = m$, $\dim N = n$), therefore 'linearizing' M around N. In general, finding such an open set is of course not possible, consider for example as submanifold the zero section of some non-trivial vector bundle. So the best we can do is consider the vector bundle structure that is given precisely by the normal bundle. Within the context of this thesis, one could say that the normal bundle is the 'normal form' of M around N (which coincidentally also removes the ambiguity around the word normal here).

What we want to do is look at this 'operation' of taking the normal bundle as a functor between categories. We start in the category of manifold pairs (M, N) where N is a closed submanifold of M. Morphisms $(M, N) \to (M', N')$ are given by smooth maps $f : M \to M'$ such that $f(N) \subseteq N'$.

Definition 1.1.2. The *normal bundle functor* ν assigns to every smooth manifold pair (M, N) the vector bundle $\nu(M, N)$. For a morphism $f : (M, N) \to (M', N')$ we obtain a vector bundle morphism $\nu(f) : \nu(M, N) \to \nu(M', N')$, as induced in the diagram below.

$$\begin{tikzcd}
TM \arrow[r, "df"] \arrow[d, "q"] & TM' \arrow[d, "q'"] \\
\nu_N = TM|_N/TN \arrow[r, "\nu(f)"] & TM'|_{N'}/TN' = \nu_{N'}
\end{tikzcd}$$

Note that the $\nu(f)$ is well-defined here since $df(TN) \subset TN'$ by assumption.

Definition 1.1.3. For a smooth map $f : (M, N) \to (M', N')$ of manifold pairs, we will call the vector bundle morphism

$$\nu(f) : \nu(M, N) \to \nu(M', N')$$

the *linear approximation of* f.

Example 1.1.4. Consider a smooth map $f : \mathbb{R} \to \mathbb{R}$ such that $f(0) = 0$. The normal bundle of $\{0\}$ in $\mathbb{R}$ is of course just $\mathbb{R}$ itself. A simple calculation shows that $v(f) : \mathbb{R} \to \mathbb{R}$ is equal to the first term in the Taylor polynomial of f, $\nu(f)(x) = \frac{df}{dx}(0)x$.

1.1.1 Normal bundle functor and tangent bundle

One can show that the normal bundle functor interacts nicely with the tangent functor (i.e. taking the tangent bundle). Particularly, we have a canonical isomorphism

$$T\nu(M, N) \simeq \nu(TM, TN), \tag{1.1}$$

where this is an isomorphism as vector bundles over $\nu(M, N)$, as well as TN (a so called double vector bundle). See Appendix A.1 for details.

This fact is important in a special case of the construction above, when $X \in \mathfrak{X}(M)$ is a vector field *tangent* to N (i.e. $X_p \in T_pN$ for all $p \in N$). In this case we can view X as a morphism $X : (M, N) \to (TM, TN)$, and obtain the morphism

$$\nu(X) : \nu(M, N) \to \nu(TM, TN).$$

Since we have $\nu(TM, TN) \simeq T\nu(M, N)$, we can then view $\nu(X)$ as vector field on the normal bundle, so $\nu(X) \in \mathfrak{X}(\nu(M, N))$. This example will be of particular importance in this paper.

Definition 1.1.5. Let $X \in \mathfrak{X}(M)$ a vector field that is tangent to N. The *linear approximation of* X is given as the induced vector field $\nu(X) \in \mathfrak{X}(\nu(M, N))$.

As we will see in Section 1.3, this idea can also be applied to other kinds of objects, for example differential forms.

1.1.2 Tubular neighborhoods

To finish this first part, we define a notion of a tubular neighborhood, that will be convenient in our setting.

Definition 1.1.6. A *tubular neighborhood embedding* is a smooth embedding $\varphi : O \subseteq \nu(M, N) \to M$, where O is an open neighborhood of N, such that the zero section is mapped to $N \subseteq$, i.e $\varphi|_N = id$. Additionally, φ should satisfy $\nu(\varphi) = id$, under the canonical identification $\nu(\nu_N, N) \simeq \nu_N$.

Remark. In a lot of works Definition 1.1.6 is given without the last condition. It makes sense to require it in our context though, as we will see in a number of instances later.

For the last part in Definition 1.1.6, we used the following lemma.

Lemma 1.1.7. *Let $\pi : E \to M$ be a vector bundle. Then for M viewed as the zero section of E, the normal bundle $\nu(E, M)$ is canonically isomorphic to E.*

Proof. This follows from the identification $TE|_M \simeq TM \oplus E$, which comes from the exact sequence

$$0 \to E \to TE|_M \to TM \to 0$$

that is split since the projection map $\pi : E \to M$ can be split by the inclusion of the zero section. Exactness can be checked pointwise, see [5]. □

While the normal bundle abstractly gives the 'linear approximation' of M around N, a tubular neighborhood is a concrete realization of this. The following theorem is a well-known result.

Theorem 1.1.8 (Tubular neigborhood theorem)**.** *Let $N \subseteq M$ be an embedded submanifold. Then there exists a tubular neighborhood embedding for N.*

Proof. For a proof using Riemannian metrics, see for example [5]. Here a tubular neighborhood is defined without the extra condition that $\nu(\varphi) = id$. A closer inspection however reveals that the exponential map used in the construction of this proof satisfies this property, so that the theorem also holds using our definition. □

1.2 The linear approximation

In the previous section we have defined the linear approximation $\nu(f)$ of a map f between manifold pairs. We have also seen that we can apply this construction to vector field that are tangent to the submanifold N. The resulting linear approximation is then a vector field on the normal bundle in a natural way. Since this construction was given very abstractly, to gain more insight we will do some explicit computations in this section.

1.2.1 In local coordinates

To start, let us calculate what the linear approximation of a vector field $X \in \mathfrak{X}(M)$, tangent to N, looks like in local coordinates on the normal bundle. On some small open neighborhood $U \subseteq M$ around a point $p \in N$, choose coordinates $(x_1, \dots x_n, y_1, \dots y_{m-n})$ where the x_i are in the direction of N, and the y_j in the normal direction. So N can in these coordinates be given as $N = \{y_i = 0\}$.

We can then express a vector field $X \in \mathfrak{X}(M)$ on this neighborhood as

$$X = \sum_{i=1}^{n} a^i(x, y) \frac{\partial}{\partial x_i} + \sum_{j=1}^{m-n} b^j(x, y) \frac{\partial}{\partial y_j} \tag{1.2}$$

where the a^i and b^j are smooth functions on the neighborhood U. The condition that X is tangent to N then means precisely that the functions $b^j(x,y)$ vanish on N, i.e. that

$$b^j(x,0) = 0 \text{ for all } x \in N.$$

Viewing X as a map $X : M \supseteq U \to TU \subseteq TM$, we can write

$$X(x,y) = (x, y, \vec{a}(x,y), \vec{b}(x,y)), \qquad \text{in the coordinates } (x_i, y_j, v_i, w_j) \text{ on } TM|_U \ .$$

We can then directly calculate $dX : TM \to T(TM)$ in these coordinates. Let us do this calculation, but to simplify notation a bit assume that N is 1-dimensional and M is 2-dimensional, and that we have global coordinates $(x,y) = (x_1, y_1)$ on M. In these coordinates, consider

$$X : M \to TM, \quad (x,y) \mapsto (x, y, a(x,y), b(x,y)) =: (X^1, X^2, X^3, X^4)$$

For $dX : TM \to T(TM)$ we then see, in coordinates (x,y,v,w) on TM:

$$\begin{aligned} dX(x,y,v,w) &= (x, y, a, b, \frac{\partial X^1}{\partial x} v + \frac{\partial X^1}{\partial y} w, \frac{\partial X^2(x,y)}{\partial x} v + \frac{\partial X^2}{\partial y} w, \\ &\qquad \frac{\partial X^3}{\partial x} v + \frac{\partial X^3}{\partial y} w, \frac{\partial X^4}{\partial x} v + \frac{\partial X^4}{\partial y} w) \\ &= (x, y, a(x,y), b(x,y), v, w, \frac{\partial a}{\partial x} v + \frac{\partial a}{\partial y} w, \frac{\partial b}{\partial x} v + \frac{\partial b}{\partial y} w) \end{aligned} \tag{1.3}$$

Now we want to obtain the induced map $\nu(X) : TM|_N/TN \to T(TM)|_{TN}/T(TN)$. First note that restricting to $TM|_N$ in coordinates means setting $y = 0$. Using that $b(x,0) = 0$ by assumption, this means that in (1.3) the 2nd and 4th coordinate become zero, so that we get indeed an element of $T(TM)|_{TN}$. Taking then the quotient by $T(TN)$, which is spanned by the 5th and 7th coordinate vector, means that those terms drop out. Here we can remark that $b(x,0) = 0$ for all x also implies that $\frac{\partial}{\partial x} b(x,0) = 0$, so that the resulting expression is completely independent of v. Concluding we obtain the well-defined map

$$\nu(X) : TM|_N/TN \to T(TM)|_{TN}/T(TN), \quad (x,w) \mapsto (x, a(x,0), w, \frac{\partial b(x,0)}{\partial y} w).$$

The identification $\nu(TM, TN) \simeq T\nu(M,N)$ is induced by the canonical map that 'flips' the 2nd and 3rd term. All in all, generalizing to dimension n we get the following result.

Proposition 1.2.1. *In local coordinates (x_i, y_j) in a neighborhood around $N \subseteq M$, one has for $X = \sum_{i=1}^n a^i(x,y) \frac{\partial}{\partial x_i} + \sum_{j=1}^{m-n} b^j(x,y) \frac{\partial}{\partial y_j} \in \mathfrak{X}(M)$ with $b^j(x,0) = 0$ for all j and x:*

$$\nu(X)_{(x,y)} = \sum_{i=1}^n a^i(x,0) \frac{\partial}{\partial x_i} + \sum_{j=1}^{m-n} \sum_{i=1}^{m-n} \frac{\partial b^j(x,0)}{\partial y_i} y^i \frac{\partial}{\partial y_j}$$

Remark. Looking at the expression in Proposition 1.2.1 above, we can already find good reasons why this could rightfully be called the linear approximation of a vector field on the normal bundle. As you can see, in the direction of N we take the 'constant part' of the coordinate functions a^i, while in the fiber direction we see again this first order term of the Taylor polynomial of the b^j's. So really we are looking at the linear approximation in the fiber direction of $\nu(M,N)$.

1.2.2 In terms of a Taylor expansion around N

We can look at $\nu(X)$ also in a different way. The normal bundle ν_N is in particular a vector bundle, so we have a scalar multiplication map $m_\lambda : \nu_N \to \nu_N$, for $\lambda \in \mathbb{R}$, at any point $x \in N$ given by

$$(m_\lambda)_x : (\nu_N)_x \to (\nu_N)_x,\ (x, y) \mapsto (x, \lambda y).$$

For λ non-zero, this map is a diffeomorphism. For now, fix a $\lambda > 0$. Consider again a vector field X, tangent to N, given in local coordinates as in (1.2). By choosing a tubular neighborhood embedding, we can view X as a vector field on the normal bundle, $X \in \mathfrak{X}(\nu(M, N))$. For such a vector field, we can then consider the pullback $(m_\lambda)^*(X)$.

Remark. Recall that in general for a diffeomorphism $F : M \to M'$ between two manifolds, the pullback of a vector field $X \in \mathfrak{X}(M')$ is given as

$$F^*(X)_p := (dF^{-1})_{F(p)}(X_{F(p)}).$$

Note that $m_\lambda^{-1} = m_{\lambda^{-1}}$, and that it is a linear map. So $(dm_\lambda)^{-1} = m_{\lambda^{-1}}$. In local coordinates then,

$$(m_\lambda)^* X_{(x,y)} = \sum_{i=1}^{n} a^i(x, \lambda y)\frac{\partial}{\partial x_i} + \lambda^{-1}\sum_{j=1}^{m-n} b^j(x, \lambda y)\frac{\partial}{\partial y_j}. \tag{1.4}$$

Note here the λ^{-1} term that appears in front of the $\frac{\partial}{\partial y_j}$ terms. We can Taylor expand $\lambda \mapsto b^j(x, \lambda y)$ in $\lambda = 0$, which gives

$$b^j(x, \lambda y) = b^j(x, 0) + \lambda \sum_i \frac{\partial b^j(x, 0)}{\partial y_i} y_i + \mathcal{O}(\lambda^2). \tag{1.5}$$

We can do same for the function $a^i(x, \lambda y)$. Then provided that $b^j(x, 0) = 0$ for all i (which is true if X is tangent to N), we can take the limit $\lambda \to 0$. Looking at 1.4 and 1.5, we then get the same expression as in Proposition 1.2.1. This gives the following result.

Proposition 1.2.2. *For a vector field X on $\nu(M, N)$, we can express*

$$(m_\lambda)^* X = \lambda^{-1} X_{[-1]} + X_{[0]} + \lambda X_{[1]} + \mathcal{O}(\lambda^2)$$

If X is tangent to N then the term $X_{[-1]}$ vanishes, and the limit $\lambda \to 0$ is well-defined. In that case,

$$\nu(X) = X_{[0]} = \lim_{\lambda \to 0} (m_\lambda)^* X.$$

Remark. Because of the previous lemma, we will also sometimes use the notation $X_{[0]}$ for the linear approximation $\nu(X)$ of a vector field X. Note that the 0 in brackets signifies the degree of homogeneity with respect to the scalar multiplication map m_λ. This can be different for other objects, for $f \in C^\infty(M)$ one would for example have $\nu(f) = f_{[1]}$ (provided the linear approximation exists). We will see this type of notation also in Section 1.5, when we discuss higher order approximations.

1.3 For other types of objects

1.3.1 Bi-vectors and general k-vectors

We can also define the linear approximation for other objects than vector fields, and give these a natural interpretation on the normal bundle in a similar way. The first example are *bi-vector* fields (or more generally k-vectors).

Given a $\theta \in \mathfrak{X}^2(M)$, in local coordinates (x_i, y_j) around N we can write

$$\theta(x,y) = \sum a^{ij}(x,y)\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial x_j} + b^{ij}(x,y)\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial y_j} + c^{ij}(x,y)\frac{\partial}{\partial y_i}\wedge\frac{\partial}{\partial y_j}.$$

To find $\nu(\theta)$, consider θ as a map $M \to \wedge^2 TM$. We will assume then that $\theta|_N = 0$, so that

$$a^{ij}(x,0) = b^{ij}(x,0) = c^{ij}(x,0) \text{ for all } x\in N, \text{ and } i,j.$$

We will get a well-defined induced map

$$\nu(\theta) : TM|_N/TN \to \wedge^2 T(TM|_N)/\wedge^2 T(TN) = \wedge^2 T(TM|_N/TN),$$

which we can view as a bi-vector on the normal bundle. Now we could again compute $\nu(\theta)$ directly in local coordinates, but this would be rather involved. So let us instead look again at the expansion using m_λ. We get

$$(m_\lambda^*)\theta_{(x,y)} = \sum_{i,j} a^{ij}(x,\lambda y)\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial x_j} + \lambda^{-1}\sum_{i,j} b^{ij}(x,\lambda y)\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial y_j} + \lambda^{-2}\sum_{i,j} c^{ij}(x,\lambda y)\frac{\partial}{\partial y_i}\wedge\frac{\partial}{\partial y_j}.$$

using that

$$\begin{aligned}
(m_\lambda)^*(\frac{\partial}{\partial y_i}\wedge\frac{\partial}{\partial y_j})(p) &= (m_\lambda^*\frac{\partial}{\partial y_i})\wedge(m_\lambda^*\frac{\partial}{\partial y_j})(p)\\
&= (dm_\lambda)^{-1}_{m_\lambda(p)}\frac{\partial}{\partial y_i}_{m_\lambda(p)}\wedge(dm_\lambda)^{-1}_{m_\lambda(p)}\frac{\partial}{\partial y_j}_{m_\lambda(p)}\\
&= \lambda^{-2}(\frac{\partial}{\partial y_i}\wedge\frac{\partial}{\partial y_j})(m_\lambda(p)).
\end{aligned}$$

We can now Taylor expand all the functions $a^{ij}, b^{ij} c^{ij}$ in $\lambda = 0$. This gives us an expansion

$$m_\lambda^*\theta = \lambda^{-2}\theta_{[-2]} + \lambda^{-1}\theta_{[-1]} + \dots$$

As for vector fields, the 'linear' term is the second one, given by

$$\theta_{[-1]}(x,y) = \sum_{i,j} b^{ij}(x,0)\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial y_j} + \sum_{i,j}\sum_k \frac{\partial c^{ij}}{\partial y_k}(x,0)y_k\frac{\partial}{\partial y_i}\wedge\frac{\partial}{\partial y_j}.$$

Concluding, one has for $\theta \in \mathfrak{X}^2(M)$,

$$\nu(\theta) = \theta_{[-1]} = \lim_{\lambda\to 0}\lambda\,(m_\lambda)^*\theta.$$

In a similar way we get for general $\eta \in \mathfrak{X}^k(M)$ that

$$\nu(\eta) = \eta_{[-k+1]} = \lim_{\lambda\to 0}\lambda^{k-1}\,(m_\lambda)^*\eta.$$

1.3.2 Differential forms

The construction also works for forms $\omega \in \Omega^k(M)$. Here we have to be a little more careful, since we are now working with the dual of the tangent space, and there is for example no (natural) embedding $TN^* \hookrightarrow TM^*$ (on the contrary actually).

Probably the most concrete way to go about defining the linearization of a 1-form ω is to interpret it as a map $\omega : TM \to \mathbb{R}$. The map $d(\omega)$ (note: the differential of maps, not the de Rham differential) then maps as $d(\omega) : T(TM) \to T\mathbb{R} \simeq \mathbb{R}^2$. To get a well-defined map $\nu(\omega) : T(TM)|_N/TN \to \mathbb{R}$, we need to require that $\omega|_{TN} = 0$, or equivalently $i^*\omega = 0$, where $i : N \to M$ denotes the inclusion. Write, again in coordinates (x_i, y_j) around N, for $\omega \in \Omega^1(M)$

$$\omega_{(x,y)} = \sum_{i=1}^{n} a^i(x,y)dx_i + \sum_{j=1}^{m-n} b^j(x,y)dy_j.$$

Then $i^*\omega = 0$ is equivalent to $a^i(x,0) = 0$ for all i and x. Now one can do a computation similar to the one for vector fields. Working everything out, one obtains:

$$\nu(\omega)_{(x,y)} = \sum_{i=1}^{n}\sum_{j=1}^{m-n} \frac{\partial a^i(x,0)}{\partial y_j} y_j dx_i + \sum_{j=1}^{m-n} b^j(x,0)dy_i. \tag{1.6}$$

In terms of the Taylor expansion with m_λ, we see

$$(m_\lambda^*\omega)_{(x,y)} = \sum_{i=1}^{n} a^i(x,\lambda y)\, dx_i + \lambda \sum_{j=1}^{m-n} b^j(x,\lambda y)\, dy_i.$$

Comparing with (1.6), we find that for $\omega \in \Omega^1(M)$

$$\nu(\omega) = \omega_{[1]} = \lim_{\lambda\to 0} \frac{1}{\lambda}\, (m_\lambda)^*\omega$$

and more generally for $\alpha \in \Omega^k(M)$, $k \geq 0$, we get the same description;

$$\nu(\omega) = \alpha_{[1]} = \lim_{\lambda\to 0} \frac{1}{\lambda}\, (m_\lambda)^*\alpha.$$

Remark. Looking at expression 1.6 for the linearization of a 1-form, the reader will note that the roles of x_i and y_j are essentially swapped when compared to the case of vector fields. This can be explained by noting that while vector fields are sections of the tangent bundle TM, forms are sections of the cotangent bundle T^*M. Now, looking at the normal bundle one notes that it comes with the following exact sequence

$$0 \longrightarrow TN \longrightarrow TM|_N \longrightarrow TM|_N/TN \longrightarrow 0$$

Dualizing gives the exact sequence

$$0 \longrightarrow (TM|_N/TN)^* \longrightarrow (TM|_N)^* \longrightarrow TN^* \longrightarrow 0$$

Since the kernel of the map $(TM|_N)^* \to TN^*$ is the set $Ann(TN)$, we have $(TM|_N/TN)^* \simeq Ann(TN)$. This set is spanned in local coordinates by the dy_i's. Contrasting this with TN, which is spanned by the elements $\frac{\partial}{\partial x_i}$, gives some intuition why the coordinates should be "swapped".

1.4 General facts

We will finish our discussion of the linear approximation ν by showing some useful facts about this operation.

Proposition 1.4.1. *The following properties are satisfied:*

(i) If $\varphi : \nu(M,N) \to M$ is a tubular neighborhood embedding, and $X \in \mathfrak{X}(M)$ is tangent to N, then

$$\nu(\varphi^* X) = \nu(X)$$

(ii) For any $\omega \in \Omega^k(\nu(M,N))$,

$$d\nu(\omega) = \nu(d\omega)$$

(iii) If additionally $X \in \mathfrak{X}(\nu(M,N))$, we have

$$\nu(\iota_X \omega) = \iota_{X_{[0]}} \omega_{[1]} = \iota_{\nu(X)} \nu(\omega)$$

(iv) If $\varphi_1 : (M,N) \to (M',N')$ and $\varphi_2 : (M',N') \to (M'',N'')$ are maps of manifold pairs, then

$$\nu(\varphi_1 \circ \varphi_2) = \nu(\varphi_1) \circ \nu(\varphi_2).$$

Proof. All of these statements can be verified in local coordinates. $\square$

Remark. Statement (i) of Proposition 1.4.1 tells us that, when determining the linear approximation of some object, we are allowed to first pull it back to the normal bundle using a tubular neighborhood embedding. This means we can from the beginning just assume that our object is already defined on the normal bundle. Note that this justifies our computations using $(m_\lambda)^*$ in Section 1.2.2.

1.5 Higher order approximations

Consider a l-form $\omega \in \Omega^l(M)$ such that $\iota^*\omega = 0$. By choosing a tubular neighborhood embedding, we can view ω as a l-form on the normal bundle, $\omega \in \Omega^l(\nu(M,N))$. Then in general we have

$$(m_\lambda)^*\omega = \lambda\omega_{[1]} + \lambda^2\omega_{[2]} + \dots$$

where as before $\omega_{[d]}$ is the part of ω that is homogeneous of degree d (with respect to m_λ). And we have seen that for the linear approximation $\nu(\omega) = \omega_{[1]} = \lim_{\lambda\to 0} \frac{1}{\lambda}(m_\lambda)^*\omega$. Now let us consider the situation where $\omega_{[1]} = 0$, so that in particular for example there are no linear terms in the local coordinate expression of ω. It then makes sense to look at the 'quadratic approximation' of ω, which would be given by

$$\omega_{[2]} = \lim_{\lambda\to 0} \frac{1}{\lambda^2}(m_\lambda)^*\omega \quad \text{(note that this limit is now well-defined).}$$

Definition 1.5.1. Let $\omega \in \Omega^l(\nu(M,N))$. The *k-th order approximation* of ω is defined, if it exists, as the limit

$$\omega_{[k]} = \lim_{\lambda\to 0} \frac{1}{\lambda^k}(m_\lambda)^*\omega$$

This limit is well-defined if $\omega_{[i]} = 0$ for all $0 \le i < k$, and then $\omega_{[k]}$ is a l-form on $\nu(M,N)$, homogeneous of degree k.

Remark. Of course, one can give a similar definition for the k-th order approximation of an l-vector field $\theta \in \mathfrak{X}^l(\nu(M,N))$ on $\nu(M,N)$.

Example 1.5.2. If $f \in C^\infty(\mathbb{R}^n)$ such that $f(0) = 0$ and also $Df(0) = 0$ (i.e. f has a critical point at 0), then the quadratic approximation of f is given as

$$f_{[2]}(x) = \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial f(0)}{\partial x_i x_j} x_i x_j$$

which is of course the second term of its Taylor expansion around 0.

Chapter 2

Euler-like vector fields

In this chapter we will see that tubular neighborhoods embeddings as we have defined them, are in one-to-one correspondence with a particular set of vector fields around N, called *Euler-like*. This forms the basis of our approach to proving general normal form theorems. Before we can define what an Euler-like vector field is, we first need to look at the *Euler vector field* on $\nu(M,N)$. We will see that this vector field satisfies some useful properties, and is also important conceptually.

2.1 The Euler vector field

Definition 2.1.1. On the normal bundle $\nu(M,N)$ we will denote for $\lambda \in \mathbb{R}$

$$m_\lambda : \nu(M,N) \to \nu(M,N)$$

the fiberwise multiplication map, coming from the vector bundle structure.

Remark. Choose local coordinates (x_i, y_i) on $\nu(M,N)$, where the x_i are in the direction of N (identified as the zero section of $\nu(M,N)$, and the y_i are in the fiber direction. Then m_λ is given as

$$m_\lambda : (x_i, y_i) \mapsto (x_i, \lambda y_i)$$

Definition 2.1.2. The *Euler vector field* (of N) is given as the unique vector field $\mathcal{E} \in \mathfrak{X}(\nu(M,N))$ with flow $\Phi^{\mathcal{E}}$ given as

$$\Phi_t^{\mathcal{E}}(x) = m_{\exp(t)}(x)$$

for all $t \in \mathbb{R}$, $x \in \nu(M,N)$.

In local coordinates one can work out that $\mathcal{E}$ is given as

$$\mathcal{E} = \sum_i y_i \frac{\partial}{\partial y}$$

Remark. In [2] and [16], the flow of a vector field is defined with a minus sign. This explains why in these two papers it is stated that the flow of $\mathcal{E}$ is given as $m_{\exp(-t)}$.

Since the flow of $\mathcal{E}$ is equal to the map m_λ, we can use the Lie derivative of $\mathcal{E}$ to determine facts about the homogeneity of functions around N. This will be very important later on.

Definition 2.1.3. For a smooth function f on $\nu(M,N)$, we say that f is *homogeneous of degree d* (with respect to m_λ) if

$$(m_\lambda)^* f = \lambda^d f \quad \text{for all } \lambda \in \mathbb{R}.$$

We can define similarly the homogeneity of vector fields and forms on $\nu(M,N)$.

Proposition 2.1.4. *For $f \in C^\infty(\nu(M,N))$,*

$$\mathcal{L}_\mathcal{E} f = d \cdot f$$

if and only if f is homogeneous of degree d, with $d \geq 0$.

Proof. By definition,

$$\begin{aligned}
\mathcal{L}_\mathcal{E} f &= \tfrac{d}{dt}\big|_{t=0} \ (\Phi_t^\mathcal{E})^* f = \big(\tfrac{d}{dt}\big|_{t=0} \ (m_{\exp(t)})^* f\big) \\
&= \tfrac{d}{dt}\big|_{t=0} \ \big(f_{[0]} + e^t f_{[1]} + e^{2t} f_{[2]} + \dots\big) \\
&= f_{[1]} + 2 f_{[2]} + \dots .
\end{aligned}$$

If f is homogeneous of degree d, then we have $f = f_{[d]}$, and the statement of the proposition clearly holds. For the converse, we can work again in local coordinates (x_i, y_j). Then

$$\tfrac{d}{dt}\big|_{t=0} \ \big((m_{exp(t)})^* f\big)(x,y) = \tfrac{d}{dt}\big|_{t=0} \ f(x, e^t y) = \sum_j y_j \frac{\partial f}{\partial y_j}(x,y).$$

So by assumption $d \cdot f = \sum_j y_j \frac{\partial f}{\partial y_j}$. Now we apply Euler's Homogeneous function theorem (Theorem 2.1.7 below), from which the statement follows. ☐

Proposition 2.1.5. *For $X \in \mathfrak{X}(\nu(M,N))$, tangent to N, $\mathcal{L}_\mathcal{E} X = [X, \mathcal{E}] = 0$ if and only if X is homogeneous of degree 1, i.e. $\nu(X) = X$.*

Proof. It is instructive to look at the expression $[X, \mathcal{E}]$ in local coordinates (x_i) on M, and consider the case where in these coordinates $N = \{0\}$. Let $X = \sum_i X^i \frac{\partial}{\partial x_i}$.Then by the formula for the Lie bracket of vector fields

$$\begin{aligned}
[X, \mathcal{E}] &= \sum_j \Big(\sum_i X^i \frac{\partial \mathcal{E}^j}{\partial x_i} - \sum_i \mathcal{E}^i \frac{\partial X^j}{\partial x_i}\Big) \frac{\partial}{\partial x_j} \\
&= \sum_j \Big(X^j - \sum_i x_i \frac{\partial X^j}{\partial x_i}\Big) \frac{\partial}{\partial x_j}
\end{aligned}$$

So we have $[X, \mathcal{E}] = 0$ precisely if $X^j = \sum_i x_i \frac{\partial X^j}{\partial x_i}$ for all j, i.e. if all the coordinate functions are linear. ☐

We used the following lemma in the proof above.

Lemma 2.1.6. *A function $f \in C^\infty(\mathbb{R}^n)$ is linear if and only if*

$$f(x) = \sum_{i=1}^n x_i \frac{\partial f(x)}{\partial x_i}.$$

Proof. Clearly letting f be linear implies the other statement. Fix $x \in \mathbb{R}^n$, and let $g : \mathbb{R} \to \mathbb{R}$ defined by $g(t) := f(e^t x)$. Then

$$g'(t) = \sum_i \frac{\partial f(e^t x)}{\partial x_i} \cdot (e^t x_i) = f(e^t x) = g(t).$$

So $g(t) = C \cdot e^t$, where since $g(0) = f(x)$ we see $C = f(x)$. So $f(e^t x) = g(t) = f(x) e^t$, so f commutes with scalar multiplication. By Lemma 2.1.9 below, this implies that f is linear. ☐

This lemma is a special case of a theorem known as *Euler's Homogeneous function theorem.* This theorem can be proved in a similar way.

Theorem 2.1.7 (Euler's Homogeneous function theorem)**.** *A function $f \in \mathbb{C}^\infty(\mathbb{R}^n)$ is homogeneous of degree d (i.e. $f(\lambda x) = \lambda^d f(x)$) if and only if*

$$\sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i} = d \cdot f.$$

2.1.1 Remark on the philosophy

Besides the useful properties of the Lie derivative of $\mathcal{E}$, this vector field is also conceptually of importance. The philosophy here is that since the flow of $\mathcal{E}$ is the map m_λ, $\mathcal{E}$ essentially captures the vector bundle structure of $\nu(M, N)$. For example, this is indicated by Theorem 2.1.8 below.

Theorem 2.1.8. *A map $\Phi : E \to F$ between two vector bundles over M is a vector bundle morphism if and only if it intertwines the scalar multiplication maps on E and F.*

Proof. Note that on both vector bundles, $m_0 = \pi$ (the projection map). So such a Φ commutes with the projection maps. The maps $\Phi_p : E_p \to F_p$ on the fibers commute with scalar multiplication and are smooth. By the lemma below, this implies that they are linear. So Φ is a morphism of vector bundles. □

Lemma 2.1.9. *A function $f \in C^\infty(\mathbb{R}^n)$ is linear if and only if it commutes with scalar multiplication.*

Proof. For an arbitrary $x \in \mathbb{R}^n$, define $\gamma : \mathbb{R} \to \mathbb{R}$ by $\gamma(t) := tf(x)$. Then $y'(0) = f(x)$. Moreover, since $tf(x) = f(tx)$, we also have that $y'(0) = (Df)_0(x)$ by the chain rule. So $f = Df_0$, so in particular it is linear. □

Following the work of Grabowski e.a, one can give an even stronger statement. The theorem below makes concrete the relation between a 'scalar multiplication map' and a vector bundle structure. The theory behind this is somewhat involved, and not that relevant for our purposes. It is best viewed within the context of *graded bundles* (a generalization of vector bundles), which will play a role in a later part of the thesis. See Appendix A.2 for an outline of some important definitions and theorems.

Theorem 2.1.10 (Grabowski, Rotkiewicz [10])**.** *An action $h : \mathbb{R} \times E \to E$ from the monoid $(\mathbb{R}, \cdot)$ comes from a vector bundle structure $\pi : E \to E_0 = h_0(E)$ if, for the curve $\mathbb{R} \to E$, $t \mapsto h(t, p)$, the 1-jet vanishes if and only if $p \in E_0$.*

2.2 Definition of Euler-like vector fields

In the first section we saw that for a vector field X tangent to N, there is the natural 'operation' of taking the linear approximation on the normal bundle, given by $\nu(X)$. With the idea in mind that the Euler vector field is characterizing for the normal bundle, it makes sense that the vector fields in the definition below will be of special interest.

Definition 2.2.1. A vector field $X \in X(M)$is called *Euler-like* if it is tangent to N, and additionally

$$\nu(X) = \mathcal{E}$$

.

Remark. The condition of being tangent to N can be replaced by the (a priori) stronger condition that $X|_N = 0$. Note that this holds true for $\mathcal{E}$ (in $\nu(M,N)$), and it follows that these definitions are equivalent.

In local coordinates (x_i, y_j) around N, X is Euler-like if it can be given as

$$X_{(x,y)} = \sum_{i=1}^{n}\sum_{k=1}^{m-n} y_k g^{ik}(x,y)\frac{\partial}{\partial x_i} + \sum_{j=1}^{m-n}\big(y_j + \sum_{k,l=1}^{m-n} y_k y_l h^{jkl}(x,y)\big)\frac{\partial}{\partial y_j}.$$

Here g^{ik} and h^{jkl} should be smooth functions on the coordinate domain. A different characterization, which can be useful in certain circumstances, is given by the proposition below.

Proposition 2.2.2. *Denote by $\mathcal{I} \subseteq C^\infty(M)$ the vanishing ideal of N. Then a vector field X is Euler-like if*

$$\mathcal{L}_X f - kf \in \mathcal{I}^{k+1} \text{ for all } f \in \mathcal{I}^k$$

Proof. Easiest in local coordinates, see [7] (the proof uses Hadamard's lemma). □

In Section 2.3 below we will see the real reason why Euler-like vector fields are interesting to us. Namely, they are in one-to one correspondence with tubular neighborhoods.

2.3 Main theorem

Theorem 2.3.1. *Let $N \subseteq M$ be an embedded submanifold. Any Euler-like vector field X for N determines a unique maximal tubular neighborhood embedding $\varphi : U \subseteq \nu(M,N) \to M$ such that $\varphi^* X = \mathcal{E}$.*

The existence argument in the proof below is taken from [2], while the argument for uniqueness is based on [16], where it was given in the case of $(\mathbb{R}^n, \{0\})$.

Proof. Let us start by showing that such a φ exists. We can choose an initial tubular neighborhood embedding

$$\psi : V \subseteq \nu(M,N) \to M.$$

We then have that $Z := \mathcal{E} - \psi^* X \in \mathfrak{X}(V)$ has linearization equal to zero. The idea is now to use the flow of this vector field to obtain the desired embedding. We start by defining

$$Z_t := \frac{1}{t}(m_t)^* Z \quad \text{for } t > 0$$

Since $m_t^* Z = (Z_{[0]} + tZ_{[1]} + \mathcal{O}(t^2))$ and as noted, $Z_{[0]} = 0$, this smoothly extends to $t = 0$. Denote by φ_t the flow of this (time-dependent) vector field (in particular $\varphi_0 = id$).

We have that $Z_t|_N = 0$, therefore the set

$$U = \{x \in V : \varphi_t(x) \text{ exists for all } 0 \le t \le 1\}$$

is an open neighborhood of N in $\nu(M,N)$. Also, since $(m_\lambda)^* Z_t = \lambda Z_{\lambda t}$ for $0 < \lambda < 1$, we get that $m_t U \subseteq U$ for all $0 \le t \le 1$ (this is of course clear for $t = 0, 1$).

We can therefore do the following computation at all points in U:

$$\begin{aligned}\tfrac{d}{dt}\varphi_t^*(\mathcal{E}-tZ_t) &= \tfrac{d}{dt}\varphi_t^*(\mathcal{E}-(m_t)^*Z)\\ &= \varphi_t^*(-[Z_t,\mathcal{E}-(m_t)^*Z]-\frac{1}{t}(m_t)^*[\mathcal{E},Z]) \quad \text{(Lemma 2.3.4)}\\ &= \varphi_t^*(-[Z_t,\mathcal{E}]+[Z_t,tZ_t]-[\mathcal{E},Z_t])=0\end{aligned}$$

Here we used that $\frac{d}{dt}(m_t)^*Y = \frac{1}{t}(m_t)^*[\mathcal{E},Y]$ for any vector field Y, which can be verified by a computation in local coordinates (compare to Prop. 2.1.5). Also we used that $(m_t)^*\mathcal{E}=\mathcal{E}$. We conclude that $\varphi_t^*(\mathcal{E}-tZ_t)$ is independent of t, hence

$$(\varphi_0)^*(\mathcal{E}) = \mathcal{E} = (\varphi_1)^*(\mathcal{E}-Z) = \varphi_1^*(\psi^*X)$$

Thus taking $\varphi := \psi\circ\varphi_1 : U \to M$, we obtain our tubular neighborhood embedding. What is left to show is that $\nu(\varphi)=id$. Since Z_t vanishes up to second order around N (with respect to coordinates on the normal bundle) for any t, we will have $\nu(\varphi_t)=id$ for all t, so the same follows for φ.

Now for uniqueness, assume that both φ_1 and φ_2 are tubular neighborhood embeddings such that $\varphi_1^*X=\varphi_2^*X=\mathcal{E}$. Consider then the composition $\varphi_2^{-1}\circ\varphi_1 : \nu(M,N)\to\nu(M,N)$ (defined on neighborhoods of N). Then we have

$$(\varphi_2^{-1}\circ\varphi_1)_*\mathcal{E}=\mathcal{E}.$$

So, by Lemma 2.3.3, $(\varphi_2^{-1}\circ\varphi_1)\circ m_\lambda = m_\lambda\circ(\varphi_2^{-1}\circ\varphi_1)$, in other words $(\varphi_2^{-1}\circ\varphi_1)$ commutes with the scalar multiplication. This means that the map is 'linear', that is we have $\nu(\varphi_2^{-1}\circ\varphi_1)=\varphi_2^{-1}\circ\varphi$. But since φ_1 and φ_2 are both tubular neighborhoods, we should have $\nu(\varphi_2^{-1}\circ\varphi_1)=id$. It follows that $\varphi_2^{-1}\circ\varphi_1=id$, and similarly $\varphi_1^{-1}\circ\varphi_2=id$, so that we must have $\varphi_2=\varphi_1$. □

Remark. A lot of normal form theorems (especially in symplectic geometry) can be proved with a 'Moser type' argument. This then involves an argument based on the Moser trick, see for example Section 7.2 in [3]. Here one finds a time-dependent vector field X_t and uses its flow to find a suitable diffeomorphism, very similar to what we do in Theorem 2.3.1. The argument is roughly speaking as follows.

Moser trick: Assume we are given a smooth family $(\omega_t)_{t\in[0,1]}$ of symplectic forms, such that

$$\tfrac{d}{dt}\omega_t = d\alpha_t \quad \text{for some } \alpha_t\in\Omega^1(M)$$

We would then like to find a smooth isotopy φ_t such that for all t, $\varphi_t^*\omega_t=\omega_0$ (typically, what one is really interested in is that $\varphi_1^*\omega_1=\omega_0$). The trick is now to try to obtain φ_t as the flow of a time-dependent vector field X_t, i.e. $\frac{d}{dt}\varphi_t = X_t\circ\varphi_t$, and $\varphi_0=id$. Say such a φ_t is given, then if $\frac{d}{dt}\varphi_t^*\omega_t=0$, the condition is certainly satisfied. With Lemma 2.3.4 and Cartan's formula, we have

$$\tfrac{d}{dt}\varphi_t^*\omega_t = \varphi_t^*(\tfrac{d}{dt}\omega_t + d\iota_{X_t}\omega_t).$$

Therefore defining X_t by $\alpha_t+\iota_{X_t}\omega_t=0$ will do the trick.

Remark. If we assume that N is a compact submanifold, then given an Euler-like vector field X, we can always multiply X by some bump function around N to make sure that X is complete. In Theorem 2.3.1 above, we can then always take U as the whole $\nu(M,N)$. This makes the notation a bit more convenient, which is why this assumption is used in [16] (here N is, somewhat ambiguously, assumed to be *closed*, which one can assume to mean compact without boundary).

If one is given some tubular neighborhood embedding, the push-forward of $\mathcal{E}$ under this map will determine an Euler-like vector field around N. Conversely, in the proof above we saw that the flow of an Euler-like vector field can be used to construct a tubular neighborhood embedding. Following [2], we can make this more precise.

The idea is to obtain φ^{-1} completely in terms of the flow of X. Since $\varphi_*\mathcal{E} = X$, it holds that $\Phi_t^X = \varphi \circ \Phi_t^{\mathcal{E}} \circ \varphi^{-1}$ Let us first look at the image of φ, which will be the domain of φ^{-1}. By definition, the flow of $\mathcal{E}$ is given by $m_{exp(t)} : \nu_N \to \nu_N$. Put differently, we have $m_t = \Phi^{\mathcal{E}}_{\log t}$. Let us define for convenience $\lambda_t := \Phi^X_{\log t}$.

Note now that since ν_N is invariant under m_t, it follows that its image under φ is invariant under λ_t, by using that $\Phi_t^X \circ \varphi = \varphi \circ \Phi_t^{\mathcal{E}}$. Note that $\lim_{t\to 0} m_t$ gives the projection map of ν_N onto its zero section, i.e. the projection to N. It follows that

$$\varphi(\nu_N) = \{p \in M : \lim_{t\to 0} \lambda_t(p) \text{ exist and lies in } N\} =: U$$

Clearly $\varphi(\nu_N)$ is contained in U, the reverse inclusion follows by uniqueness of integral curves. So we see that the image of φ is indeed determined fully by the flow of X. Now for the map φ^{-1}. We have seen that the equality

$$\lambda_t \circ \varphi(v) = \varphi \circ m_t(v) \in M$$

hold for all $v \in \nu_N$. We can differentiate with respect to t on both sides to get

$$\tfrac{d}{dt}\big|_{t=0} \; \big(\lambda_t \circ \varphi(v)\big) = (d\varphi)_{m_0(v)}\big(\tfrac{d}{dt}\big|_{t=0} \; m_t(v)\big) \in T_pM$$

where $p = \varphi(m_0(v)) \in N$. We can then pass to the quotient T_pM/T_pN (i.e. go again to the normal bundle of N), and using that $\nu(\varphi) = id$ we see

$$\tfrac{d}{dt}\big|_{t=0} \; \big(\lambda_t \circ \varphi(v)\big) \mod T_pN = \big(\tfrac{d}{dt}\big|_{t=0} \; m_t(v)\big) \mod T_pN$$

Looking at this last term, we see that this is just the element v again. Therefore, we must have

$$\varphi^{-1}(x) = (\tfrac{d}{dt}\big|_{t=0} \; \lambda_t(x)) \mod T_pN \text{ for all } x \in \varphi(\nu_N).$$

Concluding, we have proved the following Proposition.

Proposition 2.3.2. *Suppose that $X \in \mathfrak{X}(M)$ is Euler-like. Then the inverse of the associated tubular neighborhood $\varphi : U \subseteq \nu(M, N) \to M$ can be given explicitly as*

$$\varphi^{-1}(x) = \big(\tfrac{d}{dt}\big|_{t=0} \; \lambda_t(x)\big) \mod T_pN \textit{ for all } x \in \varphi(\nu_N). \tag{2.1}$$

where $\lambda_t := \Phi^X_{\log t}$

In the proofs above we used two technical lemmas, both of which are generally well-known results.

Lemma 2.3.3. *If $F : M \to N$ is a smooth map between two manifold, and $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$ are such that $F_*X = Y$, then $\Phi_t^Y \circ F = F \circ \Phi_t^X$.*

Proof. See for example [13], Proposition 9.13 ☐

Lemma 2.3.4. *Given a time-dependent vector field with flow φ_t, then for any vector field Y_t,*

$$\tfrac{d}{dt}\varphi_t^*(Y_t) = \varphi_t^*\big(\mathcal{L}_{X_t} Y_t + \tfrac{d}{dt} Y_t\big).$$

And similar for forms ω_t.

Proof. See for example [3]. ☐

Chapter 3

Examples of normal form theorems

In this section we will outline various examples of normal form theorems throughout differential geometry. We will show how to prove them using the theory of Euler-like vector fields that we have built in the previous sections.. We will see that all these proofs proceed along similar lines. We will assume the reader is familiar with the basics of symplectic geometry.

3.1 Basic theorems

We start with a proposition that follows almost directly from the definitions.

Definition 3.1.1. A vector field Y tangent to N is called *linearizable* if there exists some tubular neighborhood embedding $\varphi : \nu(M, N) \to \nu(M, N)$ such that $\varphi^*(Y) = \nu(Y)$. Such a map φ is then called a *linearization* of X.

Proposition 3.1.2. *A vector field Y tangent to N is linearizable if and only if there exists an Euler-like vector field X in some neighborhood around N such that $[Y, X] = \mathcal{L}_Y X = 0$.*

Proof. Assume we are given such a X. Let φ be the unique maximal tubular neighborhood as determined by 2.3.1, so $\varphi^*(X) = \mathcal{E}$. Then we have

$$\mathcal{L}_{\mathcal{E}}(\varphi^* Y) = \varphi^*(\mathcal{L}_X Y) = 0.$$

So $\varphi^* Y$ is linear (i.e. $\nu(\varphi^* Y) = \varphi^*(Y)$). Since $\nu(\varphi) = id$, so $\nu(\varphi^*(Y)) = \nu(Y)$, it must be equal to the linear approximation of Y.

For the other direction, assume there exists a tubular neighborhood embedding such that $\varphi^* Y = \nu(Y)$. Let $X = \varphi_* \mathcal{E}$, then X is Euler-like. Now $\varphi^*([X, Y]) = [\varphi^* X, \varphi^* Y] = [\mathcal{E}, \nu(Y)] = 0$, by Proposition 2.1.5. The map φ is a diffeomorphism on a neighborhood of N, so it follows that $[X, Y] = 0$. □

The next theorem is a famous theorem in symplectic geometry, giving the local standard form for a symplectic form.

Theorem 3.1.3 (Darboux)**.** *Let (M, ω) be a symplectic manifold of dimension $2n$. Around any point $x_0 \in M$, there exists a tubular neighborhood $\varphi : \mathbb{R}^{2n} \to M$ such that*

$$\varphi^* \omega = \omega_{std} = \sum_k dx_{2k} \wedge dx_{2k+1}.$$

Proof. By working locally, we can assume that $M = \mathbb{R}^{2n}$, $x_0 = 0$. We then want to find a tubular neighborhood φ (i.e. a diffeomorphism $\mathbb{R}^{2n} \to \mathbb{R}^{2n}$ around 0) such that $\varphi^*\omega$ has constant coefficients. From (symplectic) linear algebra we then know that there exists a linear coordinate change that puts ω into the standard form (consider ω_{x_0} as a bilinear form on $T_{x_0}M$). Using similar reasoning as in Proposition 2.1.4, $\varphi^*\omega$ is constant if

$$\mathcal{L}_{\mathcal{E}}(\varphi^*\omega) = 2\varphi^*\omega \quad \text{(note that } \omega \text{ is a 2-form)}$$

So, what we want to find is an Euler-like vector field X such that $\mathcal{L}_X\omega = 2\omega$, since then for the associated tubular neighborhood φ,

$$\mathcal{L}_{\mathcal{E}}(\varphi^*\omega) = \varphi^*(\mathcal{L}_X\omega) \ = \varphi^*(2\omega) = 2\varphi^*\omega.$$

Let now $X \in \mathfrak{X}(M)$ be determined by $\iota_X\omega = 2\alpha$, where we choose α such that $d\alpha = \omega$. The existence of such an α follows by the Poincare Lemma, using that $d\omega = 0$. The non-degeneracy of ω means that X is uniquely determined.

Using Cartan's formula, we have

$$\mathcal{L}_X\omega = d(\iota_X\omega) = 2d\alpha = 2\omega,$$

which means that we are done if we can show that this X is Euler-like. To do this, let us look at ω and α in local coordinates. Writing

$$\omega_x = \sum_{i,j} c^{ij}\, dx_i \wedge dx_j + \sum_{i,j,k} d^{ij}_k\, x_k\, dx_i \wedge dx_j + \dots, \qquad \text{where } c^{ij}, d^{ij}_k \in \mathbb{R}$$

the condition $d\alpha = \omega$ implies that (if without loss of generality we assume that α has no constant terms)

$$\alpha = \sum_{i,j} c^{ij} x_i dx_j + \text{ higher order terms }.$$

Now consider

$$\begin{aligned}\iota_{\mathcal{E}}(\sum_{i,j} c^{ij} dx_i \wedge dx_j) &= \sum_{i,j} c^{ij} x_i (dx_i \frac{\partial}{\partial x_i}) \wedge dx_j + \sum_{i,j} c^{ij} x_j dx_i \wedge (dx_j \frac{\partial}{\partial x_j}) \\ &= 2\sum_{i,j} c^{ij} x_i dx_j = 2\alpha\end{aligned}$$

We chose X such that $\iota_X\omega = 2\alpha$. Since ω is non-degenerate, X is determined by this uniquely. By the computation above, we then conclude that the first order term of X must be equal to the Euler vector field. In other words, that X must be Euler-like. $\square$

The next theorem (which is usually referred to as a lemma) forms the basis of Morse theory.

Theorem 3.1.4 (Morse Lemma for $\mathbb{R}^n$). *Let $f \in C^\infty(\mathbb{R}^n)$ such that 0 is a non-degenerate critical point of f, and assume $f(0) = 0$. Then there exists a diffeomorphism φ around 0 such that $\varphi^* f$ is quadratic on this neighborhood.*

Proof. Since $f(0) = 0 = Df$, we have that (by Taylor's theorem)

$$f(x) = \sum_{i,j} \frac{1}{2} A_{ij}(x) x_i x_j,$$

where $A(\cdot)$ is a smooth matrix valued function, with $A_{ij}(x) = A_{ji}(x)$. The partial derivatives of f are then given as

$$\frac{\partial f}{\partial x_i} = \sum_k \big(A_{ik}x_k + \frac{1}{2}\sum_l \frac{\partial A_{ik}}{\partial x_l}x_l x_k\big) := \sum_k B_{ik}(x)x_k$$

Since $A(x)$ at any x is symmetric (by how we defined it), the same holds for $B(x)$ (i.e. $B_{ik} = B_{ki}$). The non-degeneracy condition means exactly that the matrix valued function $A(x)$ is invertible near 0. Since $B(0) = A(0)$, the same is true for $x \mapsto B(x)$ in some neighborhood of 0. On this neighborhood, consider the vector field

$$X = \sum_{i,j}(A(x)B(x)^{-1})_{ij}x_i\frac{\partial}{\partial x_j}$$

Clearly, AB^{-1} is the identity matrix in its linear part, so X is Euler-like. Now,

$$\begin{aligned}
(\mathcal{L}_X f)_x = (df)_x X_x &= \sum_k B_{jk}(x)x_k\big(\sum_{i,j}(A(x)B(x)^{-1})_{ij}x_i\big)\\
&= \big(\sum_{i,j,k} B_{jk}(AB^{-1})_{ij}\, x_i x_k\big)(x)\\
&= \big(\sum_{i,j,k} B_{jk}(\sum_m A_{im}B^{-1}_{mj})x_i x_k\big)(x)\\
&= \big(\sum_{i,j,k,m} (B_{kj}B^{-1}_{jm})A_{im}x_i x_k\big)(x)\\
&= \big(\sum_{i,k,m} (BB^{-1})_{km}A_{im}x_i x_k\big)(x)\\
&= \sum_{ik} A_{ik}(x)x_i x_k = 2f(x)
\end{aligned}$$

So $\mathcal{L}_X f = 2f$. By Proposition 2.1.4, this shows that $\varphi^* f$ is homogeneous of degree 2, so is quadratic. $\square$

3.2 Group actions and equivariant versions

For our next normal form theorem, we consider a compact Lie group G acting on M. For $g \in G$, we denote the action by $a_g : M \to M$.

Definition 3.2.1. Assume $x_0 \in M$ is a fixed point of the action of a Lie group G on M, i.e. $a_g(x_0) = x_0$ for all $g \in G$. Then the *linearization* of this action is given by the differentials $(da_g)_{x_0} : T_{x_0}M \to T_{x_0}M$ for every $g \in G$.

Definition 3.2.2. A smooth action of a Lie group G on a manifold M with a fixed point $x_0 \in M$ is called *linearizable* if around $\{x_0\}$ there exist a tubular neighborhood embedding $\varphi : T_{x_0}M \to M$ that is G-equivariant, i.e. intertwines the action $a_g : M \to M$ of each element $g \in G$ with its linearization $\nu(a_g) = (da_g)_{x_0}$.

$$\begin{tikzcd}
(M, x_0) \arrow[r, "a_g"] & (M, x_0) \\
T_{x_0}M \arrow[u, "\varphi"] \arrow[r, "(da_g)_{x_0}"'] & T_{x_0}M \arrow[u, "\varphi"']
\end{tikzcd}$$

Theorem 3.2.3 (Bochner's Linearization theorem)**.** *Let G be a compact Lie group acting on M, with fixed point x_0. Then the action is linearizable around x_0.*

Proof. We claim that the theorem is proven if we can find a G-invariant Euler-like vector field X. By Lemma 2.3.3, the flow of such a vector field commutes with the group action. Equation 2.1 then gives for the induced tubular neighborhood φ

$$\begin{aligned}
\varphi^{-1}(a_g(x)) &= \tfrac{d}{dt}\big|_{t=0} \ \Phi^X_{log(t)}(a_g(x)) \\
&= \tfrac{d}{dt}\big|_{t=0} \ a_g \circ \Phi^X_{log(t)}(x) \\
&= (da_g)_{x_0} \circ \tfrac{d}{dt}\big|_{t=0} \ \Phi^X_{log(t)}(x) \\
&= (da_g)_{x_0} \varphi^{-1}(x).
\end{aligned}$$

So it satisfies the requirements.

To find such a vector field, the idea is to start with any Euler-like vector field X (for $N = \{x_0\}$) and then average it over the group action. So let $X \in \mathfrak{X}(M)$ be an Euler-like vector field. Define then

$$X^G := \int_G (a_g)_* X dg$$

where dg is the normalized *Haar measure*, which exists because G is compact. For any $h \in G$,

$$(a_h)_* X^G = (a_h)_* \int_G (a_g)_* X dg = \int_G (a_h)_*(a_g)_* X dg = \int_G (a_{gh})_* X dg = \int_G (a_g)_* X dg$$

using the left-invariance of the Haar measure (see Lemma 3.2.8 below). So X^G is G-invariant. Now the integration over G commutes with the linear approximation (we can locally differentiate under the integral sign), so consider the integrand

$$(a_g)_* X = (a_g)_*(\mathcal{E} + X_{[1]} + \text{ higher order terms}).$$

If we want the find the linearization of this expression, only the first order term will be of importance. So let us calculate

$$((a_g)_*\mathcal{E})_p := (da_g(p))_{a_g^{-1}(p)}\mathcal{E}_{a_g^{-1}(p)} = (da_g(p))_{a_g^{-1}(p)}\sum_i (a_g^{-1}(p))^i \frac{\partial}{\partial y_i}.$$

The linearization of this is, using the description in local coordinates, where $x_0 = 0$,

$$\begin{aligned}\nu((a_g)_*\mathcal{E})_y &= 0 + \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial (da_g)_{a_g^{-1}(x_0)}(a_g^{-1}(x_0))^j}{\partial y_i} y_i \frac{\partial}{\partial y_j} \\ &= (da_g)_{a_g^{-1}(x_0)} \circ (da_g^{-1})_{x_0}\big(\sum_i y_i \frac{\partial}{\partial y_i}\big) = \sum_i y_i \frac{\partial}{\partial y_i} = \mathcal{E}_y\end{aligned}$$

So we get that $\nu(X^G) = \int_G \mathcal{E} dg = \mathcal{E}$, so X^G is Euler-like. □

Looking closely at the proof of Theorem 3.2.3, we have actually proved the following.

Proposition 3.2.4. *Assume G is a compact Lie group acting on M, with a fixed point x_0. Denote the action of $g \in G$ by $a_g : M \to M$. If X is any Euler-like vector field for $\{x_0\}$, then*

$$X^G := \int_G (a_g)_* X dg$$

is a G-invariant, Euler-like vector field.

This 'trick' allows us to easily prove G-equivariant versions of the previous normal form theorems. We first give the following definition.

Definition 3.2.5. Let G be a compact Lie group acting on M, such that a submanifold $N \subseteq M$ is invariant under the group action. A *G-equivariant tubular neighborhood embedding* is a tubular neighborhood embedding $\varphi : O \subseteq \nu(M,N) \to M$ such that the following diagram commutes for all $g \in G$:

$$\begin{tikzcd} M \arrow[r, "a_g"] & M \\ \nu(M,N) \arrow[u, "\varphi"] \arrow[r, "\nu(a_g)"'] & \nu(M,N) \arrow[u, "\varphi"'] \end{tikzcd}$$

The proof of the next theorem can be given by a rather straightforward generalization of the proof Theorem 3.2.3.

Theorem 3.2.6 (Equivariant tubular neighborhood theorem)**.** *Let G be a compact Lie group acting on M, such that a submanifold $N \subseteq M$ is invariant under the group action. Then there exists a G-equivariant tubular neighborhood embedding for N.*

Proof. The theorem is as before proven if we can find a G-invariant Euler-like vector field. This follows as in Theorem 3.2.3 by considering the description of Equation 2.1. Given a such a vector field X, one obtains that for all $g \in G$,

$$\varphi^{-1}(a_g(x)) = (da_g)_{x_0} \circ \tfrac{d}{dt}\big|_{t=0}\ \Phi^X_{\log(t)}(x) \bmod T_pN.$$

from which it follows that $\varphi \circ \nu(a_g) = a_g \circ \varphi$. Now we pick any Euler-like vector field X on M, and let $X^G = \int_G (a_g)_* X dg$. Then X^G is G-invariant, and we claim that it is still Euler-like. This can be verified by the same computation in local coordinates as in Theorem 3.2.3, except that derivatives are now only taken in the fiber direction. Since $X|_N = 0$, the argument still works. □

To give one other example, consider the G-equivariant version of the Morse Lemma.

Theorem 3.2.7 (Equivariant Morse Lemma on $\mathbb{R}^n$)**.** *Assume G is a compact Lie group acting on $\mathbb{R}^n$, with fixed point x_0. Let $f \in C^\infty(\mathbb{R}^n)$ such that x_0 is a non-degenerate critical point of f with $f(x_0) = 0$, and assume furthermore that f is G-invariant, i.e. $f \circ a_g = f$ for all $g \in G$. Then we can find a G-equivariant tubular neighborhood in which f can put into quadratic form.*

Proof. As in the usual Morse Lemma, we can find an Euler-like $X \in \mathfrak{X}(M)$ such that $\mathcal{L}_X f = 2f$. Now let X^G be defined as in Proposition 3.2.4, then we still see $\mathcal{L}_{X^G} f = 2f$ (apply d to both sides of $f \circ a_g = f$). So the tubular neighborhood determined by X^G pulls back f to quadratic form, and since by construction it is G-invariant this tubular neighborhood will be G-equivariant (compare to the proof of Theorem 3.2.6). □

We used the following lemma when working with the Haar measure. Recall that on a compact group G, the Haar measure is defined as the unique left-invariant density dg such that $\int_G dg = 1$. See for example [19].

Lemma 3.2.8. *Let G be Lie group, denote by $l_g : G \to G$ the left multiplication with an element $g \in G$. For a left-invariant density λ on G, and $f \in C_c(G)$, we then have for all g in G:*

$$\int_G \big((l_g)^* f\big)\lambda = \int_G f\lambda. \tag{3.1}$$

Here $(l_g)^* f := f \circ l_g$

Proof. Since λ is left-invariant, we can write $(l_g)^*(f)\lambda = (l_g)^*(f)(l_g)^*(\lambda) = (l_g)^*(f\lambda)$. Since l_g is a diffeomorphism of G, $\int_G f d\lambda = \int (l_g)^*(f d\lambda)$ and the result follows. □

3.3 Weinstein's Lagrangian neighborhood theorem

We will end this first exposition of normal form theorems by showing another famous theorem in symplectic geometry, the Weinstein's Lagrangian neighborhood theorem.

Theorem 3.3.1 (Weinstein's Lagrangian neighborhood theorem)**.** *Let (M, ω) be a symplectic manifold, and $L \subseteq M$ a Lagrangian neighborhood. Then there exists a neighborhood U of L and a symplectomorphism $\varphi : U \to V \subseteq T^*L$ that maps L to the zero section of T^*L. Here T^*L is equipped with the canonical symplectic form of the cotangent bundle.*

Let (M, ω) be a symplectic manifold, and assume that $N \subseteq M$ is a Lagrangian submanifold, so that $\dim N = \frac{1}{2}m$ and the pullback of ω to N is zero, i.e. $\iota^*\omega = 0$. Choosing local coordinates (x_i, y_j), we know that ω must have the following form around N.

$$\omega_{(x,y)} = \sum_{i,j} a^{ij} dx_i \wedge dx_j + \sum_{i,j} b^{ij} dx_i \wedge dy_j + \sum_{i,j} c^{ij} dy_i \wedge dy_j$$

The assumption $\iota^*\omega =$ implies then that $a^{ij}(x, 0) = 0$ for all i, j. As we have done a number of times in Section 2, we can look at the linearization $\nu(\omega) = \omega_{[1]}$ in these coordinates. We see

$$\omega_{[1]} = \frac{\partial a^{ij}(x, 0)}{\partial y_k} y_k (dx_i \wedge dx_j) + b^{ij}(x, 0) dx_i \wedge dy_j$$

We have $\omega_{[1]} \in \Omega^2(\nu(M, N))$, but a priori it is not clear that this is again a symplectic form. First we note that $d(\omega_{[1]}) = (d\omega)_{[1]} = 0$, so it is a closed form. For $\omega_{[1]}$ to be symplectic, what is left to check is if $\omega_{[1]}$ is non-degenerate. Non-degeneracy is an open condition, so it suffices to check this only for points $p \in N$, where $\omega_{[1]}$ is given as

$$(\omega_{[1]})_p = \sum_{i,j} b^{ij}(x,0) dx_i \wedge dy_j \quad \text{for } p = (x,y) \in N.$$

Compare this to

$$\omega_p = \sum_{i,j} b^{ij}(x,0) dx_i \wedge dy_j + \sum_{i,j} c^{ij}(x,0) dy_i \wedge dy_j \quad \text{for } p \in N.$$

Let now X be a vector field tangent to N, so in local coordinates $X = \sum_i a^i \frac{\partial}{\partial x_i}$. Then, looking at the two expressions above, we have that at all $p \in N$, and for all $Y \in \mathfrak{X}(M)$,

$$(\omega_{[1]})_p(X_p, Y_p) = \omega_p(X_p, Y_p).$$

Since ω is non-degenerate, there thus exist some Y such that $(\omega_{[1]})_p(X_p, Y_p) \neq 0$.

Now let Y' be a vector field normal to N, so in local coordinates $Y' = \sum_j b^j \frac{\partial}{\partial y_j}$. Since N is Lagrangian, we have that $TN^\omega = TN$. In particular $Y'|_N \notin TN^\omega$, so for $p \in N$ there must exist some X', tangent to N, with $\omega_p(X'_p, Y'_p) \neq 0$. Since then we have again $(\omega_{[1]})_p(X'_p, Y'_p) = \omega'_p(X'_p, Y'_p) \neq 0$, this combined with the above shows non-degeneracy of $\omega_{[1]}$ at all $p \in N$.

It follows that $\omega_{[1]}$ is also non-degenerate at points in an open neighborhood of N. Then we can use the fact that $\omega_{[1]}$ is homogeneous with respect to the scalar multiplication to get that it is non-degenerate on the entire normal bundle (we have $(\omega_{[1]})_{(x,\lambda y)} = \lambda(\omega_{[1]})_{(x,y)}$).

Following a remark in [16], we also have the following. As we have seen above $\omega_{[1]}$ is a linear symplectic form on $\nu(M,N)$. Restricting this form to a bilinear form on $T\nu(M,N)|_N \simeq TN \oplus \nu(M,N)$ gives a non-degenerate pairing between $\nu(M,N)$ and TN. This induces a canonical isomorphism $\nu(M,N) \to T^*N$, and one can check that this map is a symplectomorphism between $(\nu(M,N), \omega_{[1]})$ and (T^*N, ω_{can}).

By the discussion above, we have that Weinstein's Lagrangian neighborhood theorem is equivalent to the theorem below.

Theorem 3.3.2. *Let (M,ω) be a symplectic manifold, and $N \subseteq M$ a Lagrangian neighborhood. Then there exist a tubular neighborhood embedding φ such that*

$$\varphi^*\omega = \omega_{[1]}$$

Proof. Choose as usual an initial tubular neighborhood around N so that ω can be consider as a 2-form on the normal bundle. We claim that we can find a primitive α (i.e. a 1-form such that $d\alpha = \omega$) by defining (the reader might notice that the construction below is similar as that in the proof of the Poincare Lemma):

$$\alpha = \int_0^1 \frac{1}{t}(m_t)^* \iota_{\mathcal{E}}\omega \, dt.$$

We can consider here the integral at $t = 0$ since both ω and $\mathcal{E}$ vanish on N, and m_0 is equal to the projection to N. To see that $d\alpha = \omega$, note that

$$\begin{aligned} d\alpha &= d\int_0^1 \frac{1}{t}(m_t)^* \iota_{\mathcal{E}}\omega \, dt \\ &= \int_0^1 \frac{1}{t} m_t^* d(\iota_{\mathcal{E}}\omega) \, dt \\ &= \int_0^1 \frac{1}{t} m_t^* \mathcal{L}_{\mathcal{E}}\omega \, dt \end{aligned}$$

using that we can (locally) differentiate under the integral sign, and the fact that $d\omega = 0$ (when applying Cartan's formula). Recall that the flow of $\mathcal{E}$ is given by $m_{\exp(t)}$. By applying a change of coordinates $t = \exp(s)$, $dt = \exp(s)ds$, we get

$$\begin{aligned}\int_{-\infty}^{0} \frac{e^s}{e^s} m^*_{\exp(s)}(\mathcal{L}_\mathcal{E}\omega) \cdot ds &= \int_{-\infty}^{0} m^*_{\exp(s)}(\tfrac{d}{dt}\big|_{t=0}\ m^*_{\exp(t)})\omega ds \\ &= \int_{-\infty}^{0} \tfrac{d}{dt}\big|_{t=0}\ (m_{\exp(s)\exp(t)})^*\omega ds \\ &= \int_{-\infty}^{0} \tfrac{d}{ds}(m_{\exp(s)})^*\omega ds \\ &= m_1^*\omega - m_0^*\omega = \omega\end{aligned}$$

So α is indeed a primitive of ω . Now clearly α pulls back to zero on N (since this is true for ω), so its linear approximation is well-defined. Using that $m_t^*\omega_{[1]} = t\omega_{[1]}$ and Prop. 1.4.1, we have

$$\alpha_{[1]} = \nu(\int_0^1 \frac{1}{t}(m_t)^*\iota_\mathcal{E}\omega\, dt) = \int_0^1 \frac{1}{t}(m_t)^*\iota_{\nu(\mathcal{E})}\nu(\omega)\, dt = \int_0^1 \iota_{m_t^*\mathcal{E}}\big(\frac{1}{t}(m_t)^*\omega_{[1]}\big)\, dt = \int_0^1 \iota_\mathcal{E}\omega_{[1]}\, dt,$$

so that $\alpha_{[1]} = \iota_\mathcal{E}\omega_{[1]}$ (note that taking the linear approximation "commutes" with the integration). Now define $X \in \mathfrak{X}(M)$ by requiring

$$\alpha = \iota_X\omega.$$

Since ω is symplectic, X exists and is determined uniquely. Taking the linear approximation in the equation above we see that $\alpha_{[1]} = \iota_{X_{[0]}}\omega_{[1]}$. Since $\omega_{[1]}$ is also symplectic, and we already saw that $\alpha_{[1]} = \iota_\mathcal{E}\omega_{[1]}$, we must have $X_{[0]} = \nu(X) = \mathcal{E}$, so X is Euler-like.

So X defines a tubular neighborhood embedding $\varphi : U \subseteq \nu(M, N) \to M$. For this φ, we can do the by now familiar computation:

$$\mathcal{L}_\mathcal{E}(\varphi^*\omega) = \varphi^*(\mathcal{L}_X\omega) = \varphi^*(d\iota_X\omega) = \varphi^*(d\alpha) = \varphi^*\omega.$$

This means $\varphi^*\omega$ is linear, so must it must be equal to $\omega_{[1]}$.

$\square$

Chapter 4

Normal forms in Poisson geometry: Conn's linearization theorem

In this section we will give a proof of an important normal form theorem in Poisson geometry. We will first introduce some basic notions in this field, after that we will talk about the theorem itself. Our goal is to prove part of the theorem using Euler-like vector fields, in the same way we did in the previous section.

4.1 Basic notions in Poisson geometry

We will start by giving a short introduction to Poisson Geometry. This is based in large part on [9].

Definition 4.1.1. Let M a manifold. A *Poisson bracket* on M is a binary operation $\{.,.\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ satisfying for all $f, g, h \in C^\infty(M)$:

1. $\{f, g\} = -\{g, f\}$ (skew-symmetry)
2. $\{f, ag + bh\} = a\{f, g\} + b\{f, h\}$ for all $a, b \in \mathbb{R}$ (linearity)
3. $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$ (Jacobi identity)
4. $\{f, gh\} = g\{f, h\} + \{f, g\}h$. (Leibniz identity)

Because of the Leibniz identity, the bracket $\{\cdot, \cdot\}$ is determined in local coordinates entirely by how it acts on the coordinate functions, i.e. by expressions of the form

$$\pi_{ij} := \{x_i, x_j\}.$$

One then has locally that

$$\{f, g\}|_U = \sum_{i,j} \pi_{ij} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}.$$

The Leibniz identity moreover means that for any $f \in C^\infty(M)$ the operation $\{f, \cdot\}$ is a *derivation*, so it corresponds to a unique vector field X_f for which $\{f, g\} = \mathcal{L}_{X_f}(g)$ (recall that there is a 1-1 correspondence between derivations on $C^\infty(M)$ and vector fields). This X_f is called the *Hamiltonian vector field* for f.

4.1.1 Poisson bivector

We can alternatively characterize a Poisson bracket in terms of a bi-vector field, i.e. a smooth section of $\Lambda^2 TM$. For this, let us first look a little closer at the spaces $\mathfrak{X}^k(M)$ of k-vector fields. Recall first that a differential k-form $\omega \in \Omega^k(M)$ can be viewed as a k-linear, alternating map

$$\omega : \mathfrak{X}^1(M) \times \ldots \times \mathfrak{X}^1(M) \to C^\infty(M).$$

A differential form is a section of $\wedge^k T^*M$, while a k-vector is a section of $\wedge^k TM$. Noting this duality, a k-vector $\theta \in \mathfrak{X}^k(M)$ can be viewed as a map (k-linear, alternating)

$$\theta : \Omega^1(M) \times \ldots \times \Omega^1(M) \to C^\infty(M).$$

Since 1-forms are determined by expressions of the type df for $f \in C^\infty(M)$, given a Poisson bracket $\{\cdot,\cdot\}$ we can define a bi-vector $\pi \in \mathfrak{X}^2(M)$ by

$$\pi(df, dg) = \{f, g\}$$

In local coordinates, we then have

$$\pi_x = \sum_{i<j} \pi^{ij}(x) \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}.$$

Conversely, given a bi-vector field π one could define a bracket by setting $\{f, g\} := \pi(df, dg)$. This bracket however will in general not satisfy the Jacobi identity, so we need an additional condition for it to be a Poisson bracket. With some work, one could find such a condition on the local structure functions π_{ij}, but it turns out there is a more convenient description available.

4.1.2 Schouten-Nijhuis bracket

To do this, we will generalize the Lie bracket to multi-vectors. What would we want for such an operation? After some deliberation, the theorem below gives the natural conditions to ask for, and these uniquely determine the operation.

Theorem 4.1.2. *There exists a unique bilinear operation* $[\cdot,\cdot] : \mathfrak{X}^k \times \mathfrak{X}^l(M) \to \mathfrak{X}^{k+l-1}(M)$ *satisfying*

1. *For $k = l = 1$, it is the usual Lie bracket.*
2. *For $k = 1$, $l = 0$, one has $[X, f] = \mathcal{L}_X(f)$ (just as for the Lie bracket in the case $l = 1$).*
3. *$[\theta, \eta] = -(-1)^{(k-1)(l-1)}[\eta, \theta]$ for $\theta \in \mathfrak{X}^k(M), \eta \in X^l(M)$ (graded skew-symmetry)*
4. *$[\theta, \eta \wedge \xi] = [\theta, \eta] \wedge \xi + (-1)^{(k-1)l}\theta \wedge [\eta, \xi]$ for $\theta \in \mathfrak{X}^k(M), \eta \in X^l(M)$ and $\xi \in \mathfrak{X}^m(M)$ (graded Leibniz)*

Proof. We can give an explicit formula (as in [9], (1.11)) by setting for $\theta \in \mathfrak{X}^l(M)$, $\eta \in X^m(M)$

$$[\theta, \eta] = \theta \bullet \eta + (-1)^{(k-1)(l-1)} \eta \bullet \theta, \tag{4.1}$$

where

$$\theta \bullet \eta(df_1, \ldots, df_{k+l-1}) = \sum_\sigma (-1)^\sigma \theta(d(\eta(df_{\sigma(1)}, \ldots, df_{\sigma(k)})), df_{\sigma(k+1)}, \ldots, df_{\sigma(k+l-1)})$$

and we sum over all $(k, l-1)$ shuffles (note the resemblance with how one might define the wedge product of forms). Then $[\theta, \eta] \in \mathfrak{X}^{k+l-1}(M)$, and one can check that this satisfies the requirements. Bi-linearity and the graded Leibniz together with the first condition then give uniqueness (the operation is determined by what it does on 1-vectors). $\square$

Definition 4.1.3. The operation $[\cdot,\cdot] : \mathfrak{X}^k \times \mathfrak{X}^l(M) \to \mathfrak{X}^{k+l-1}(M)$ is called the *Schouten-Nijhuis* bracket.

Proposition 4.1.4. *For $\theta \in \mathfrak{X}^k(M), \eta \in \mathfrak{X}^l(M), \xi \in \mathfrak{X}^m(M)$, the Schouten-Nijhuis bracket satisfies the* graded Jacobi identity

$$(-1)^{(k-1)(m-1)}[[\theta,\eta],\xi] + (-1)^{(l-1)(k-1)}[[\eta,\xi],\theta] + (-1)^{(l-1)(m-1)}[[\xi,\theta],\eta] = 0.$$

Theorem 4.1.5 (Prop. 1.2.4 in [9])**.** *Let $\pi \in \mathfrak{X}^2(M)$ be a bivector associated to some Poisson bracket $\{\cdot,\cdot\}$. For $f,g,h \in C^\infty(M)$, one has*

$$\frac{1}{2}[\pi,\pi](df,dg,dh) = \{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\}.$$

In particular, we have that $\{\cdot,\cdot\}$ satisfies the Jacobi identity if and only if $[\pi,\pi] = 0$.

4.1.3 Different viewpoint: the map $\pi^\sharp$

There is another way to look at Poisson structures, that will put what we are doing in a more general context. Given a bivector $\pi \in \mathfrak{X}^2(M)$, one gets a induced map of vector bundles

$$\pi^\sharp : T^*M \to TM$$

which is determined by requiring for $\alpha_x \in T^*_xM$,

$$\beta_x(\pi^\sharp(\alpha_x)) = \pi_x(\alpha_x,\beta_x) \text{ for all } \beta_x \in T^*_xM.$$

This also induces a map on sections, also denoted $\pi^\sharp$:

$$\pi^\sharp : \Omega^1(M) \to \mathfrak{X}^1(M).$$

Under this map, one can show that

$$\pi^\sharp(df) = X_f.$$

What does the 'Poisson condition' $[\pi,\pi] = 0$ mean for the map $\pi^\sharp$? For a useful characterization, we introduce another new bracket operation; $[\cdot,\cdot]_\pi : \Omega^1(M) \times \Omega^1(M) \to \Omega^1(M)$, given by

Definition 4.1.6.

$$[\alpha,\beta]_\pi := \mathcal{L}_{\pi^\sharp\alpha}(\beta) - \mathcal{L}_{\pi^\sharp(\beta)}(\alpha) - d(\pi(\alpha,\beta)).$$

Note that in the definition above we use the map $\pi^\sharp$ to 'take Lie derivatives' along the 1-forms α and β.

Proposition 4.1.7. *For a bi-vector $\pi \in \mathfrak{X}^2(M)$, the following are equivalent:*

(i) $[\pi,\pi] = 0$ *(where $[\cdot,\cdot]$ is the Schouten bracket)*

(ii) $[\cdot,\cdot]_\pi$ *satisfies the Jacobi identity*

The triple $(T^*M,\ \pi^\sharp,\ [\cdot,\cdot]_\pi)$ is an example of what is called a *Lie algebroid.* These are defined as follows.

Definition 4.1.8. A *Lie algebroid* over M is a triple $(A, \rho, [\cdot,\cdot]_A)$ with

- A a vector bundle over M
- $[\cdot,\cdot]_A : \Gamma(A) \times \Gamma(A) \to \Gamma(A)$ a Lie bracket on the space of sections of A
- $\rho : A \to TM$ a vector bundle morphism, called the *anchor.*

satisfying (another 'Leibniz rule'):

$$[X, fY]_A = \mathcal{L}_{\rho(X)} fY + f[X,Y]_A$$

for $X, Y \in \Gamma(A)$, $f \in C^\infty(M)$.

One of the ideas of a Lie algebroid is that it replaces the tangent bundle TM of a manifold with the vector bundle A. The Lie bracket for vector fields is replaced by the bracket on $\Gamma(A)$, while the anchor map allows one to still 'take Lie derivatives' along these sections. In the special case of Poisson structures, this gives rise to the idea of Poisson Geometry as *contravariant geometry.* That is, geometry where the tangent bundle is replaced by the cotangent bundle. Note that here there is always the choice of a Poisson bi-vector π, there is no canonical option. For more background on Lie algebroids the interested reader could look at [4], which is also what most of our treatment to come is based on.

Example 4.1.9. Some easy examples of Lie algebroids are

- $(TM,\ [\cdot,\cdot],\ id)$, the usual tangent bundle with the Lie bracket for vector fields.
- $(\mathfrak{g},\ [\cdot,\cdot],\ 0),\ M = \{pt\}$, where $(\mathfrak{g}, [\cdot,\cdot])$ is a Lie algebra.
- $(T^*M,\ [\cdot,\cdot]_\pi,\ \pi^\sharp)$, the Poisson Lie algebroid as we discussed above.

4.2 Poisson cohomology

With the idea of contravariant geometry in mind, we could wonder if there is a way find a 'contravariant version' of the de Rham cohomology. Recall that for a manifold M with tangent space TM the de Rham cohomology is given as the cohomology of the cochain complex $(\Omega^k(M), d)$, where the differential d can be given as the following formula:

$$\begin{aligned} d\omega(X_1, \dots X_{k+1}) = &\sum_{i=0}^{k} (-1)^i L_{X_i}(\omega(X_1, \dots, \hat{X}_i, \dots X_{k+1})) \\ &+ \sum_{i<j}^{k} (-1)^{i+j} \omega([X_i, X_j], X_1, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots X_{k+1}), \end{aligned} \tag{4.2}$$

for $\omega \in \Omega^k(M), X_i \in \mathfrak{X}^1(M)$. For a contravariant version of this, we would like to replace the spaces $\Omega^k(M)$ with the spaces $\mathfrak{X}^k(M)$ (instead of section of $\wedge^k T^*M$, we should consider sections of $\wedge^k TM$). To define then a differential using a similar formula as (4.2), we would need to have a notion of 'taking the Lie derivate along a 1-form', as well as a Lie bracket on the space $\Omega^1(M)$. But this is exactly what the Lie algebroid $(T^*M,\ [\cdot,\cdot]_\pi,\ \pi^\sharp)$ gives us.

Definition 4.2.1. Let (M, π) a Poisson manifold. The *Poisson differential* $d_\pi : \mathfrak{X}^k(M) \to \mathfrak{X}^{k+1}(M)$ is given by:

$$\begin{aligned} d_\pi\theta(\alpha_1, \dots \alpha_{k+1}) = &\sum_{i=0}^{k}(-1)^i L_{\pi^\sharp(\alpha_i)}(\theta(\alpha_1, \dots, \hat{\alpha_i}, \dots \alpha_{k+1})) \\ &+ \sum_{i<j}^{k}(-1)^{i+j}\theta([\alpha_i, \alpha_j]_\pi, \alpha_1, \dots, \hat{\alpha_i}, \dots, \hat{\alpha_j}, \dots \alpha_{k+1}). \end{aligned} \tag{4.3}$$

Proposition 4.2.2. *The Poisson differential d_π can be given in terms of the Schouten bracket as*

$$d_\pi(\theta) = [\pi, \theta].$$

Because of this Proposition, we have in particular that $d_\pi^2(\theta) = [\pi, [\pi, \theta]] = 2[[\pi, \pi], \theta] = 0$ (here we used the graded Jacobi identity for the second equality). So d_π is indeed a differential, and we can define a cohomology.

Definition 4.2.3. For (M, π) we define its *Poisson cohomology* $H_\pi^\bullet(M)$ as the homology of the complex $(\mathfrak{X}^k(M), d_\pi)$. Specifically,

$$H_\pi^k(M) = \ker(d_\pi^k)/\mathrm{Im}(d_\pi^{k-1}).$$

Example 4.2.4. For $k = 0$, the differential $d_\pi : X^0(M) = C^\infty(M) \to \mathfrak{X}(M)$ is given as

$$f \mapsto \pi^\sharp(df) = X_f$$

So we find that $H_\pi^0(M) = \{C \in C^\infty(M) \mid \{C, f\} = 0,\ \forall f \in C^\infty(M)\}$. This is known as the space of *Casimirs* of (M, π).

4.2.1 Lie algebroid cohomology

It should by now not come as a surprise that the Poisson cohomology fits in as a special case of a more general construction with Lie algebroids.

Definition 4.2.5. For a Lie algebroid $(A, [\cdot, \cdot]_A, \rho)$ we define the set of *k-chains* $\mathcal{C}^k(A)$ as

$$\mathcal{C}^k(A) = \{\omega : \Gamma(A) \times \ldots \times \Gamma(A) \to C^\infty(M) \mid \omega \text{ is } k\text{- linear, alternating}\}$$

The *Lie algebroid cohomology* $H^\bullet(A)$ of $(A, [\cdot, \cdot]_A, \rho)$ is then given as the homology of the chain complex $(\mathcal{C}^k(A), d_A)$, where the differential $d_A : \mathcal{C}^k(A) \to \mathcal{C}^{k+1}(A)$ is defined by

$$\begin{aligned} d_A\omega(a_1, \dots a_{k+1}) = &\sum_{i=0}^{k}(-1)^i L_{\rho(a_i)}(\omega(a_1, \dots, \hat{a}_i, \dots a_{k+1})) \\ &+ \sum_{i<j}^{k}(-1)^{i+j}\omega([a_i, a_j]_A, a_1, \dots, \hat{a}_i, \dots, \hat{a}_j, \dots a_{k+1}), \end{aligned} \tag{4.4}$$

Example 4.2.6. Consider the Lie algebroid $(\mathfrak{g}, [\cdot, \cdot], 0)$ over $M = \{pt\}$, where $\mathfrak{g}$ is some Lie algebra. Then because $\Gamma(\mathfrak{g}) = \mathfrak{g}$,

$$\mathcal{C}^k(\mathfrak{g}) = \{\omega : \mathfrak{g} \times \ldots \times \mathfrak{g} \to \mathbb{R} : \omega \text{ is } k\text{-linear, alternating,}\}$$

while $d : \mathcal{C}^k(\mathfrak{g}) \to \mathcal{C}^{k+1}(\mathfrak{g})$ is given by

$$d\omega(a_1, \ldots, a_{k+1}) = \sum_{i<j}^{k} \omega([a_i, a_j], a_1, \ldots, \hat{a_i}, \ldots, \hat{a_j}, \ldots a_{k+1})$$

This complex is known as the Chevallier-Eilenberg complex, and its cohomology is the *Lie algebra cohomology.*

4.2.2 Cohomology with coefficients in a vector bundle

Given a vector bundle $E \to M$, one can consider cohomologies with coefficients in the section of E. This requires the choice of a connection, i.e. a bilinear map

$$\nabla : \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E),\ \nabla(X, s) := \nabla_X(s)$$

that satisfies

$$\nabla_{fX}(s) = f\nabla_X(s),\ \nabla_X(f \cdot s) = \nabla_X(s) + (L_X f)s.$$

Moreover, this connection should be *flat*, meaning that

$$\nabla_X \circ \nabla_Y - \nabla_X \circ \nabla_Y - \nabla_{[X,Y]} = 0.$$

This allows us to define a differential on the complex

$$\begin{aligned}\Omega^k(M, E) &= \Gamma(\wedge^k T^*M \otimes E) \\ &= \{\omega : \mathfrak{X}^1(M) \times \ldots \times \mathfrak{X}^1(M) \to \Gamma(E) \mid C^\infty(M)\ k\text{-linear, skew-symmetric}\}\end{aligned}$$

using a by now familiar looking formula:

$$\begin{aligned}d_\nabla\omega(X_1, \ldots X_{k+1}) &= \sum_{i<j}^{k} (-1)^{i+j}\omega([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots X_{k+1}) \\ &+ \sum_{i=0}^{k} (-1)^i \nabla_{X_i}(\omega(X_1, \ldots, \hat{X}_i, \ldots X_{k+1}))\end{aligned} \tag{4.5}$$

Note here that for $d_\nabla \circ d_\nabla = 0$ to hold, we need the condition that ∇ is flat (see for example Prop. 1.30 in [5]).

In a completely similar way, we can also define the *Lie algebroid cohomology with coefficients in a representation.* Given a Lie algebroid $(A, \rho, [\cdot, \cdot]_A)$ over M, and a vector bundle $E \to M$, we require the choice of a *flat A-connection.* That is, a map

$$\nabla : \Gamma(A) \times \Gamma(E) \to \Gamma(E)$$

satisfying the same conditions as a normal connection, but with the Lie bracket on vector fields replaced by $[\cdot, \cdot]_A$, and using the anchor map when necessary.

Example 4.2.7. Consider again the situation as in Example 4.2.6, where the Lie algebroid is just given as a Lie algebra $\mathfrak{g} \to \{pt\}$. A vector bundle E over $\{pt\}$ is of course then just a vector space, while a $\mathfrak{g}$-connection should be a bilinear map

$$\nabla : \mathfrak{g} \times E \to E.$$

The condition that ∇ is flat then means we should have

$$\nabla_{[X,Y]}(e) = \nabla_X \circ \nabla_Y(e) - \nabla_Y \circ \nabla_X(e)$$

for all $e \in E$, $X, Y \in \mathfrak{g}$. If we view ∇ as a map $\nabla : \mathfrak{g} \to \mathfrak{gl}(E), X \mapsto \nabla_X(\cdot)$ this is exactly saying that ∇ is a *Lie algebra representation.*

Example 4.2.8. In the example above we can take $E = \mathfrak{g}$, and for our flat connection we could then take the adjoint representation, $ad : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}),\ ad_X = [X, \cdot]$. The k-chains are then given as

$$\mathcal{C}^k(\mathfrak{g}, \mathfrak{g}) = \{\rho : \mathfrak{g} \times \ldots \times \mathfrak{g} \to \mathfrak{g} \mid \rho\ k\text{- linear, alternating}\}$$

The differential d then also has a very concrete description, we have

$$\begin{aligned} d\rho(X_1, \ldots X_{k+1}) = &\sum_{i<j}^{k} (-1)^{i+j} \rho([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots X_{k+1}) \\ &+ \sum_{i=0}^{k} (-1)^i [X_i, \rho(X_1, \ldots, \hat{X}_i, \ldots X_{k+1})]. \end{aligned} \tag{4.6}$$

4.3 Linear Poisson structures and the Isotropy Lie algebra

In this final section about Poisson structures we will look at the important concept of the *linearization* of a Poisson structures, which relates this subject to the rest of the thesis. We are given a Poisson manifold (M, π) and a point $x_0 \in M$ that is a zero of π. The idea is now that on the space $T^*_{x_0}M$ there exists a natural Lie algebra structure. Below, we give three different ways of defining a Lie bracket on this space. These however are all equivalent, which can be verified by some straightforward computations.

Let us consider the situation locally, so we can assume $M = \mathbb{R}^n$ and $x_0 = 0$. As note before, π can then be given as

$$\pi(x) = \sum_{i,j} \pi_{ij}(x) \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}.$$

By assumption, we have $\pi(0) = 0$, so we can expand as

$$\pi(x) = \sum_{i,j,k} \frac{\partial \pi_{ij}}{\partial x_k}(0)\, x_k \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} + \text{ higher order terms}$$

Note that in the language of the previous section we have here

$$\nu(\pi) = \sum_{i,j,k} \frac{\partial \pi_{ij}}{\partial x_k}(0)\, x_k \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}.$$

Denote

$$c^k_{ij} = \frac{\partial \pi_{ij}}{\partial x_k}(0). \tag{4.7}$$

Let $\mathfrak{g} = T^*_{x_0}M$. We claim that the c^k_{ij} form *structure constants* for a Lie algebra structure on $\mathfrak{g}$, defined on the basis $\{dx_1|_0, \ldots, dx_n|_0\} =: \{e_1, \ldots e_n\}$ by

$$[e_i, e_j] = \sum_k c_{ij}^k e_k.$$

For this to be a Lie algebra, this bracket has to satisfy the Jacobi identity, which in terms of the structure constants can be formulated as

$$\sum_l c_{il}^m c_{jk}^l + c_{jl}^m c_{ki}^l + c_{kl}^m c_{ij}^l = 0 \text{ for all } i, j, k, m.$$

We know that the bracket $\{\cdot,\cdot\}$ defined by π satisfies the Jacobi identity, and a rather tedious computation in local coordinates shows that this is equivalent to

$$\sum_{l=1}^n \pi_{il} \frac{\partial \pi_{jk}}{\partial x_l} + \text{ cyclic terms } = 0 \quad \text{for all i,j,k.}$$

Applying then the partial derivative $\frac{\partial}{\partial x_m}$ on the equation above, a straightforward (but again tedious) computation shows that Jacobi identity for the structure constants is indeed satisfied.

Alternatively, the Lie bracket on $\mathfrak{g} = T^*_{x_0}M$ can be given as

$$[(df)_{x_0}, (dg)_{x_0}] := d\{f, g\}_{x_0} = \nu(\pi)(df, dg) \tag{4.8}$$

where we note that the last equality is true since the normal bundle to $\{x_0\}$ is just the whole of $TM|_{x_0}$. We of course should check if this expression is independent of the choice of f and g. However, this will also follow by consider one final other way to obtain this Lie algebra structure. It turns out that this construction works more generally for any Lie algebroid. We will need the following observation:

Lemma 4.3.1. *Let $(A, \rho, [\cdot,\cdot]_A)$ be some Lie algebroid over M, and let $x_0 \in M$. Then for $\alpha, \beta \in \mathcal{C}^1(A)$ (i.e. $\alpha, \beta : \Gamma(A) \to C^\infty(M)$) such that $\alpha_{x_0}, \beta_{x_0} \in \ker(\rho_{x_0})$ (where $\rho_{x_0} : A_{x_0} \to T_{x_0}M$), the expression*

$$[\alpha, \beta]_A(x_0) \in \ker(\rho_{x_0}$$

only depends on $\alpha_{x_0}, \beta_{x_0}$. In particular, it induces a Lie bracket on $\ker(\rho_{x_0}) \subseteq A_{x_0}$.

Proof. Consider $\beta + f\gamma$, where $\gamma \in \mathcal{C}^1(A)$ and $f \in C^\infty(M)$ such that $f(x_0) = 0$. Then

$$[\alpha, \beta + f\gamma]_A(x_0) = [\alpha, \beta]_A(x_0) + f(x_0)[\alpha, \gamma]_A(x_0) + \big(\mathcal{L}_{\rho(\alpha)}(f)\beta\big)(x_0) = [\alpha, \beta]_A(x_0).$$

Where we used that $\alpha_{x_0} \in \ker(\rho_{x_0})$, so that $\big(\mathcal{L}_{\rho(\alpha)}(f)\beta\big)(x_0) = 0$. $\square$

If x_0 is a zero of π, then for the map $\pi^\sharp$ associated to the Lie algebroid $(TM^*, \pi^\sharp, [\cdot,\cdot]_\pi)$ we have that $\ker(\pi^\sharp_{x_0}) = T_{x_0}M^*$, so $[\cdot,\cdot]_\pi$ induces a Lie algebra structure on $T^*_{x_0}M = \mathfrak{g}$.

As we mentioned before, these three ways of endowing $\mathfrak{g}$ with a Lie brackets are all equivalent. We can then give the following definition.

Definition 4.3.2. Given a Poisson manifold (M, π) and $x_0 \in M$ such that $\pi(x_0) = 0$, the *isotropy Lie algebra* is the space $\mathfrak{g} := T^*_{x_0}M$ endowed with the Lie bracket given by

$$[df_{x_0}, dg_{x_0}] = [df, dg]_\pi(x_0) = d\{f, g\}_{x_0}$$

So we know that the linearization of π induces a Lie algebra structure on $T^*_{x_0}M$. One could wonder of the opposite also holds: does a Lie bracket on $T^*_{x_0}M$ induce a Poisson bracket on $T_{x_0}M$? It turns out that this is indeed the case.

Theorem 4.3.3 (Proposition 7.3 in [12]). *Let V be a vector space. The construction above gives a one-to-one correspondence*

$$\{\textit{Linear Poisson brackets on } V\} \longleftrightarrow \{\textit{Lie algebra structure on } V^*\}$$

4.4 Conn's linearization theorem

If x_0 is a zero of a Poisson bracket, then we have the canonical structure of the isotropy Lie algebra on the space $\mathfrak{g} = T^*_{x_0}M$. Moreover (and equivalently), there is a canonical linear Poisson bracket on $\mathfrak{g}^* = T_{x_0}M$, defined by the linear approximation $\pi_{lin} = \nu(\pi)$. An interesting question one could ask is how much this linear Poisson structure resembles the Poisson bracket around x_0.

Definition 4.4.1. A Poisson structure (M, π) is called *linearizable* around a point $x_0 \in M$ with $\pi_{x_0} = 0$ if there exists some tubular neighborhood φ around x_0 such that $\varphi^*\pi = \nu(\pi)$.

The theorem that we will present below shows that a Poisson structure is linearizable at a point x_0 if its isotropy Lie algebra $\mathfrak{g}_{x_0}$ at that point satisfies a certain condition.

Definition 4.4.2. A Lie algebra $\mathfrak{g}$ is called *semi-simple of compact type* if there exist a simply connected compact Lie group G that has $\mathfrak{g}$ as its Lie algebra

Remark. As the name suggests, there are also separate notions of a *semi-simple Lie algebra* and a *compact Lie algebra* . However, it is not important for the story here to go deeper into this. But we leave the reader with the assurance that a Lie algebra that is both semi-simple and compact, is also semi-simple of compact type as defined above. The interested reader could look at [8], in particular Corollary 3.6.3. Recall also Lie's third fundamental theorem, which states that every finite-dimensional Lie algebra can be given as the Lie algebra of some unique simply connected Lie group (see for example [13], Theorem 20.21).

Theorem 4.4.3 (Conn's linearization theorem)**.** *Let (M, π) be a Poisson structure, and $x_0 \in M$ a zero of π. If the isotropy Lie algebra $\mathfrak{g}_{x_0}$ is semi-simple of compact type, then π is linearizable at x_0.*

Following the paper [6] by Crainic and Fernandes, the proof can be split into four distinct parts. The first part, which is the one relevant to us, shows that π is linearizable under certain cohomological conditions. The rest of the proof then consist of showing that these conditions are indeed satisfied if $\mathfrak{g}$ is semi-simple of compact type. We refer the interested reader to the paper mentioned above for these parts. We will restrict ourselves to the following theorem, which is where the theory of Euler-like vector fields can be used.

In the formulation of the theorem, the *local Poisson cohomology* $H^\bullet_\pi(M, x_0)$ is used. The idea is that this is basically $H^\bullet_\pi(M)$, except that instead of the whole M we can look at an arbitrarily small neighborhood of x_0. To be precise, we can give the following definition.

Definition 4.4.4. The *local Poisson cohomology* $H^\bullet_\pi(M, x_0)$ is defined as the Poisson cohomology of the germ of (M, π). That is, it is the categorical limit $\varinjlim H^k_\pi(U)$ over a decreasing filter of neighborhoods U.

Remark. In principle, we could just formulate the theorem by requiring the vanishing of the whole $H^\bullet_\pi(M)$, but that would be a far too strong condition. Note that requiring $H^2_\pi(M, x_0) = 0$ means that for any cochain $\theta \in H^2_\pi(V)$, where $V \subseteq M$ is a neighborhood of x_0, there is some neighborhood U of x_0 so that this θ is a boundary in $H^2_\pi(U)$, as well as for all neighborhoods $U' \subseteq U$.

Theorem 4.4.5. *Let (M, π) a Poisson manifold, and let x_0 be a zero of π. Assume that both $H^1(\mathfrak{g}_{x_0})$ and $H^1(\mathfrak{g}_{x_0}, \mathfrak{g}_{x_0})$ vanish. If $H^2_\pi(M, x_0) = 0$, then $\{\cdot, \cdot\}$ is linearizable at x_0.*

Proof. In the same fashion as the previous normal form theorems, we will look for a suitable Euler-like vector field X and then use its uniquely determined tubular neighborhood φ as the desired diffeomorphism. We can work locally by choosing coordinates, and view π as an bivector on the normal bundle (which is now just $T_{x_0}M$). For any bivector π on the normal bundle, $(m_\lambda)^*(\nu(\pi)) = \lambda^{-1}\nu(\pi)$. From a similar computation as in Proposition 2.1.4 we conclude that π is linear if and only if $\mathcal{L}_{\mathcal{E}}\pi = -\pi$. So we will need the vector field X to satisfies $\mathcal{L}_X\pi = -\pi$. If so, then we calculate

$$\mathcal{L}_{\mathcal{E}}(\varphi^*\pi) = \varphi^*(\mathcal{L}_X\pi) = -\varphi^*(\pi).$$

It follows that $\varphi^*(X)$ is linear, and therefore as we have seen before it must be equal to the linear approximation of X.

Our task is now to find this Euler-like vector field X. Since always $d_\pi\pi = 0$, the assumption that that $H^2_\pi(M, x_0) = 0$ already gives us that there must be some $Z \in \mathfrak{X}(M)$ (in a neighborhood of x_0) such that

$$d_\pi Z = -\mathcal{L}_Z\pi = \pi.$$

Claim. *We must have $Z_{x_0} = 0$.*

Let us work out what the equation $[\pi, Z] = -\pi$ looks like in local coordinates. Writing $Z = \sum_i z_i \frac{\partial}{\partial x_i}$, $\pi = \sum_{i,j} \pi_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$, we see

$$\begin{aligned}
[\pi, Z](df, dg) &= \pi(Z(df), dg) - \pi(dg, Z(dg)) - \mathcal{L}_Z(\pi(df, dg)) \\
&= \sum_{k,j} \pi_{kj} \Big(\frac{\partial(\sum_i z_i \frac{\partial f}{\partial x_i})}{\partial x_k} \frac{\partial g}{\partial x_j}\Big) - \sum_{i,k} \pi_{ik} \Big(\frac{\partial f}{\partial x_i} \frac{\partial(\sum_j z_j \frac{\partial g}{\partial x_j})}{\partial x_k}\Big) - \mathcal{L}_Z\Big(\sum_{i,j} \pi_{ij} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}\Big) \\
&= \sum_{i,j,k} \pi_{kj} \frac{\partial z_i}{\partial x_k} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} - \sum_{i,j,k} \pi_{ik} \frac{\partial z_j}{\partial x_k} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} + \sum_{i,j,k} z_k \frac{\partial \pi_{ij}}{\partial x_k} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}
\end{aligned}$$

where we note for the third equality that all second order derivatives cancel each other out. Now given that $\pi_{x_0} = 0$, the first two summands in the expression above will vanish at x_0, so that

$$[\pi, Z]_{x_0} = \sum_{i,j,k} z_k(x_0) \frac{\partial \pi_{ij}}{\partial x_k}(x_0) \Big(\frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}\Big)_{x_0} = 0 = -\pi_{x_0} \tag{4.9}$$

(since by assumption $[\pi, Z] = -\pi$). Using again the notation $c^k_{ij} = \frac{\partial \pi_{ij}}{\partial x_k}(x_0)$, as in Section 4.3, we have that for $Z_{x_0} \in T_{x_0}M$, viewed as a map $T^*_{x_0}M \to \mathbb{R}$ (i.e. in $\mathcal{C}^1(\mathfrak{g})$):

$$d_{\mathfrak{g}}(Z_{x_0})(e_i, e_j) = Z_{x_0}([e_i, e_j]) = \sum_k z_k(x_0) c^k_{ij} = 0$$

This holds for all i, j, because of (4.9). So Z_{x_0} is a boundary in $C^1(\mathfrak{g})$. The assumption $H^1(\mathfrak{g}) = 0$ then implies $Z_{x_0} = 0$ (note that $\mathcal{C}^0(\mathfrak{g}) = \{0\}$).

So we can conclude that the linear approximation $\nu(Z) = Z_{[0]}$ is well-defined. Now, in general we will not have $Z_{[0]} = \mathcal{E}$. However, we will show that Z differs from being Euler-like by at most a boundary in the Poisson cohomology.

To see this, let Y be any Euler-like vector field on M, so we can write $Y = \mathcal{E} + Y_{[1]} + \dots$. Let $R = Z - Y$, and consider

$$d_\pi(R) = d_\pi(Z - Y) = \pi - d_\pi(Y) = \pi - [\pi, Y]$$

A direct calculation shows that $[\pi, \mathcal{E}] = -\mathcal{L}_{\mathcal{E}}\pi = \pi_{[-1]} + \pi_{[1]} + \ldots$, so the right hand side of the equation vanishes up to second order. Therefore, it follows that

$$\nu(d_\pi(R)) = 0.$$

Let us again compute this in local coordinates. Write $R_x = \sum_{i=1}^n r^i(x)\frac{\partial}{\partial x_i}$. Note that the linear approximation ν in the current situation is just the sum of all partial derivatives at x_0. Fixing a coordinate x_m, we have

$$\begin{aligned}\frac{\partial}{\partial x_m}([\pi, R])(x_0) &= \frac{\partial}{\partial x_m}\Big(\sum_{i,j,k}(\pi_{kj}\frac{\partial r_i}{\partial x_k} - \pi_{ik}\frac{\partial r_j}{\partial x_k} + r_k\frac{\partial \pi_{ij}}{\partial x_k})(\frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j})(x_0)\\ &= \sum_{i,j,k}\Big(\frac{\partial \pi_{kj}}{\partial x_m}\frac{\partial r_i}{\partial x_k} - \frac{\partial \pi_{ik}}{\partial x_m}\frac{\partial r_j}{\partial x_k} + \frac{\partial r_k}{\partial x_m}\frac{\partial \pi_{ij}}{\partial x_k}\Big)(x_0)\frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}\Big|_{x_0} = 0\end{aligned}$$

where we note that all terms involving second derivatives vanish since we have $\pi_{x_0} = 0$, $R_{x_0} = 0$.

Define $\rho : \mathfrak{g} \to \mathfrak{g}$ by $(\rho)_{ij} = \frac{\partial r_i}{\partial x_j}(x_0)$, then $\rho \in \mathcal{C}^1(\mathfrak{g}, \mathfrak{g})$ (as in Example 4.2.6). Note that $\nu(R)_x = \sum_i \frac{\partial r_i}{\partial x_j}(x_0)x_j\frac{\partial}{\partial x_i}$. We can calculate:

$$\begin{aligned}d_{\mathfrak{g}}\rho(e_i, e_j) &= [\rho(e_i), e_j] - [e_i, \rho(e_j)] + \rho([e_i, e_j])\\ &= \sum_k[\rho_{ik}e_k, e_j] - \sum_k[e_i, \rho_{jk}e_k] + \rho(\sum_k c_{ij}^k e_k)\\ &= \sum_k \rho_{ik}[e_k, e_j] - \sum_k \rho_{jk}[e_i, e_k] + \rho(\sum_k c_{ij}^k e_k)\\ &= \sum_k \rho_{ik}\sum_m c_{kj}^m e_m - \sum_k \rho_{jk}\sum_m c_{ik}^m e_m + \sum_m\sum_k \rho_{km}c_{ij}^k e_m\\ &= \sum_{k,m}\big(c_{kj}^m\rho_{ik} - c_{ik}^m\rho_{jk} + \rho_{km}c_{ij}^k\big)e_m\end{aligned}$$

Carefully comparing the two equations above we conclude that $d_{\mathfrak{g}}\rho = 0$, so that ρ is a boundary in $\mathcal{C}(\mathfrak{g}, \mathfrak{g})$. From the assumption $H^1(\mathfrak{g}, \mathfrak{g}) = 0$ it follows then that there exists an element $v \in \mathfrak{g}$ such that $d_{\mathfrak{g}}v = \rho$. Here recall that $d_{\mathfrak{g}}v$ is given by $d_{\mathfrak{g}}v((df)_{x_0}) = [v, (df)_{x0}] = \nu(\pi)(dh, df)$.

Since $v \in \mathfrak{g} = T^*_{x_0}M$, we can write $v = (dh)_{x_0}$ for some $h \in C^\infty(M)$. Let now $X = Z - d_\pi(h)$. Then we have $d_\pi X = \pi$, and $\nu(X) = \nu(Z) - d_{\nu(\pi)}\nu(h)$. Now looking at the expressions for ρ and R, and the definition of the isotropy Lie algebra, we see that $d_{\nu(\pi)}\nu(h) = \nu(R) = \nu(Z-Y) = \nu(Z) - \mathcal{E}$. We conclude that $\nu(X) = \mathcal{E}$, so that X has the required properties, and we are done.

$\square$

Chapter 5

Weighted Euler-like vector fields

In this final part of the thesis, we will take a look at a generalization of the theory of Euler-like vector fields. This is based on the last section in Meinrenken's paper [16].

Remark. The main goal of this section will be to give an exposition of the work done in [16]. We will attempt to work out the underlying ideas, and to provide some more details. At the end, we will indicate a possible extension to the theory, not found in the paper, but here work remains to be done.

5.1 Motivating example: isotropic embedding theorem

Let us begin by looking again at the Lagrangian neighborhood theorem, this time formulated more explicitly in coordinates.

Theorem 5.1.1 (Lagrangian neighborhood, local version)**.** *Let (M, ω) a symplectic manifold of dimension $2n$, with N a Lagrangian submanifold. Then around any point $p \in N$ we can find local coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$, such that in these coordinates: $N = \{y_1 = \ldots = y_n = 0\}$ and*

$$\omega = \sum_{i=1}^{n} dx_i \wedge dy_i.$$

Consider $m_\lambda : \nu(M, N) \to \nu(M, N)$, we see that ω then satisfies

$$(m_\lambda)^* \omega = \lambda \omega,$$

i.e. it is homogeneous of degree 1. This is of course what allowed us to prove this theorem using Euler-like vector fields. There is a similar theorem in symplectic geometry (the Isotropic Embedding theorem), that gives a standard form for ω around general isotropic submanifolds. The local version is as follows.

Theorem 5.1.2 (Isotropic embedding theorem, local version)**.** *Let (M, ω) a symplectic manifold of dimension $2n$, with N a isotropic submanifold of dimension $k < n$. Then around any point $p \in N$ we can find local coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$, such that in these coordinates: $N = \{x_{k+1} = \ldots = x_n = y_1 = \ldots = y_n = 0\}$ and*

$$\omega = \sum_{i=1}^{n} dx_i \wedge dy_i = \sum_{i=1}^{k} dx_i \wedge dy_i + \sum_{i=k+1}^{n} dx_i \wedge dy_i.$$

In this last equality we are just splitting the summation into two parts. There is an important difference when comparing this theorem to Theorem 5.1.1, which stems from the fact that a fiber of the normal bundle is now spanned at each point by the tangent vectors $\{\frac{\partial}{\partial x_{k+1}}, \ldots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_n}\}$. Because of this, we have

$$(m_\lambda)^*\omega = \lambda \sum_{i=1}^{k} dx_i \wedge dy_i + \lambda^2 \sum_{i=k+1}^{n} dx_i \wedge dy_i.$$

So the 'normal form' here is not homogeneous (of any degree). However, we do have clearly one linear part and one quadratic one, both in a disjoint set of coordinates. Note that the coordinates y_i in the quadratic part are exactly those belonging to the *symplectic normal bundle*, defined as $TN^\omega/TN \subseteq \nu(M, N)$. Now what if instead of the usual scalar multiplication m_λ, we considered in local coordinates the map given by

$$k_\lambda : (x_1, \ldots, x_n, y_1, \ldots y_k, y_{k+1}, \ldots, y_n) \mapsto (x_1, \ldots x_n, \lambda^2 y_1, \ldots \lambda^2 y_k, \lambda y_{k+1}, \ldots, \lambda y_n)$$

Essentially, what we are doing then is giving these coordinates different weights. Then the normal form could be expressed by saying it is homogeneous of degree 2 under this new multiplication map, i.e. $(k_\lambda)^*\omega = \lambda^2\omega$.

In this chapter, we are going to use this idea of replacing the scalar multiplication with a weighted version. This leads to a more general version of the theory. An important concept in this will be the *weighted normal bundle*.

5.2 The situation in $\mathbb{R}^n$

To start, let us look at what happens in $\mathbb{R}^n$ if we replace our standard scalar multiplication with a weighted version. Inspiration for this section is taken from a talk given by Y. Loizides at the Friday Fish Seminar.

5.2.1 Weighted Euler-like

Definition 5.2.1. Let $\mathbf{w} = (w_1, \ldots w_n) \in \mathbb{N}^n$, called a 'weight vector'. The *weighted multiplication map* $k_\lambda : \mathbb{R}^n \to \mathbb{R}^n$ is given as

$$k_\lambda(x_1, \ldots x_n) = (\lambda^{w_1} x_1, \ldots, \lambda^{w_n} x_n)$$

We will denote $\mathbb{R}^n$ equipped with this weighted multiplication as $(\mathbb{R}^n, \mathbf{w})$.

We could then also consider the notion of the *weighted Euler vector field*

Definition 5.2.2. On $(\mathbb{R}^n, \mathbf{w})$, the *weighted Euler vector field* is defined as the unique vector field $\mathcal{E}_\mathbf{w}$ with flow given by

$$\Phi_t^{\mathcal{E}_\mathbf{w}} = k_{exp(t)}.$$

In coordinates, this means that $\mathcal{E}_\mathbf{w}$ can be given as

$$\mathcal{E}_\mathbf{w}(x) = \sum_{i=1}^{n} w_i x_i \frac{\partial}{\partial x_i}$$

In analogy with the non-weighted case, we could then define the following.

Definition 5.2.3. A vector field $X \in \mathfrak{X}(\mathbb{R}^n)$ is called *weighted Euler-like* (for $\mathbf{w}$) if its *weighted linear approximation* is equal to $\mathcal{E}_{\mathbf{w}}$. By this we mean that

$$\lim_{\lambda \to 0} (k_\lambda)^* X = \mathcal{E}_{\mathbf{w}}$$

A weighted Euler-like vector field determines a tubular neighborhood (on $\mathbb{R}^n$), in much the same way as a 'regular' one. This is shown in the next lemma.

Lemma 5.2.4. *Let $X \in \mathfrak{X}(\mathbb{R}^n)$ be weighted Euler-like for $\mathbf{w}$. Then there exists a diffeomorphism φ around 0 such that $\varphi(0) = 0$, $D\varphi(0) = id$, and $\varphi^* X = \mathcal{E}_{\mathbf{w}}$.*

Proof. The proof is very similar to the proof of Lemma **??**, the one for the non-weighted case. We can define, for $t \neq 0$, a time-dependent vector field

$$Z_t = \frac{1}{t} k_t^* (\mathcal{E}_{\mathbf{w}} - X)$$

which then smoothly extends to $t = 0$. Let φ_t be the flow of this vector field.

Then the same computation as in Theorem 2.3.1 shows that $\frac{d}{dt} \varphi_t^* (\mathcal{E}_{\mathbf{w}} - tZ_t) = 0$. This means that we can again take $\varphi = \varphi_1$ as the diffeomorphism with the required properties. $\square$

Note that the argument above also works when some of the weights w_i are 0. The flow φ_t will then be constant in the corresponding directions.

Remark. The diffeomorphism φ is determined uniquely (up to choice of domain) if one requires that $\lim_{\lambda \to 0} (k_\lambda)^{-1} \circ \varphi \circ (k_\lambda) = id$. This could be called the weighted linear approximation of φ. See [16].

Using the Lemma, we can give a proof of the Submersion theorem for functions $f \in C^\infty(\mathbb{R}^n)$.

Proposition 5.2.5. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be smooth function such that $f(0) = 0$, and assume that $Df(0) \neq 0$ (i.e. f is a submersion). Then there exists a diffeomorphism φ around neighborhoods of 0 in $\mathbb{R}^n$, such that $f \circ \varphi : (x_1, \ldots x_n) \mapsto x_1$.*

Proof. Assume without loss of generality that $\frac{\partial f}{\partial x_1}(0) \neq 0$. Then on some neighborhood around 0, $(\frac{\partial f}{\partial x_1}(x))^{-1} := 1 / \frac{\partial f}{\partial x_1}(x)$ is well-defined. Let $\mathbf{w} = (1, 0, \ldots, 0)$. Then the vector field X given as

$$X(x) = f(x) (\frac{\partial f}{\partial x_1}(x))^{-1} \frac{\partial}{\partial x_1}$$

is weighted Euler-like for $\mathbf{w}$. Note also that $\mathcal{L}_X f = f$. So it determines a diffeomorphism φ around $0 \in \mathbb{R}^n$ such that $\varphi^* X = \mathcal{E}_{\mathbf{w}}$. Then

$$\mathcal{L}_{\mathcal{E}_{\mathbf{w}}} (\varphi^* f) = \varphi^* (\mathcal{L}_X f) = \varphi^* f$$

By Lemma 5.2.6 below, from this we can conclude that $\varphi^* f = f \circ \varphi = x_1 \cdot g(x_2, \ldots x_2)$, where $g : \mathbb{R}^n \to \mathbb{R}$ is a smooth function depending only on the coordinates $x_2, \ldots, x_n$. Since φ is a diffeomorphism, we still have $D(f \circ \varphi) \neq 0$, from which we conclude that $g(0) \neq 0$. So let $\psi(x_1, \ldots, x_n) = (\frac{x_1}{g(x_2, \ldots, x_n)}, x_2, \ldots, x_n)$ defined in a possibly smaller neighborhood of 0. Then we see $f \circ \psi \circ \varphi(x_1, \ldots, x_n) = x_1$. $\square$

5.2.2 Filtration of the ring of function

We now want to investigate which functions are homogeneous with respect to this new, weighted multiplication k_λ. Denote

$$\mathcal{J}_{(i)} = \{f \in C^\infty(\mathbb{R}^n) : k_\lambda^* f = \lambda^i f\} \subseteq C^\infty(\mathbb{R}^n).$$

Lemma 5.2.6. *Let $f \in C^\infty(\mathbb{R}^n)$. If*

$$k_\lambda^* f(x_1, \dots x_n) = f(\lambda^{w_1} x_1, \dots \lambda^{w_n} x_n) = \lambda^d f(x_1, \dots x_n)$$

holds for some $d \in \mathbb{N}$, then f is polynomial function in the coordinates x_i with $w_i > 0$.

Proof. We calculate

$$\frac{\partial}{\partial x_i} f(\lambda^{w_1} x_1, \dots, \lambda^{w_n} x_n) = \lambda^{w_i} (\frac{\partial f}{\partial x_i})(\lambda^{w_1} x_1, \dots \lambda^{w_n} x_n) = \frac{\partial}{\partial x_i}(\lambda^d f(x_1, \dots, x_n)) = \lambda^d (\frac{\partial f}{\partial x_i}(x_1, \dots, x_n))$$

So therefore

$$(\frac{\partial f}{\partial x_i})(\lambda^{w_1} x_1, \dots, \lambda^{w_n} x_n) = \lambda^{(d-w_i)} (\frac{\partial f}{\partial x_i})(x_1, \dots, x_n)$$

In other words, $\frac{\partial f}{\partial x_i}$ is homogeneous (with respect to k_λ) of degree $d - w_i$. We can do this calculation for all partial derivatives, and iterating this will give that for some order of partial derivatives the degree of homogeneity becomes negative. But then these derivatives should vanish, otherwise taking the limit $\lambda \to 0$ is not possible. We conclude that there exist an $N \in \mathbb{N}$ such that all N-th order partial derivatives of f vanish. This implies that f is polynomial, by Taylor's theorem. $\square$

Remark. A polynomial f such as in Lemma 5.2.6 is sometimes called a *quasi homogeneous polynomial.*

Definition 5.2.7. We define a filtration $C^\infty(\mathbb{R}^n) = C^\infty(\mathbb{R}^n)_{(0)} \supseteq \dots \supseteq C^\infty(\mathbb{R}^n)_{(i)} \supseteq \dots$, associated to a weighting $\mathbf{w}$ by

$$C_{(i)} = C^\infty(\mathbb{R}^n)_{(i)} := \langle x_1^{s_1} \dots x_n^{s_n} : \mathbf{s} \cdot \mathbf{w} \geq i \rangle = \langle f : f \in \mathcal{J}_k,\ k \geq i \rangle$$

for $i \in \mathbb{N}$ (here $\langle \cdots \rangle$ denotes the ideal generated in $C^\infty(\mathbb{R}^n)$). The last equality follows by Lemma 5.2.6.

Note that regardless of the weighting $\mathbf{w}$, the first filtration subset $C_{(1)}$ will be equal to the vanishing ideal around zero, $\mathcal{I}_0$. Using the 'standard' weighting $\mathbf{w} = (1, \dots, 1)$, the higher order filtration degrees will exactly be powers of this ideal, $C_{(k)} = \mathcal{I}_0^k$. With a different weighting this idea still holds true in a sense, except that certain coordinate functions x_i are counted with a higher multiplicity.

Example 5.2.8. Consider $\mathbb{R}^2$, with coordinates $\{x, y\}$ and weight $\mathbf{w} = (1, 2)$. Then $\mathcal{J}_1 = \langle x \rangle$, $\mathcal{J}_2 = \langle x^2, y \rangle$, $\mathcal{J}_3 = \langle x^3, xy \rangle, \dots$. We get:

$$\begin{aligned}
C_{(0)} &= C^\infty(\mathbb{R}^2) \\
C_{(1)} &= \langle x, y \rangle \\
C_{(2)} &= \langle y, x^2 \rangle \\
C_{(3)} &= \langle xy, x^3, y^2 \rangle \\
C_{(4)} &= \langle y^2, x^4, x^2 y \rangle \\
C_{(5)} &= \dots
\end{aligned}$$

Remark. The 'greater or equal to'-sign used in in Definition 5.2.7 is necessary mostly because we want $C_{(1)}$ to be this vanishing ideal around 0. If we would use an equal-sign instead, then coordinate functions with weight higher than 1 would not appear here.

The idea of giving a weighting $\mathbf{w}$ was that some directions are given a higher weight than others. We can actually recover this information from the associated filtration $C_{(i)}$. For this, what we would have to do is look at how long a coordinate function x_i 'survives' in the filtration. In other words, find the highest k such that $x_i \in C_{(k)}$. Going the other way around, for each $k \in \mathbb{N}$ we could try to find all coordinate functions that are in $C_{(k)}$. So for $i \geq 1$, consider

$$\mathcal{F}_i := C^\infty(\mathbb{R}^n)_{(i)}/(C^\infty(\mathbb{R}^n)_{(i)} \cap \mathcal{I}_0^2)$$

Example 5.2.9. In the previous example, we would have

$$\begin{aligned}\mathcal{F}_2 &= C_{(2)}/(C_{(2)} \cap \mathcal{I}_0^2)\\ &= \langle y, x^2\rangle/(\langle y, x^2\rangle \cap \langle x^2, xy, y^2\rangle)\\ &= \langle y\rangle.\end{aligned}$$

In general, the $\mathcal{F}_i$ will precisely be generated by representatives of those coordinate functions of weighting degree greater or equal to i. The intersection with $\mathcal{I}_0$ is used to quotient out all other coordinate functions, since these will appear as a square or higher in $C_{(i)}$.

Note that by construction we have $\mathcal{F}_i \subseteq \mathcal{I}_0/\mathcal{I}_0^2 = (\mathbb{R}^n)^*$, the dual space of $\mathbb{R}^n$. Being a linear subspace of $(\mathbb{R}^n)^*$, there must exist a (unique) subspace F_{-i+1} of $\mathbb{R}^n$ such that $\mathcal{F}_i$ is the annihilator of this subspace.

Definition 5.2.10. For $i \in \mathbb{N}$, let $F_{-i+1} \subseteq \mathbb{R}^n$ be the subspace determined by

$$\mathcal{F}_i = \mathrm{Ann}(F_{-i+1}) = \{f \in (\mathbb{R}^n)^* \mid f(x) = 0 \; \forall x \in F_{-i+1}\}.$$

Proposition 5.2.11. *The subspace F_{-k+1} is spanned by exactly those basis vectors e_i such that $w_i \leq k-1$.*

Proof. This follows from how the subspace $F_{-(k-1)}$ is defined. By definition, $\mathbb{F}_k = C_{(k)}/(C_{(k)} \cap \mathcal{I}_0^2)$, which will consist of those x_i's that appear in $C_{(k)}$ not as a square of higher, i.e. that have weight greater or equal to k. The annihilator of this is of course spanned by the remaining ones (under the identification of the dual), so those with weight less then k. □

Let $r = \max\{w_i\}$. We then get a filtration of $\mathbb{R}^n$, given as

$$\mathbb{R}^n = F_{-r} \supseteq F_{-r+1} \supseteq \ldots \supseteq F_0 = \{0\}.$$

Now let us return to the filtration $C_{(i)}$. Any filtration of an algebra defines an *associated graded algebra.*

Definition 5.2.12. Given a filtration $\{A_{(i)}\}$ of a real algebra A, the *associated graded algebra* $gr(A)$ is defined as

$$gr(A) = \bigoplus_i gr(A)_i = \bigoplus_i A_{(i-1)}/A_{(i)}.$$

The multiplication is given by $[f]_i \cdot [g]_j = [f \cdot g]_{i+j}$.

Remark. As a caution to the reader, in Definition 5.2.12 the filtration does not appear as part of the notation. Of course, the resulting graded algebra *will* depend on it.

For $C^\infty(\mathbb{R}^n)$, we get the graded algebra $gr(C^\infty_{\mathbb{R}^n})$, where one sees by Lemma 5.2.6 that each $gr(C^\infty_{\mathbb{R}^n})_k$ consists precisely of all functions that are homogeneous of degree k with respect to the weighted multiplication map k_λ.

$$gr(C^\infty_{\mathbb{R}^n})_k = \{f \in C^\infty_{\mathbb{R}^n} \mid k_t^* f = t^k f\}.$$

As a general fact, the graded algebra $gr(A)$ is canonically isomorphic to A as an $\mathbb{R}$- module (i.e. as vector spaces). However, it is in general *not* isomorphic as an algebra. It is isomorphic if A was a graded vector space to begin with, and the filtration agrees with this grading.

Example 5.2.13. $\mathbb{R}^n$ with trivial weighting, gives the usual graded polynomial algebra.

5.2.3 The sheaf $C^\infty_{\mathbb{R}^n}$

For what is to come, it will be necessary to consider the *sheaf* $C^\infty_{\mathbb{R}^n} = C^\infty(\mathbb{R}^n)(\cdot)$ instead of just the algebra $C^\infty(\mathbb{R}^n)$. A reader unfamiliar with this language of sheaves could for example look at [18], Section III. For now though, it is enough to know that for any open set $U \subseteq \mathbb{R}^n$, $C^\infty_{\mathbb{R}^n}(U)$ consists of all smooth functions defined on U, i.e. is equal to $C^\infty(U)$. In the same way as above, we can also define a *filtration of sheaves* of $C^\infty_{\mathbb{R}^n}$ by setting $C^\infty_{\mathbb{R}^n} = C^\infty_{\mathbb{R}^n,(0)} \supseteq C^\infty_{\mathbb{R}^n,(1)} \supseteq \ldots$, where then

$$C^\infty_{\mathbb{R}^n,(i)}(U) = \langle x_1^{s_1} \ldots x_n^{s_n} : \mathbf{s} \cdot \mathbf{w} = i \rangle,$$

where this is now the ideal generated in $C^\infty_{\mathbb{R}^n}(U)$. For any open U, we now can then consider again the associated graded algebra,

$$gr(C^\infty_{\mathbb{R}^n}(U)) = \bigoplus_k gr(C^\infty_{\mathbb{R}^n}(U))_k$$

If the filtration comes from a weight vector $\mathbf{w}$, we will denote this *sheaf of graded algebras* by

$$gr_{\mathbf{w}}(\mathbb{R}^n) = gr_{\mathbf{w}}(\mathbb{R}^n)(\cdot).$$

5.3 Filtrations and Gelfand duality for manifolds

In the previous section we saw that a weighted multiplication map on $\mathbb{R}^n$ naturally gives rise to a filtration of $C^\infty(\mathbb{R}^n)$. We also obtained a graded algebra $gr(C^\infty(\mathbb{R}^n))$, isomorphic (as an algebra) to the algebra of polynomial functions. Now the question is how to proceed in the general situation of a submanifold $N \subseteq M$. First, we could look at the case of a trivial weighting, $\mathbf{w} = (1, \dots, 1)$. Somehow, we should then obtain a structure that is just the usual normal bundle with its scalar multiplication m_λ.

We will see how this works exactly later, but first we should introduce two theorems. These, in some sense, give a different way of looking at manifolds and vector bundles. Denote by $\mathrm{Hom}_{\mathrm{alg}}(C^\infty(M), \mathbb{R})$ the set of all algebra morphisms from $C^\infty(M)$ to $\mathbb{R}$. If $p \in M$, then the evaluation map $ev(p) : C^\infty(M) \to \mathbb{R}$, $f \mapsto f(p)$ is such a morphism. It turns out that this is actually all of them.

Theorem 5.3.1. *If M is a smooth manifold, then the map*

$$ev : M \to Hom_{alg}(C^\infty(M), \mathbb{R})$$

is a bijection. Moreover, if $Hom_{alg}(C^\infty(M), \mathbb{R})$ is equipped with the Gelfand topology it is a homeomorphism.

More details and a proof are provided in the Appendix. A basic fact about manifolds is that two smooth atlases are equivalent if and only if they induce the same smooth functions. This means that we can recover the smooth structure of M on $\mathrm{Hom}_{\mathrm{alg}}(C^\infty(M), \mathbb{R})$. For $f \in C^\infty(M)$, let $Ev_f : \mathrm{Hom}_{\mathrm{alg}}(C^\infty(M), \mathbb{R}) \to \mathbb{R}$ be given by $Ev_f(\Phi) := \Phi(f)$. Since all maps Φ are of the form ev_p for $p \in M$, under the bijection $M \to \mathrm{Hom}_{\mathrm{alg}}(C^\infty(M), \mathbb{R})$ requiring that all Ev_f are smooth is just saying that all f should be smooth. So the smooth structure can be recovered in this way.

Now consider a vector bundle $E \to M$. By the above theorem, $E = \mathrm{Hom}_{\mathrm{alg}}(C^\infty(E), \mathbb{R})$, but we can actually make this description simpler. Let us denote by $C^\infty_{pol}(E)$ all functions that are polynomial on the fibers E_p.

Theorem 5.3.2. *If $\pi : E \to M$ is a vector bundle over a smooth manifold M, then the map*

$$ev : E \to Hom_{alg}(C^\infty_{pol}(E), \mathbb{R})$$

is a homeomorphism.

We can then recover the multiplication map $m_\lambda : E \to E$ by noting that there exist a natural scalar multiplication on $C^\infty_{pol}(E)$, given by multiplying a degree k polynomial by λ^k. This commutes with the evaluation maps ev_p, so we also get a scalar multiplication on $\mathrm{Hom}_{\mathrm{alg}}(C^\infty_{pol}(E), \mathbb{R})$, which is just the usual multiplication on E under the identification. The projection onto M can be recovered by identifying the degree 0 polynomials with $C^\infty(M)$.

Remark. The identification of Theorem 5.3.2 probably seems somewhat mysterious. One can keep in mind that statement of the Serre-Swan theorem, that provides a one-one correspondence between E and its space of global section $\Gamma(E)$. Again, more details are provided in the Appendix.

Now let us look at the normal bundle $\nu(M, N) \to N$. What are the fiberwise polynomial functions here?

Definition 5.3.3. Let $N \subseteq M$ be a submanifold. By $\mathcal{I}_N$ we will denote the vanishing ideal of N in $C^\infty(M)$

$$\mathcal{I}_N := \{f \in C^\infty(M) \mid f(x) = 0 \;\forall x \in N\}.$$

Then we define a filtration $C^\infty(M) = A_0 \supseteq A_1 \supseteq \ldots$ of $C^\infty(M)$ by letting

$$A_k = \mathcal{I}_N^k$$

.

Proposition 5.3.4. *The graded algebra $gr(A)$ associated to the filtration $C^\infty(M) \supseteq \mathcal{I}_N \supseteq \mathcal{I}_N^2 \supseteq \ldots$ is isomorphic (as a graded algebra) to the algebra of fiberwise polynomial functions on $\nu(M,N)$, so*

$$gr(A) \simeq C^\infty_{pol}(\nu(M,N))$$

Proof. In local coordinates (x_i, y_j) around N, we have that $\mathcal{I}_N = \langle y_1, \ldots y_n \rangle$, as an ideal in $\mathbb{C}^\infty(M)$. Writing $gr(A) = C^\infty(M)/\mathcal{I}_n \oplus \mathcal{I}_N/\mathcal{I}_N^2 \oplus \ldots$, we see that this first term corresponds to functions that are fiberwise constant on $\nu(M,N)$, the second term are the fiberwise linear ones, etc. $\square$

Corollary 5.3.5. *If $gr(A)$ is as above, then $\nu(M,N)$ can be realized as*

$$\nu(M,N) = Hom_{alg}(gr(A), \mathbb{R})$$

Now look back at Example 5.2.13, where we showed that on $\mathbb{R}^n$ with the trivial weight vector $\mathbf{w} = (1, \ldots, 1)$ we obtain as associated graded algebra the polynomial algebra $C^\infty_{pol}(\mathbb{R}^n)$. In the proof of the Proposition above we see something very similar happening, if we think of the y_j's as having weight 1. Intuitively, if we now want to introduce a weighted structure around N we would want to give some y_j's a different weight w_j. This would corresponds to some other filtration of $C^\infty(M)$, but ideally the graded algebra should (at least as an algebra) still be the polynomial algebra. This is what inspires Definition 5.4.2 in the next section.

5.4 The (1,2) -weighting

We will now focus on the case where $w_i = 1, 2$ for all i. In this situation a (relatively) simple definition can be given. Note that for the example of the isotropic embedding theorem, this kind of weighting suffices.

Let us start with a definition.

Definition 5.4.1. A *(1,2)- weighting* of $N \subseteq M$ is the choice of a vector subbundle

$$F \subseteq \nu(M, N).$$

Having chosen such a subbundle, we can determine a filtration of the sheaf $C^\infty(M)$ in the following way. Consider the vanishing ideal sheaf $\mathcal{I} := \mathcal{I}_N$ of N, and additionally the ideal sheaf $\mathcal{J}$, where

$$\mathcal{J}(U) = \{f \in C^\infty(U) \mid f|_{U\cap N} = 0, df|_{TU\cap\tilde{F}} = 0\}$$

Here by $\tilde{F} \subseteq TM|_N$ we denote the pre-image of F in $TM|_N$. Then the filtration $C^\infty(M) = \tilde{A}_0 \supseteq \tilde{A}_1 \supseteq \ldots$ is given as

$$\tilde{A}_{2l+1} = \mathcal{I}\mathcal{J}^l, \;\; \tilde{A}_{2l} = \mathcal{J}^l.$$

So $\tilde{A}_k(U)$ is spanned by products $f_1 \cdots f_a g_1 \cdots g_b$ with $f_i \in \mathcal{I}$, $g_j \in \mathcal{J}$ with $a + 2b \geq k$ (note that $\mathcal{I}^2 \subseteq \mathcal{J}$). We can look at this in local coordinates. For this, choose coordinates (x_i, y_j, z_l) such that N is given by the vanishing of all y_j and z_l, while $\tilde{F} \cap TU$ is additionally given by the vanishing of the differentials dz_l. This can be done for example by choosing a submanifold $N \subseteq \Sigma \subseteq M$ such that $\nu(M, \Sigma) = F$. Then the z_l can be chosen such that Σ is given by the vanishing of these. Within such coordinates, one sees

$$\tilde{A}_k(U) = \langle y_{i_1} \cdots y_{i_a} z_{l_1} \cdots z_{l_b} \mid a + 2b \geq k\rangle.$$

Comparing this to the situation in $\mathbb{R}^n$, specifically Definition 5.2.7, what we are doing here is giving the y_j 's weight 1, the z_l's weight 2 and the x_i's weight 0.

The filtration $\{\tilde{A}_{(k)}\}$ defines an associated graded sheaf of algebras $gr(\tilde{A})$ (as we have seen before), given by

$$gr(\tilde{A}) = \bigoplus_k \tilde{A}_{(k)}/\tilde{A}_{(k+1)}.$$

By looking at this $gr(\tilde{A})$ in local coordinates as above, we see that as an algebra it is still isomorphic to $C^\infty_{pol}(\nu(M, N))$. However, importantly, the grading is now different.

Following [16], we can now define the *weighted normal bundle.*

Definition 5.4.2. Given a $(1, 2)$ -weighting on $N \subseteq M$, determining a graded algebra $gr(\tilde{A})$ as above, we define the *weighted normal bundle* $\nu_{\mathcal{W}}(M, N)$ as

$$\nu_{\mathcal{W}}(M, N) := \mathrm{Hom}_{\mathrm{alg}}(gr(\tilde{A}), \mathbb{R})$$

On this, we have the projection map $\pi : \nu_{\mathcal{W}}(M, N) \to N$ which is induced by the inclusion $C^\infty(N) \to gr(\tilde{A})$.

In the definition above, note that always $gr(\tilde{A})_0 = C^\infty(M)/\mathcal{I}_N \simeq C^\infty(N)$. Since, as we remarked before, the algebra $gr(\tilde{A})$ is isomorphic to $C^\infty_{pol}(\nu(M, N)) \simeq gr(A)$, we see that as the underlying set we have just $\nu_{\mathcal{W}}(M, N) = \nu(M, N)$. However, there is a difference beyond that. We can consider the scalar multiplication on $gr(\tilde{A})$ given by multiplication with λ^k on $gr(\tilde{A})_k$, which induces a scalar multiplication

$$k_\lambda : \nu_{\mathcal{W}}(M,N) \to \nu_{\mathcal{W}}(M,N).$$

This k_λ will (in general) be a different map compared to the usual m_λ on $\nu(M,N)$. It makes $\nu_{\mathcal{W}}(M,N)$ into a *graded bundle* (and in general not a vector bundle).

Definition 5.4.3. A *graded bundle with weight vector* $\mathbf{w}$ *and degree* d is a smooth fiber bundle $p : F \to M$ with typical fiber $\mathbb{R}^n$, that admits an atlas of local trivializations $\psi : p^{-1}(U) \to U \times \mathbb{R}^n$ such that the transition functions $U \cap U' \to \mathrm{Diff}(\mathbb{R}^n, \mathbb{R}^n)$ are automorphisms of the standard graded space $(\mathbb{R}^n, \mathbf{w})$. In other words, given two trivializations ψ, ψ' the composition $\psi' \circ \psi^{-1} : U \cap U' \times \mathbb{R}^n \to U \cap U' \times \mathbb{R}^n$ should be given as

$$\psi' \circ \psi^{-1}(x, v) = (x, \tau(x)v),$$

where $\tau(x) : \mathbb{R}^n \to \mathbb{R}^n$ is an automorphism of $(\mathbb{R}^n, \mathbf{w})$ for all x.

See Appendix A.2 for some more details. The fact that $\nu_{\mathcal{W}}(M,N)$ is a graded bundle can be concluded from Theorem A.2.6.

To better understand the map k_λ, we have the following result.

Proposition 5.4.4. *Let (M, N, F) be a $(1,2)$-weighting. The choice of a submanifold $N \subseteq \Sigma \subseteq M$, such that $\nu(M, \Sigma) = F$, determines a isomorphism of graded bundles*

$$R_\Sigma : \nu_{\mathcal{W}}(M,N) \to F \oplus \nu(M,N)/F =: gr(\nu(M,N)).$$

Here the scalar multiplication on $gr(\nu(M,N))$ is given as multiplication by λ on F, and λ^2 on $\nu(M,N)/F$.

Proof. See [16]. This can also be seen in local coordinates (x_i, y_j, z_l) as we used earlier. ☐

Remark. Note that isomorphism in Proposition 5.4.4 above is not canonical, it depends on the choice of Σ. Also, concretely realizing the quotient $\nu(M,N)/F$ requires the choice of a metric. Therefore, simply defining the weighted normal bundle as $gr(\nu(M,N))$ is not desirable.

5.4.1 Further constructions

Having defined the weighted normal bundle, one can now proceed along the same lines as the unweighted case, and consider constructions like the weighted linear approximation. We will give a brief outline of this here, based on the work in [16]. For details, we also refer to that paper.

First, one has to show that the construction of the weighted normal bundle is again functorial. In this case, that means that if $f : (M, N, F) \to (M', N', F')$ is a map of manifold pairs, with the additional condition that $\nu(f)(F) \subseteq F'$, then there should be a induced isomorphism of graded bundles

$$\nu_{\mathcal{W}}(f) : \nu_{\mathcal{W}}(M,N) \to \nu_{\mathcal{W}}(M',N').$$

Then also we would like the property, similar to the usual normal bundle, that

$$\nu_{\mathcal{W}}(\nu_{\mathcal{W}}(M,N),N) \simeq \nu_{\mathcal{W}}(M,N)$$

where this is a canonical isomorphism of graded bundles (note that for this we have to view F as also a subbundle of $\nu_{\mathcal{W}}(M,N)$). We can then define a notion of *weighted tubular neighborhood embedding.*

Definition 5.4.5. A *weighted tubular neighborhood embedding* is an embedding $\varphi : U \subseteq \nu_{\mathcal{W}}(M, N) \to M$ of an open neighborhood of N in $\nu_{\mathcal{W}}(M, N)$, such that $\varphi|_N = id$, $\nu(\varphi)(F) \subseteq F$ and

$$\nu_{\mathcal{W}}(\varphi) = id.$$

In analogy with the situation in $\mathbb{R}^n$, we can define the *weighted Euler vector field* $\mathcal{E}_{\mathbf{w}} \in \mathfrak{X}(\nu_{\mathcal{W}}(M, N))$ as the vector field with flow $k_{exp(t)}$. A vector field $X \in \mathfrak{X}(M)$ that is tangent to N, and additionally 'preserves F', is called *weighted Euler-like* if $\nu_{\mathcal{W}}(X) = X_{[0]} = \mathcal{E}_{\mathbf{w}} \in \mathfrak{X}(\nu_{\mathcal{W}}(M, N))$. And then we have the following theorem.

Theorem 5.4.6. *A weighted Euler-like vector field X for a $(1, 2)$-weighting (M, N, F) determines a unique maximal weighted tubular neighborhood embedding φ, such that $\varphi^* X = \mathcal{E}_{\mathbf{w}}$.*

With Theorem 5.4.6, one can give a proof of the Isotropic Embedding theorem. This proof is very similar to the Lagrangian case, if one translates everything into the weighted situation. As we discussed in Section 5.1, we want to choose weight 1 on the symplectic normal TN^ω/TN, and weight 2 in the remaining normal directions. This means we should choose the $(1, 2)$- weighting with $F = TN^\omega/TN$. We will use the following claim without proof, and refer to the paper instead.

Claim. Let (M, ω) a symplectic manifold, and $N \subseteq M$ an isotropic submanifold. Then the weighted approximation $\omega_{[2]}$, for weighting $F = TM/TN^\omega \subseteq \nu(M, N)$, is a well-defined, symplectic form on $\nu_{\mathcal{W}}(M, N)$.

The theorem can then be formulated as follows.

Theorem 5.4.7. *Let (M, ω) a symplectic manifold, and $N \subseteq M$ an isotropic submanifold. Consider the $(1, 2)$-weighting defined by $F = TN^\omega/TN \subseteq \nu(M, N)$. Then there exists a* weighted *tubular neighborhood embedding φ around N with*

$$\varphi^*\omega = \omega_{[2]}.$$

Proof. We will give an outline of the proof. Working locally, we can (similar as to the proof of Theorem 3.3.2) define

$$\alpha = \int_0^1 \frac{1}{t}(k_t)^* \iota_{\mathcal{E}_{\mathbf{w}}}\omega \, dt.$$

Then $d\alpha = \omega$. Also, one checks that the second order (weighted) approximation $\alpha_{[2]}$ is well-defined. Define a vector field X by setting

$$\iota_X \omega = 2\alpha.$$

Then it holds that

$$\iota_{X_{[0]}}\omega_{[2]} = \iota_{\mathcal{E}_{\mathbf{w}}}\omega_{[2]},$$

so that $X_{[0]} = \mathcal{E}_{\mathbf{w}}$, i.e. X is weighted Euler-like. The weighted tubular neighborhood φ determined by X then satisfies

$$\mathcal{L}_{\mathcal{E}_{\mathbf{w}}}\varphi^*\omega = \varphi^*(\mathcal{L}_X\omega) = 2\varphi^*(d\alpha) = 2\varphi^*\omega.$$

And we conclude $\varphi^*\omega = \omega_{[2]}$. □

5.5 Possible alternative definition

With the definition of the weighted normal bundle as we have just given, one could see two problems. First of all, the construction using the $\mathrm{Hom}_{\mathrm{alg}}(\cdot, \mathbb{R})$-functor is not very geometric, and it takes quite some work to see what is really going on. Secondly, it is not immediately clear how one would generalize it to weightings other than $(1, 2)$. To solve these two issues, we propose an alternative definition that more directly incorporates the fact that we should obtain a graded bundle in the end. As of now, this is purely a suggestion, the theory that would use this definition has not been fully developed (yet). Some inspiration is again taken from the talk by Y. Loizides.

Definition 5.5.1. Given a manifold M, a *general weighting* with respect to $\mathbf{w} \in \mathbb{R}^{n+m}$ is a filtration $C^\infty(M) = A_0 \supseteq A_1 \supseteq \dots$ of the sheaf of smooth sections, and an atlas $\mathcal{U} = \{(U, \Phi_U)\}$ with the property that $\Phi_U : U \to V \subseteq \mathbb{R}^n$ induces an isomorphism of graded algebras

$$gr(A)(U) \to gr_{\mathbf{w}}(\mathbb{R}^{n+m})(V), \quad [f] \mapsto [f \circ \Phi_U],$$

for all U where the filtration is not trivial.

Lemma 5.5.2. *Given a weighting of M, we have that the first filtration degree $A_1 \subseteq C^\infty(M)$ is equal to the vanishing ideal sheaf of some submanifold N.*

Proof. This is true locally, for if $\{A_i(U)\}$ is the trivial filtration then N is the empty set, while if the filtration is non-trivial then the submanifold can be recovered in local coordinates using the algebra isomorphism. Specifically, in these local coordinates it is given by the vanishing of all coordinates that do not have weight 0. □

Example 5.5.3. In $(\mathbb{R}^n, \omega)$, with all weights $w_i > 0$, and as atlas just the identity map, the submanifold of Lemma 5.5.2 would be $\{0\}$.

The idea is that with this definition we can obtain the structure of a graded bundle on N. This is based on the proposition below, which says that a graded bundle is essentially determined by its transition functions. Note that the construction given in this proposition is similar to a method that is often used to create a vector bundle structure.

Proposition 5.5.4. *Let M a smooth manifold, and suppose we are given an open cover $\mathcal{U} = \{U_a\}$, and for each a, b a smooth map $\tau_{ab} : U_a \cap U_b \to Aut_{\mathbf{w}}(\mathbb{R}^n)$ (where $Aut_{\mathbf{w}}(\mathbb{R}^n)$ denotes the space of graded automorphisms for $(\mathbb{R}^n, \mathbf{w})$). If moreover these functions satisfy*

$$\tau_{ab} \circ \tau_{bc} = \tau_{ac} \quad \text{on } U_a \cap U_b \cap U_c,$$

then there exists a graded bundle structure $F \to M$ with transition functions given by the τ_{ab}.

Proof. The argument is virtually the same as for vector bundles, F can be realized as

$$\Big(\bigsqcup_a U_a \times \mathbb{R}^n\Big)/\sim,$$

with for $p \in U_a, q \in U_b$, $(p, v) \sim (q, w)$ if and only if $p = q$, $w = \tau_{ab}(p)v$.

□

Consider a general weighting on M, and pick two opens U, V with non-empty intersection. Let N be the submanifold that is given by the vanishing of the first filtration degree. Under $\Phi_U : U \to \mathbb{R}^{n+m} = \mathbb{R}^n \times \mathbb{R}^m$, N is then mapped to $\mathbb{R}^n \times \{0\}$, the 'part with weight 0'. The idea is now to obtain a graded bundle structure over N, with fibers R^m. For this, we want to find a transition function on $(U \cap N) \cap (V \cap N)$.

The composition $\Phi_V \circ \Phi_U^{-1}$ induces a morphism of graded algebras

$$gr_{\mathbf{w}}(\mathbb{R}^{n+m})(U') \to gr_{\mathbf{w}}(\mathbb{R}^{n+m})(U''),$$

where U', U'' are two opens in U. Both these algebras are generated (as algebras over $C^\infty(U'), C^\infty(U'')$ respectively) by the coordinate functions $\{x_1, \dots x_n, y_1, \dots y_{m-n}\}$, where the x_i have weight 0. The isomorphism should map generators to generators, and since it preserves weight it should have the following form on generators:

$$\begin{aligned} x_i &\mapsto f(x_1, \dots x_n) : \mathbb{R}^n \to \mathbb{R} \\ y_i &\mapsto \sum_j f_j(x_1, \dots x_n) y_{j_1} \dots y_{j_k} \end{aligned}$$

From this, we obtain transition functions $\tau_{UV} : (U \cap N) \cap (V \cap N) \to Aut_{\mathbf{w}}(\mathbb{R}^m)$ by setting,

$$(\tau_{UV}(p)v)_i = \sum_j f_j(\Phi_U(p)_1, \dots \Phi_U(p)_n) v_{j_1} \dots v_{j_k}. \tag{5.1}$$

We can then use these transition functions to build a graded bundle, leading to the next definition.

Definition 5.5.5. Given a general weighting of M with respect to $\mathbf{w}$, where A_1 is the vanishing ideal sheaf of a submanifold N, the *general weighted normal bundle*

$$\nu_{\mathbf{w}}(M, N) \to N$$

is the graded bundle induced by the transition functions as in Equation 5.1.

Example 5.5.6. $A_k = \mathcal{I}_N^k$, with atlas a trivializing atlas for $\nu(M, N)$. Then the transition functions are linear (they are exactly the vector bundle transition functions), and we simply get back the normal bundle.

Assume we are given $(1, 2)$-weighting (M, N, F), with the associated filtration $\tilde{A}_{(k)}$ of $C^\infty(M)$. Let $k := \dim F$. Choose as an atlas a covering of trivializing open sets for the normal bundle, such that the charts $\varphi : \pi^{-1}(U) \to U \times \mathbb{R}^m$ consistently map F to the first k coordinates. Then one finds that the graded bundle constructed as in Definition 5.5.5 agrees with the definition for the weighted normal bundle we gave earlier.

5.6 Final remarks

5.6.1 Frobenius theorem

One well-known normal form theorem that we have not yet looked at is the Frobenius theorem, which is an important theorem when studying foliations.

Definition 5.6.1. A distribution $D \subseteq TM$ is called *involutive* if for all $X, Y \in D$, also $[X, Y] \in D$.

Definition 5.6.2. A distribution D of dimension k is said to be *completely integrable* if around any point $p \in M$ we can find coordinates $(x_1, \ldots, x_n)$ on an open U such that D is spanned by the first k coordinate vector fields $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_k}$.

Theorem 5.6.3 (Frobenius). *Let $D \subseteq TM$ be a smooth distribution. If D is involutive, then it is completely integrable.*

Our hope is that this theorem is also provable using the theory of Euler-like vector fields. We expect that this would involve using the weighted version of theory. Looking at Definition 5.6.2, a natural choice might be to, in local coordinates, assign weight 1 to the coordinates $x_1, \ldots, x_k$ and weight 0 to the rest. But this remains to be worked out.

A starting point could be to consider a special type of codimension-one distribution, given as the kernel of a nowhere-vanishing 1-form. Then involutivity has a simple characterization, given by the proposition below.

Proposition 5.6.4. *Let D be a defined as $D = \ker \alpha$, where $\alpha \in \Omega^1(M)$ is nowhere vanishing. Then D is involutive if and only if $\alpha \wedge d\alpha = 0$.*

5.6.2 Co-isotropic embedding

Let (M, ω) a symplectic manifold, and assume that N is a co-isotropic submanifold. In particular $\dim N \geq \frac{1}{2} \dim M$. We could then formulate a local version for the Co-isotropic embedding theorem, similar to the isotropic case.

Theorem 5.6.5 (Co-isotropic embedding theorem, local version). *Let (M, ω) a symplectic manifold of dimension $2n$, with N a co-isotropic submanifold of dimension $k > n$. Then around any point $p \in N$ we can find local coordinates $(x_1, \ldots x_n, y_1, \ldots y_n)$, such that in these coordinates $N = \{y_{k-n} = \ldots = y_n = 0\}$ and*

$$\omega = \sum_{i=1}^{n-k} dx_i \wedge dy_i + \sum_{i=n-k+1}^{n} dx_i \wedge dy_i.$$

If ω is given in this way, then

$$(m_\lambda)^*\omega = \sum_{i=1}^{n-k} dx_i \wedge dy_i + \lambda \sum_{i=n-k+1}^{n} dx_i \wedge dy_i.$$

so we see a linear part and a constant one. Now there is a problem, because if we want to choose weights such that the equation above becomes homogeneous, the only choice we have is to assign weight 0 to all $y_{n-k+1}, \ldots, y_n$ (note that these are the coordinates normal to N). But then the Euler vector field $\mathcal{E}_{\mathbf{w}} = 0$, so that in particular the zero vector field is Euler-like. Therefore, the theory seems to break down in this particular case. Inherently, the problem is here that for this method to work we would want to manipulate coordinates also in the direction tangent to N (the coordinates $y_1, \ldots, y_{n-k}$). But with our method of using Euler-like vector fields we can only change things in the direction normal to N.

Appendix A

A.1 Tangent and normal bundle functor

Lemma A.1.1. *Given a manifold pair (M, N), we have the following isomorphism of double vector bundles over N and TN*

$$\nu(TM, TN) \simeq T\nu(M, N).$$

Where the double vector bundle structures are given as

$$\begin{tikzcd} \nu(TM,TN) \arrow[r] \arrow[d, "\nu(q_N)"] & TN \arrow[d, "q_N"] \\ \nu(M,N) \arrow[r] & N \end{tikzcd}$$

and

$$\begin{tikzcd} T\nu(M,N) \arrow[r, "dp"] \arrow[d] & TN \arrow[d] \\ \nu(M,N) \arrow[r, "p"] & N \end{tikzcd}$$

Proof. We consider the double tangent bundle $TTM = T(TM)$. There are two ways to view this as a vector bundle over TM. There is the usual projection of a tangent bundle $pr_{TM} : T(TM) \to TM$, but we can also consider the differential $d(pr_M) : T(TM) \to TM$ of $pr_M : TM \to M$. These two fit in the following commuting diagram, giving TTM a double vector bundle structure

$$\begin{tikzcd} TTM, \arrow[r, "d(pr_M)"] \arrow[d, "pr_{TM}"] & TM \arrow[d] \\ TM \arrow[r] & M \end{tikzcd}$$

We claim that there is a map $J : TTM \to TTM$ (called the *canonical involution*) such that $d(pr_M) \circ J = pr_{TM}$, and which interchanges the two vector bundle structures. Choose a trivializing neighborhood U of M, then we can choose local coordinates $(x, v, (y, w))$ on TTM. We have pr_{TM} : $(x, v, (y, w)) \mapsto (x, v) \in TM$, while since $pr_M : (x, v) \mapsto x \in M$, we see $d(pr_M) : (x, v, y, w) \mapsto (x, y)$. So locally we should have

$$J : (x, v, y, w) \mapsto (x, y, v, w).$$

See [15], Section 9, for a proof that this map is globally well-defined, and that it has the required properties.

This J restricts to an isomorphism $T(TM|_N) \to (TTM)|_{TN}$ (both viewed as submanifolds of TTM). Moreover, when viewing also TTN as a submanifold of TTM, the restriction of J becomes the canonical involution of TTN. Since TTN is also a submanifold of both $T(TM|_N)$ and $(TTM)|_{TN}$, J induces a vector bundle morphism when passing to the quotient for the normal bundle

□

A.2 Graded bundles

We first give the main definitions. This is taken from [1].

Definition A.2.1. The *standard graded space* $(\mathbb{R}^n, \mathbf{w})$ of *degree d* is $\mathbb{R}^n$ together with a weight vector $\mathbf{w} = (w_1, \dots w_n)$, with $d = \max\{w_i\}$, equipped with an action $h : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n, h(t, x) = h_t(x)$ of the monoid $(\mathbb{R}, \cdot)$ given as

$$h_t(x_1, \dots x_n) = (t^{w_1} x_1, \dots, t^{w_n} x_n)$$

Definition A.2.2. A *graded space* $(M, \mathbf{w})$ (of degree $d = \max\{w_i\}$, where $\mathbf{w} = (w_1, \dots w_n)$ is a manifold M of dimension n equipped with an action $h : \mathbb{R} \times M \to M$ of $(\mathbb{R}, \cdot)$, such that there exists a diffeomorphism $\Phi : M \to \mathbb{R}^n$ to the standard graded space $(\mathbb{R}^n, \mathbf{w})$, that intertwines the two actions.

Definition A.2.3. A *graded bundle with weight vector* $\mathbf{w}$ *and degree d* is a smooth fiber bundle $p : F \to M$ with typical fiber $\mathbb{R}^n$, that admits an atlas of local trivializations $\psi : p^{-1}(U) \to U \times \mathbb{R}^n$ such that the transition functions $U \cap U' \to \mathrm{Diff}(\mathbb{R}^n, \mathbb{R}^n)$ are automorphisms of the standard graded space $(\mathbb{R}^n, \mathbf{w})$.

In other words, given two trivializations ψ, ψ' the composition $\psi' \circ \psi^{-1} : U \cap U' \times \mathbb{R}^n \to U \cap U' \times \mathbb{R}^n$ should be given as

$$\psi' \circ \psi^{-1}(x, v) = (x, \tau(x)v),$$

where $\tau(x) : \mathbb{R}^n \to \mathbb{R}^n$ is an automorphism of $(\mathbb{R}^n, \mathbf{w})$ for all x.

Given a graded bundle $F \to M$ as above, clearly every fiber F_x is a graded space with weight vector $\mathbf{w}$, and F comes with a monoid action $h : \mathbb{R} \times F \to F$ that in local coordinates is given as the standard graded action $h : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ for $(\mathbb{R}^n, \mathbf{w})$. Under this action, $M = h_0(F)$.

And these are the theorems relating graded bundles to a 'scalar multiplication' , i.e. an action of the monoid $(\mathbb{R}, \cdot)$. See [11] and [10] for the proofs. Let E be some manifold.

Theorem A.2.4. *An action $h : \mathbb{R} \times E \to E$ from the monoid $(\mathbb{R}, \cdot)$ such that there exist an $0^E \in E$ with $h_0(v) = 0^E$ for all $v \in E$, comes from a vector space structure on E if and only if*

$$\frac{\partial h(0, v)}{\partial t} = 0 \iff v = 0^E$$

Theorem A.2.5. *An action $h : \mathbb{R} \times E \to E$ from the monoid $(\mathbb{R}, \cdot)$ comes from a vector bundle structure $\pi : E \to E_0 = h_0(E)$ if, for the curve $\mathbb{R} \to E$, $t \mapsto h(t, p)$, the 1-jet vanishes if and only if $p \in E_0$.*

Theorem A.2.6. *Any action $h : \mathbb{R} \times E \to E$ from the monoid $(\mathbb{R}, \cdot)$ comes from a graded bundle structure*

$$\pi : E \to E_0 = h_0(E)$$

of degree d, where $d \in \mathbb{N}$ is the lowest integer N such that for the curve $\mathbb{R} \to E$, $t \mapsto h(t, p)$, the N-jet vanishes if and only if $p \in E_0$.

Proposition A.2.7. *Any action $h : \mathbb{R} \times M \to M$ from the monoid $(\mathbb{R}, \cdot)$ such that there is a fixed $h_0(M) = 0^M \in M$ comes from the structure of a graded space on M.*

A.3 Gelfand duality for manifolds

The following is based on the book "C^∞ Differentiable Spaces" by Navarro Gonzalez and Sancho de Salas, [17].

Definition A.3.1. Given a real algebra A, we define

$$\mathrm{Hom}_{\mathrm{alg}}(A, \mathbb{R}) = Spec_r(A)$$

as the set of all $\mathbb{R}$ -algebra morphisms from A to $\mathbb{R}$.

Given an element $\Phi \in \mathrm{Hom}_{\mathrm{alg}}(A, \mathbb{R})$, its kernel will be an ideal of A, denoted by $\mathfrak{m}$ with $A/\mathfrak{m} \simeq \mathbb{R}$. In particular, it is a *maximal ideal* of A.

Let $f \in A$, then we can define a map $\hat{f} : \mathrm{Hom}_{\mathrm{alg}}(A, \mathbb{R}) \to \mathbb{R}$ by $\hat{f}(x) = x(f)$. The *Gelfand topology* on $\mathrm{Hom}_{\mathrm{alg}}(A, \mathbb{R})$ is defined as the smallest topology in which all maps $\hat{f}$ are continuous. If $A = C^\infty(M)$ for M a smooth manifold, then any $p \in M$ defines an element $ev_p \in \mathrm{Hom}_{\mathrm{alg}}(C^\infty(M), \mathbb{R})$ by letting

$$ev_p(f) = f(p) \quad \text{for } f \in C^\infty(M).$$

The map $ev : M \to \mathrm{Hom}_{\mathrm{alg}}(C^\infty(M), \mathbb{R})$ is then given by $p \mapsto ev_p$.

Theorem A.3.2. *If M is a smooth manifold then the map $ev : M \to Hom_{alg}(C^\infty(M), \mathbb{R})$ is a bijection. Moreover, if $Hom_{alg}(C^\infty(M), \mathbb{R})$ is equipped with the Gelfand topology it is a homeomorphism.*

Proof. First we show injectivity. Let $p, q \in M$ with $p \neq q$. Let K be a compact neighborhood of p that does not contain q. Then we know there exist an $f \in C^\infty(M)$ with $f(p) = 1$, and $f \equiv 0$ outside K (argue with partitions of unity). So we conclude that $ev_p \neq ev_q$.

For surjectivity, assume $\Phi \in \mathrm{Hom}_{\mathrm{alg}}(C^\infty(M), \mathbb{R})$, and let $\mathfrak{m} = \ker \Phi$. We want to show that $\mathfrak{m} = \mathfrak{m}_x := \{f \in C^\infty(M) : f(x) = 0\}$ for some $x \in M$. Since M is a manifold, we can choose a compact exhaustion of M, i.e. a sequence $\{K_n\}$ of compact sets such that $K_n \subseteq \mathrm{int}(K_{n+1})$ and $M = \bigcup K_n$. Again using partitions of unity we can find $f_n \in C^\infty(M)$ such that $0 \leq f_n \leq 1$, $f_n \equiv 0$ on K_n and $f_n \equiv 1$ outside $\mathrm{int}(K_{n+1})$. Let $f = \sum_n f_n$, then $f \geq n$ on $M \backslash K_{n+1}$ for any $n \in \mathbb{N}$, and it follows that the level sets $f^{-1}(a) \subseteq M$ are compact for any $a \in \mathbb{R}$, since they will be contained within some K_n. Now let $b = \Phi(f) \in \mathbb{R}$, so that $f - b \in \mathfrak{m}$. Now assume for the sake of contradiction that

$$\bigcap_{g \in \mathfrak{m}} g^{-1}(0) = \emptyset$$

(so that in particular $\mathfrak{m} \neq \mathfrak{m}_x$). Then since $f^{-1}(b)$ is compact, there must exist some $g_1, \ldots g_m \in C^\infty(M)$ (finitely many) such that $g_1^{-1}(0) \cap \ldots \cap g_m^{-1}(0) \cap (f - b)^{-1}(0) = \emptyset$. Consider then

$$h := g_1^2 + \ldots + g_m^2 + (f - b)^2 \in \mathfrak{m}.$$

By construction, h is nowhere zero, and so gives an invertible element in the ideal $\mathfrak{m}$. But that would mean $\mathfrak{m} = \mathrm{Hom}_{\mathrm{alg}}(C^\infty(M), \mathbb{R})$, which is impossible. So we conclude that there exist some $x \in \bigcap_{g \in \mathfrak{m}} g^{-1}(0)$, and we see then $\mathfrak{m} \subseteq \mathfrak{m}_x$. By maximality, we conclude $\mathfrak{m} = \mathfrak{m}_x$. For the proof that ev is a homeomorphism we refer to the book. $\square$

Remark. Theorem A.3.2 can be compared to the Gelfand-Naimark theorem, that relates a (compact) topological space X to its set $C(X)$ of continuous functions. The correspondence between M and $\mathrm{Hom}_{\mathrm{alg}}(C^\infty(M), \mathbb{R})$ could thus be referred to as 'Gelfand duality for manifolds'.

Now let $E \to M$ be some vector bundle over M. The theorem above gives us the correspondence $E = \mathrm{Hom}_{\mathrm{alg}}(C^\infty(E), \mathbb{R})$, but we will see that we can do better than that. This is based on the paper [14].

Definition A.3.3. Given a vector bundle $E \to M$, by $C^\infty_{pol}(E)$ we will denote the fiberwise polynomial functions on E.

Theorem A.3.4. *If $\pi : E \to M$ is a vector bundle over a smooth manifold M, then the map*

$$ev : E \to Hom_{alg}(C^\infty_{pol}(E), \mathbb{R})$$

is a homeomorphism.

Proof. For injectivity let $p, q \in E$ with $p \neq q$. If also $\pi(p) \neq \pi(q)$, then $ev_p \neq ev_q$ by the same argument as in Theorem A.3.2 (note that we can identify $C^\infty(M)$ with the constant functions in $C^\infty_{pol}(E)$").

If we do have $p, q \in E_x$ (where $x = \pi(p) = \pi(q)$), then let $U \ni x$ be some trivializing neighborhood of x, with $\varphi : E|_U \to U \times \mathbb{R}^k$ its vector bundle chart. Write

$$\varphi(p) = (x, v) \quad \varphi(q) = (x, w).$$

Let $\Psi : \mathbb{R}^k \to \mathbb{R}^k$ be an *affine linear* map such that $\Psi(v)) = e_1$, $\Psi(\varphi^{-1}(w)) = e_2$ (clearly Ψ exists). Now let $f := pr_1 \circ \Psi \circ \varphi : E|_U \to \mathbb{R}$, then we can extend f to $\tilde{f} : E \to \mathbb{R}$ by multiplying it with a function that is zero outside a compact neighborhood in U, and 1 around x. One checks that $\tilde{f} \in C^\infty_{pol}(E)$ (note that the bump function only depends on the base coordinates), and we see $f(p) = 1, f(q) = 0$. So again $ev_p \neq ev_q$.

For surjectivity, let again $\Phi \in \mathrm{Hom}_{\mathrm{alg}}(C^\infty_{pol}(E), \mathbb{R})$ and let $\mathfrak{m}$ be its kernel. By a similar argument as in Theorem A.3.2, we have that for *finite* $g_1, \ldots g_m \in \mathfrak{m}$,

$$\bigcap_i g_i^{-1}(0) \neq \emptyset.$$

Fact: We can find an embedding $i_E : E \to \mathbb{R}^{2n} \times \mathbb{R}^{2n}$ over $i_M : M \to \mathbb{R}^{2n}$ with coordinate functions $((\chi_1, \ldots \chi_{2n}), (\xi_1, \ldots \xi_{2n}))$ such that the χ_i are smooth functions on M, and the ξ_i's are fiberwise affine linear. This follows by using the Whitney embedding theorem, combined with the fact that for each vector bundle E there exists another vector bundle F such that $E \oplus F \simeq M \times \mathbb{R}^N$ for some N.

Let now

$$f_i := \chi_i - \Phi(\chi_i), \quad F_j = \xi_j - \Phi(\xi_j),$$

so $f_i, F_j \in \mathfrak{m}$ for all $1 \leq i, j \leq 2n$. We know that there is an element $b \in \left(\bigcap f_i^{-1}(0)\right) \cap \left(\bigcap F_j^{-1}(0)\right)$. For any such b, $\chi_i(b) = \Phi(\chi_i)$, so we see

$$i_E(b) = ((\Phi(\chi_1), \ldots, \Phi(\chi_{2n})), (\Phi(\xi_1), \ldots, \Phi(\xi_{2n}))).$$

Since i_E is injective, we conclude therefore that $\left(\bigcap f_i^{-1}(0)\right) \cap \left(\bigcap F_j^{-1}(0)\right) = \{b\}$. For any other $f \in \mathfrak{m}$ we then also get $f(b) = 0$ by considering $f^{-1}(0) \cap \{b\} = f^{-1}(0) \cap \left(\bigcap f_i^{-1}(0)\right) \cap \left(\bigcap F_j^{-1}(0)\right) \neq \emptyset$, therefore $b \in f^{-1}(0)$. So $\mathfrak{m} \subseteq \mathfrak{m}_b$, so by maximality $\mathfrak{m} = \mathfrak{m}_b$, and we conclude $\Phi = ev_b$.

For the proof of that the map is actually a homeomorphism we refer to the paper. $\square$

Remark. We can understand the theorem above in the following context. All the arrows in the diagram below indicate 1-1 correspondences.

$$\begin{tikzcd}
\{\text{vector bundles over M}\} \arrow[r, leftrightarrow, "\text{Serre-Swan theorem}"] \arrow[d, leftrightarrow, dotted] & \{\text{fin. gen. projective } C^\infty(M) \text{ - modules}\} \arrow[d, leftrightarrow, "\text{Serre's theorem}"] \\
\{\text{total spaces of algebraic vector bundles}\} \arrow[r, leftrightarrow] & \{\text{locally free sheaves of } C^\infty(M)\text{-modules}\}
\end{tikzcd}$$

The horizontal arrow on the bottom is a construction that is used in Algebraic Geometry. If we start with a vector bundle $E \to M$, one gets the following sequence of objects in the corresponding diagram.

$$\begin{tikzcd}
E \arrow[r, leftrightarrow] \arrow[d, leftrightarrow, dotted] & \Gamma(E) \arrow[d, leftrightarrow] \\
Spec_r(Sym(\mathcal{E}^*)) \arrow[r, leftrightarrow] \arrow[d, equal] & \mathcal{E}(\cdot) = \Gamma(E)(\cdot) \\
Hom_{alg}(C^\infty_{pol}(E), \mathbb{R}) &
\end{tikzcd}$$

For the equality $Spec_r(Sym(\mathcal{E}^*)) = Hom_{alg}(C^\infty_{pol}(E)$, note in particular that for any vector space V, the symmetric algebra $Sym(V)$ is isomorphic to the polynomial algebra of V in a chosen basis. The condition in the Serre-Swan theorem that $\Gamma(E)$ is a finitely generated projective module, corresponds to the property that for every vector bundle E, there exists another vector bundle F such that $E \oplus F$ is trivial. This fact was used in the theorem above, and now this provides an explanation of why this was really necessary.

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