Classification of real division algebras and which n-spheres can be topological groups

Stefan Franssen
4035844

August 1, 2015

Contents

Chapter 1

Introduction

In this thesis we are going to classify division algebras, and in a related fashion, classify which spheres can be groups. The classification of division algebras will be done in 3 phases. First we will classify the associative division algebras. Then we will classify the composition algebras. In order to classify all the division algebras we will show that if $\mathbb{R}^n$ is a division algebra then S^{n-1} is a Hopf space. And then we will show that S^{n-1} is a Hopf space if and only if $n = 1, 2, 4, 8$. This allows us to classify the division algebras. For this thesis we first study division algebras and follow the proofs stated in [10] for the Frobenius theorem. Then we will take a closer look at composition algebras and follow [8] closely. Then from the introduction of vector bundles and further we closely follow [5]. There are extra references to more background information. See for example [1].

1.1 Preliminary knowledge

We will assume that the reader has understanding of introduction to topology, for example in [3] and group theory, and hence we assume the reader knows the following concepts: spheres, real numbers, complex numbers and quaternions. Further more we assume the reader has background in algebraic topology as on the level of [4].

Chapter 2

Algebras

We start by introducing the definitions of an algebra, and some theorems about algebras over the real field. We are mostly intrested in division algebras. These are algebras where every non zero element has an inverse element. At the end of this chapter we start our work on the first classification theorem. This is the Frobenius classification theorem which states that all associative real division algebras are isomorphic to either $\mathbb{R}, \mathbb{C}, \mathbb{H}$. After this is done we will introduce a generalization of normed algebras, the concept of composition algebras, and then classify all composition algebras over a field. This work is highly related to the following question which has historically been important: Let K be a field with characteristic other than 2. The only values of $n \in \mathbb{N} - \{0\}$ for which the statement is true are $1, 2, 4, 8$: there exists n bilinear functions $z_i : K^n \times K^n \to K$ such that for all $x, y \in K^n$ the following equation holds:

$$(\sum_{i=1}^{n} x_i^2)(\sum_{i=1}^{n} y_i^2) = \sum_{i}^{n} z_i(x,y)^2 \tag{2.1}$$

This is the Hurwitz problem, which we will examine in the third chapter. At the end of the third chapter we have a list of all normed division algebras over a field K, and we have classified all associative real division algebras. Historically Hurwitz problem is related to the question which natural numbers could be written as a sum of squares. That Hurwitz problem has a solution when $n = 4$ is a mayor ingredient in the proof that every natural number can be written as a sum of 4 square numbers, since this allows us to prove the statement for all odd primes, and then generalize it to all whole numbers greater than 2. Since 1 and 2 are sums of 4 squares ($1^2 + 0 + 0 + 0$ and $1^2 + 1^2 + 0 + 0$) this then allows us to show the statement for every whole number. Thus we only need to prove that you can write every odd prime p as the sum of 4 squares. For more details in the proof, we will refer to the dictaat of Frits Beukers on number theory [2].

2.1 Fields

A field K is a set S with two associative operations $+, \cdot$. These operations have to obey the following axioms:

- $(S, +)$ is an Abelian group.
- $(S - \{0\}, \cdot)$ is an Abelian group.
- $+$ and $\cdot$ obey the distributive law:

$$x(y + z) = xy + xz \tag{2.2}$$

$$(y + z)x = yx + zx \tag{2.3}$$

2.1.1 Examples of fields

We will give some example of fields.

- $\mathbb{Q}$, the rational numbers form a field.
- $\mathbb{R}$, the real numbers form a field.
- $\mathbb{C}$, the complex numbers form a field.
- $\mathbb{Z}/p$ for p prime is a field. If we have $[n], [m] \in \mathbb{Z}/p$, where $[x]$ are the natural numbers congruent to x modulo p, we can talk about addition and multiplication in the following way:

$$[n] + [m] = [n + m] \tag{2.4}$$

$$[n] \cdot [m] = [nm] \tag{2.5}$$

  From the extended Euclidean algorithm we can find inverses for every number which is not a multiple of p, and the equivalence class of numbers which are a multiple of p form the unit element in $\mathbb{Z}/p$. Hence $\mathbb{Z}/p$ is a field.

- As an extension of $\mathbb{Z}/p$ we have the Galois fields of order p^n, for p prime and $n \in \mathbb{N}$. These form a field as well. Every finite field of characteristic p prime is isomorphic to the Galois field of order p^n for some $n \in \mathbb{N}$.
- Given a field K the rational functions over K form a field. I.e. $\{P/Q | P, Q \in K[x]\}$ is an field.
- Given a field K and a set S, the maximal set F of functions $f : S \to K$, such that if there exists an $x \in S : f(x) = 0$ then $f(y) = 0 \; \forall y \in S$ with pointwise addition and multiplication.

2.2 Algebra

Definition 1. *An algebra A over a field K is a vector space over K together with a bilinear product $m : A \times A \to A$. We will denote $m(a, b) = a \cdot b$ for $a, b \in A$.*

We will create a list of examples of algebras which we will keep using as examples to illustrate definitions in the rest of the coming two chapters. When we will define properties we will refer back to these examples and examine which of these examples have these properties.

- $\mathbb{R}$ endowed with the real product.
- $\mathbb{R}^2 = \mathbb{C}$ with the complex product. I.e. $(x + yi)(z + wi) = xz - yw + (xw + yz)i$, stated otherwise, for two pairs of real numbers (a, b) and (c, d) their product is $(ac - bd, ad + bc)$. Addition then works pointwise $(a, b) + (c, d) = (a + c, b + d)$.
- $\mathbb{R}^2$ with the split complex product. I.e. $(x + yi)(z + wi) = xz + yw + (wx + yz)i$, we can again restate this in the form of product of pairs of real numbers: $(a, b)(c, d) = (ac + db, ad + bc)$, while addition again works pointwise.
- $\mathbb{R}^4$ with the quaternionic product. The algebra is generated by $\{1, i, j, k\}$ with the following multiplication relation: $i^2 = j^2 = k^2 = ijk = -1$. If we look at the product we can restate it as the product of two pairs of complex numbers: $(a, b)(c, d) = (ac - d^*b, da + bc^*)$, and addition works pointwise.

- $\mathbb{R}^4$ with the split quaternionic product. The algebra is again generated by $\{1, i, j, k\}$ but now the relations are $i^2 = -1$, $j^2 = k^2 = 1$. We have the following relations:

$$ij = k = -ji$$
$$jk = -i = -kj$$
$$ki = j = -ik$$

- $\mathbb{R}^8$ with the product of the Cayley octonions. It is an extension of the quaternions by doubling the dimension again. A possible definition of the multiplication is for pairs of quaternion numbers (a, b) and (c, d) the product is $(ac - d^*b, da + bc^*)$, and addition is defined pointwise.

- $\mathbb{R}^n$ with the piece-wise product $(x_1, .., x_n) \cdot (y_1, ..., y_n) = (x_1y_1, ..., x_ny_n)$.

- $\mathbb{R}^3$ with the cross product $a \times b$.

Definition 2. *If m is associative we call A an associative algebra. In explicit the following holds for all $x, y, z \in A$:*

$$(x \cdot y) \cdot z = x \cdot (y \cdot z) \tag{2.6}$$

Examples

- The real multiplication is associative.
- The complex multiplication is associative.
- The split complex product is associative.
- The quaternion product is associative.
- The split quaternion product is associative.
- The octonions product is not associative.
- $\mathbb{R}^n$ with the piece wise product is associative.
- $\mathbb{R}^3$ with the cross product $a \times b$ is not associative since $j \times (j \times k) = j \times i = k \neq (j \times j) \times k = 0$.

Definition 3. *We call an algebra A commutative if and only if $m(a, b) = m(b, a)$ for all $a, b \in A$.*

Examples

- The real multiplication is commutative.
- The complex multiplication is commutative.
- The split complex product is commutative.
- The quaternion product is not commutative: $ij = -ji$.
- The split quaternion product is not commutative $ij = -ji$.
- The octonions product not commutative since it contains the quaternions as a sub-algebra and hence it is not commutative.

- $\mathbb{R}^n$ with the piecewise product is commutative.
- $\mathbb{R}^3$ with the cross product $a \times b$ is not commutative since $a \times b = -b \times a$.

Definition 4. *We call an algebra A unital if there exists an identity e such that $m(x, e) = m(e, x) = x$ for all x.*

Lemma 1. *Any associative unital algebra A is a ring.*

Proof. Notice that the vector structure provides an addition operator $+$, and by bilinearity m right- and left-distributes over $+$. There exists an identity and the multiplication is associative. Thus A obeys the ring axioms. □

Definition 5. *Given two algebras A, B over a field K, a K-algebra homomorphism is an K-linear map $f : A \to B$ such that $f(xy) = f(x)f(y)$ for all $x, y \in A$.*

Definition 6. *A K-algebra isomorphism is a bijective K-algebra homomorphism.*

Definition 7. *A norm on an algebra A is a function $A \to \mathbb{R}_{\geq 0}$ which obeys the following relations:*

- $|x| = 0 \Leftrightarrow x = 0$.
- $|xy| = |x||y|$ *for all* x, y.
- $|x + y| \leq |x| + |y|$ *for all* x, y.

Examples

- The real multiplication are a normed algebra with as norm the function $\sqrt{x^2}$.
- The complex multiplication forms a normed algebra with $|x + yi| = \sqrt{x^2 + y^2}$.
- The split complex product forms a normed algebra with $|x + yi| = \sqrt{x^2 + y^2}$.
- The quaternion product has a norm: $|x + yi + zj + wk| = \sqrt{x^2 + y^2 + z^2 + w^2}$.
- The split quaternion product has a norm $|(x, y)| = \sqrt{|x|^2 + |y|^2}$.
- The octonions product has also a norm $|(x, y)| = \sqrt{|x|^2 + |y|^2}$.
- $\mathbb{R}^n$ with the piecewise product is not equipped with a norm when $n > 1$, which we will show in chapter 3.
- $\mathbb{R}^3$ with the cross product forms no normed algebra, we will show this in chapter 3.

Definition 8. *We call an algebra A alternative if the following is true for all $x, y \in A$*

$$x(yx) = (xy)x \tag{2.7}$$

$$(xx)y = x(xy) \tag{2.8}$$

$$(xy)y = x(yy) \tag{2.9}$$

(2.10)

All of our examples of algebras except the cross product are alternative.

Definition 9. *We call an algebra A quadratic if for all $x \in A$ there exists $\lambda, \mu \in K$ such that the following holds: $x^2 = \lambda x + \mu e$.*

All of our examples of algebras except the last two are quadratic.

2.2.1 Sub-algebras and ideals

Definition 10. *A sub-algebra B of an algebra A is a linear subspace B such that the product of any two elements of B is again an element of B. Thus a sub-algebra is a subset of elements that is closed under addition, multiplication and scalar multiplication.*

Corollary 2. *Any sub-algebra is also an algebra.*

Proof. Let B be a subalgebra of A. We have a product m on A, this can be restricted to a product on B:

$$m|_{B\times B}(x,y) = m(x,y) \quad \forall x,y \in B \tag{2.11}$$

Then B is a vector space over K since it is a linear subspace of A, together with a bilinear multiplication map $m|_{B\times B}$, since m is bilinear and we restrict it to a linear subspace. This restriction is also closed under multiplication hence get a map $m_B : B \times B \to B$ which obeys all the axioms. □

We can also define left and right ideals I. They have a somewhat stronger property that multiplication by any element of the algebra A with an element in I is again an element of I.

Definition 11. *An left/right ideal of an F-algebra A is a linear subspace I such that for all $x \in I$ we have that for all $y \in A$ we have respectively $x \cdot y \in I$ or $y \cdot x \in I$.*

Notice that every left or right ideal is a sub-algebra. We speak of an ideal if the space is both a left and right ideal.

2.3 Operations on algebras

Given an ideal I of an algebra A, there exists a quotient algebra A/I. This quotient has again a bilinear multiplication m on a vector space. The ideal induces an equivalence relation $\sim$ on A, given $a,b \in A$ we write $a \sim b$ iff $a - b \in I$. This induces equivalence classes on A, and we have addition and multiplication on these equivalence classes: $[a]+[b] := [a+b]$ and $[a]\cdot[b] = [a\cdot b]$. It is clearly well defined since if we have $\tilde{a}, \tilde{b}$ alternative representations for $[a]$ and $[b]$, we get $[\tilde{a}]+[\tilde{b}] = [\tilde{a}+\tilde{b}]$. Since $a - \tilde{a}$ and $b - \tilde{b}$ are both elements of I these are congruent to zero, and hence we can add them under the equivalence relation. This yields

$$[\tilde{a} + \tilde{b}] = [\tilde{a} - \tilde{a} + a + \tilde{b} - \tilde{b} + b] = [a + b] \tag{2.12}$$

And this shows that addition is well defined. For multiplication the following shows that it is well defined: Let $a, b \in A$ and $a', b' \in I$ then

$$\begin{aligned}
[a + a'] \cdot [b + b'] &= [(a + a') \cdot (b + b')] \\
&= [a \cdot b + a' \cdot b + a \cdot b' + a' \cdot b'] \\
&= [a \cdot b]
\end{aligned}$$

The last step is true since I is closed under multiplication of elements of A and hence $a' \cdot b, a \cdot b'$ and $a' \cdot b'$ are all elements of I and thus congruent to zero. This shows that A/I exists and multiplication and addition on A/I are well defined.
Direct products of A and B is again an algebra since we can define $m((\alpha_1, \beta_1), (\alpha_2, \beta_2))$ as$(m_A(\alpha_1, \alpha_2), m_B(\beta_1, \beta_2))$.
Tensor products of algebras form again an algebra since we can define $(\alpha_1 \otimes \beta_1)(\alpha_2 \otimes \beta 2) = (\alpha_1\alpha_2) \otimes (\beta_1\beta_2)$.

2.4 Division algebras

We begin with the definition of a division algebra.

Definition 12. *A division algebra D is an algebra that does not only contain zero, and for every non zero elements $a, b \in D$ there exists precisely one element $x \in D$ with $a = bx$ and precisely one element $y \in D$ with $a = yb$.*

Lemma 3. *If A is an associative algebra, then A is a division algebra if and only if there exists a non zero identity and every non zero element a has a multiplicative inverse.*

Proof. If A is an associative division algebra, then we need to show that there exists a non-zero identity and every non-zero element has an multiplicative inverse.
To show that there exist an unit we use that for every non-zero element x there exists an element e_x such that $x = e_x x$. Notice that $e_x e_x = e_x$, since $x = e_x x = e_x(e_x x) = (e_x e_x)x$ since A is associative. There only exists 1 such element by our hypothesis, thus $e_x = e_x e_x$. Now we are going to show that for every $y \in A$ the following holds: $e_x y = y$. Let $y \in A$. If $y = 0$ then $e_x y = e_x 0 = 0$ hence our statement is true. Now suppose y is non-zero. Then there exists a $z \in A$ such that $y = xz$. Then $e_x y = e_x(xz) = (e_x x)z = xz = y$. Hence $e_x y = y$. Thus e_x is a left unit. We need to show that $xe_x = x$. In order to do that we first show another relation: e_x is non-zero since $x = e_x x$ for some x non-zero, and $0x = 0 \neq x$. Thus e_x is non-zero. This allows us to create 2 extra elements: x_R and x_L. By our hypothesis there exists x_R and x_L such that $e_x = x_L x = x x_R$ since x and e_x are both non-zero.
Then we will show that $x_R = X_L e_x$:

$$\begin{aligned} x_R &= e_x x_R \\ &= (x_L x)x_R \\ &= x_L(x x_R) \\ &= x_L e_x \end{aligned}$$

This then allows us to show that $xe_x = x$:

$$\begin{aligned} xe_x &= x(x_L x) \\ &= (x x_L)x \\ &= (x x_L)(e_x x) \\ &= (x(x_L e_x))x \\ &= (x x_R)x \\ &= e_x x \\ &= x \end{aligned}$$

Thus $xe_x = x$. From this we will also show that $ye_x = y$ for all $y \in A$. Let $y \in A$, if $y = 0$ then $ye_x = 0e_x = 0 = y$. If y is non-zero there exists $z \in A$ such that $y = zx$. Then $ye_x = (zx)e_x = z(xe_x) = zx = y$. Hence $ye_x = y$. This shows that e_x is the unit in A, which we from now on denote by e.
Now we are going to show that left and right inverses are the same. Let $y \in A$ non-zero. Then there exists y_L, y_R such that $e = y_L y = y y_R$. Then

$$\begin{aligned} y_R &= e y_R \\ &= (y_L y)y_R \\ &= y_L(y y_R) \\ &= y_L e \\ &= y_L \end{aligned}$$

Hence every non-zero element y has an inverse y^{-1} such that $yy^{-1} = y^{-1}y = e$.

If A is an associative algebra with unit and every non-zero element has a multiplicative inverse then observe that for every $a, b \in A$ we find that $a = b(b^{-1}a)$ and $a = (ab^{-1})b$ by associativity. Hence A is a division algebra. □

2.4.1 Frobenius theorem

We will require the Cayley-Hamilton theorem before we prove the main theorem of this section. We will not prove the Cayley-Hamilton theorem.

Theorem 4 (Cayley-Hamilton)**.** *Every square matrix M over a commutative ring R satisfies its own characteristic polynomial p, i.e. $p(M) = 0$.*

With this theorem we will prove the next theorem which classifies the associative finite dimensional division algebras.

Theorem 5 (Frobenius)**.** *If D is a finite dimensional associative division algebra of the real numbers, then D is isomorphic to one of the following:* $\mathbb{R}, \mathbb{C}, \mathbb{H}$.

Corollary 6. *If D is a finite dimensional associative commutative division algebra of the real numbers, then D is isomorphic to $\mathbb{R}$ or $\mathbb{C}$.*

Proof. We have a natural inclusion of $\mathbb{R}$ in D since m is bilinear over $\mathbb{R}$, and we have an identity element e. Then $m(\lambda, e)$ is a natural inclusion of $\mathbb{R}$ if we let λ vary over $\mathbb{R}$. Hence we can speak of $\mathbb{R} \subset D$.

We first consider the elements such that their square is a non positive real number, i.e.: $V = \{z | z^2 \in \mathbb{R}_{\leq}\}$. We will show that this set has codimension 1 and that $D = \mathbb{R} \oplus V$.

Let m be the dimension of D. Fix $\alpha \in D$. We can see α as an linear operation of a vector space $m(\alpha, \cdot)$. Then α has a characteristic polynomial $p(x)$. By the fundamental theorem of algebra this polynomial can be written as a product of $(x - z_j)$ with $z_j \in \mathbb{C}$. We make a distinction if a root of this polynomial is in $\mathbb{R}$ or in $\mathbb{C} - \mathbb{R}$. We then get:

$$p(x) = (x - t_1)..(x - t_r)(x - z_1)(x - \bar{z_1})..(x - z_s)(x - \bar{z_s}) \tag{2.13}$$

By the Cayley-Hamilton theorem we know that $p(\alpha) = 0$. Since D is a division algebra, it has no nilpotent elements, and thus one of the following should be zero

- $\alpha - t_j$ for some j. Then $\alpha = t_j$ and thus real.
- $(\alpha - z_j)(\alpha - \bar{z_j})$. Then this expression forms a minimal polynomial of α. Notice that $p(x)$ has the same zeroes as the minimal polynomial of α. Since $p(x)$ is a characteristic polynomial of α it follows that $p(x) = (x - z_j)(x - \bar{z_j})^k$ for some $k \geq 1$. We can now rewrite $(x - z_j)(x - \bar{z_j}) = x^2 - 2Re(z_j)x + |z_j|^2$. The coefficient of x^{2k-1} is the trace of α up to a sign. But we can also see that this is $Re(z_j)$. Hence the trace of α is zero if and only if $Re(z_j) = 0$. But this is also equivalent with $\alpha^2 = -|z_j|^2 \leq 0$. Thus V is the subset for which we have that the trace is zero. Hence V is a vector space, and its the kernel of a non-zero linear form. Hence V has codimension 1.

Thus we have concluded that there exists a vector space V such that this is a codimension 1 vector space, and we can form $D = \mathbb{R} \oplus V$.

We will now define an inner product on V:

Define $\langle \alpha, \beta \rangle = \frac{-ab-ba}{2}$ for $\alpha, \beta \in V$. We will show that this is a positive definite bilinear symmetric real form. First we will show it is real. Since V is a vector space, $\alpha + \beta$ is in V. Since for any element in V their square is a real number, which is smaller or equal to zero. Then $(\alpha + \beta)^2 - \alpha^2 - \beta^2 = ab + ba$, hence $ab + ba$ is the sum of real numbers, and thus real. Also

$\langle \alpha, \alpha \rangle = -\alpha^2$ which is non negative and only zero if $\alpha = 0$. Furthermore it is bilinear, and hence it is an inner product on V. Let W be a subspace of V that generates D, and that is minimal with this property. We can chose an orthonormal basis for V since we have an inner product, hence can we talk about lengths and angles. We will label the elements of this basis as $e_1, .., e_n$. If we use -1 times the inner product we find that these basis elements obey the following relationships:

$$e_i^2 = -1 \tag{2.14}$$

$$e_i e_j = -e_i e_j \tag{2.15}$$

- If $n = 0$ then V contains only zero and thus $D = \mathbb{R}$.
- If $n = 1$ then V contains one element e_1, and then D is generated by 1 and e_1 with $e_1^2 = -1$. Hence we can identify e_1 with i and see that D is isomorphic to $\mathbb{C}$.
- if $n = 2$ then $e_1^2 = e_2^2 = -1$ and $e_1 e_2 = -e_2 e_1$. This is a basis for the quaternions hence in this case D is isomorphic with $\mathbb{H}$.
- if $n > 2$ then we can define $u = e_1 e_2 e_n$. Then $u^2 = e_1 e_2 e_n e_1 e_2 e_n = -e_n e_2 e_1 e_1 e_2 e_n = 1$. Hence we can write $0 = u^2 - 1 = (u-1)(u+1)$. Since there are no zero divisors we see that $u = \pm 1$ and hence $e_n = \pm e_1 e_2$. Hence we can remove e_n and keep a generating basis. Thus this basis was not minimal. Therefore we cannot have $n > 2$.

□

Chapter 3

Classification of all composition algebras

In this chapter we will prove Hurwitz theorem. In order to do that, we will be creating a lot of machinery needed in the proof. At first we will take a closer look to the properties of the quadratic form. Then we will derive general properties of composition algebras. In the end we will use some of these properties to describe a process which creates new algebras from an existing algebra. This will turn out to generate all possible algebras. We then show this process stops after applying this doubling operation 3 times. Then we will have 4 algebras associated to 1 field. This chapter is highly algebraic, and in order to gain the most from this chapter, it is advised to keep a list of the statements of the theorems which we have proved so far. This allows you get get through the proofs in the least time, and yield the most information gained.

3.1 Quadratic forms

Definition 13. *Given a field K and a vector space V over K, a quadratic form on V is a map $N : V \to K$ satisfying*

$$N(\lambda x) = \lambda^2 N(x) \quad \forall \lambda \in K, v \in V \tag{3.1}$$

and the associated map $\langle \cdot, \cdot \rangle_N : V \times V \to K$ given by

$$\langle x, y \rangle = N(x + y) - N(x) - N(y) \tag{3.2}$$

is bilinear.

As an immediate consequence we see that $N(0) = 0$, and $\langle \cdot, \cdot \rangle$ is symmetric. For the rest of this chapter we will assume that every bilinear map is symmetric unless stated otherwise.

3.1.1 Quadratic forms on $\mathbb{R}^n$

We will classify all quadratic forms on $V = \mathbb{R}^n$. The claim is that they are given by $N(X) = x^T Ax$ for some symmetric matrix A and every symmetric matrix A induces an symmetric form. The first step we undertake is showing that given a symmetric map A we can define a quadratic form N induced by A.

$$N(\lambda x) = (\lambda x)^T A(\lambda x) = \lambda^2 x^T Ax = \lambda^2 N(x) \tag{3.3}$$

Hence the first property holds. Now the second property holds since

$$\begin{aligned}
\langle x, y\rangle &= N(x+y) - N(x) - N(y) && (3.4)\\
&= (x+y)^T A(x+y) - x^T Ax - y^T Ay && (3.5)\\
&= x^T Ax + y^T Ax + y^t Ax + y^T Ay - x^T Ax - y^T Ay && (3.6)\\
&= x^T Ay + y^T Ax && (3.7)\\
&= 2x^T Ay && (3.8)
\end{aligned}$$

hence $\langle\cdot,\cdot\rangle$ is bilinear. Thus N is an quadratic form.

Now we want to show the converse. We know that for all bilinear forms $\langle\cdot,\cdot\rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ there exists a symmetric matrix A such that $\langle x, y\rangle = x^T Ay$. The quadratic form then is recovered from $N(x) = \frac{\langle x,x\rangle}{2}$, since $\langle x, x\rangle = N(2x) - 2N(x) = 2N(x)$. Now suppose we have an A and B such that $N(x) = x^T Ax = x^T Bx$. Then we can subtract $x^T Bx$ everywhere and get $x^T(A - B)x = 0$ for all x. Hence $A - B = 0$ and therefore $A = B$. This proof immediately yields the following proposition

Proposition 7. *If the characteristic of K is other than 2 all quadratic forms N on V can be recovered from the bilinear form $\langle\cdot,\cdot\rangle$ via*

$$N(x) = \frac{1}{2}\langle x, x\rangle \tag{3.9}$$

If the characteristic of K is two however, $\langle x, x\rangle = 0$ for all $x \in V$

Proof.

$$\langle x, x\rangle = N(2x) - 2N(x) = 2N(x) \tag{3.10}$$

if the characteristic of K is not 2 then we can divide by 2 and find the expression first claimed, otherwise we get the second claim. □

We will now recall some basic definitions from linear algebra. We call two vectors $x, y \in V$ orthogonal with respect to $\langle\cdot,\cdot\rangle$ if and only if $\langle x, y\rangle = 0$. We call two subsets $U_1, U_2 \in V$ orthogonal iff all vectors in U_1 are orthogonal to all vectors in U_2. We will use $x \perp U$ as a notation for $\{x\} \perp U$. The orthogonal complement $U^\perp$ is $\{x \in V | x \perp U\}$.

Proposition 8. *Given a bilinear form $\langle\cdot,\cdot\rangle : V \times V \to K$ and a subset $U \subset V$ then $U^\perp$ is a subspace of V and $U \subset (U^\perp)^\perp$.*

Proof. We will show that $U^\perp$ is a subspace of V:

- $\langle 0, x\rangle = 0$ for all $x \in V$ hence $0 \in U^\perp$.
- assume that x and y lie in $U^\perp$, then $\langle x, u\rangle + \langle y, u\rangle = \langle x + y, u\rangle = 0$ for all $u \in U$ hence $U^\perp$ is closed under addition.
- Let $x \in U^\perp$, then $\langle \lambda x, u\rangle = \lambda\langle x, u\rangle = \lambda 0 = 0$ hence $U^\perp$ is closed under scaler multiplication

This shows that $U^\perp$ is a linear subspace of V. Now since $x \in U$ then $x \perp U^\perp$ by definition of $U^\perp$. But that is also the definition of being in $(U^\perp)^\perp$, hence $x \in (U^\perp)^\perp$. □

Definition 14. *A bilinear form $\langle\cdot,\cdot\rangle$ is called non-degenerate if the only vector in V orthogonal to all other vectors is 0. Equivalently*

$$\langle x, y\rangle = 0 \quad \forall y \in V \Rightarrow x = 0 \tag{3.11}$$

We will call a quadratic form non-degenerate if its associated bilinear form is non-degenerate.

If the restriction of $\langle\cdot,\cdot\rangle$ to a subspace U of V is non-degenerate we call U non-singular. We will denote $\langle\cdot,\cdot\rangle||U \times U$ and $\perp_U$ will denote the restricted form and the restricted orthogonality, respectively.

We will now show a handy property of non-degenerate forms:

Proposition 9. *If the form $\langle \cdot, \cdot \rangle$ is non-degenerate and $\langle a, y \rangle = \langle b, y \rangle$ for all y then $a = b$.*

Proof.

$$\begin{aligned} &\langle a, y \rangle = \langle b, y \rangle \quad \forall y \in V \\ &\Leftrightarrow \langle a - b, y \rangle = 0 \quad \forall y \in V \\ &\Leftrightarrow a - b = 0 \\ &\Leftrightarrow a = b \end{aligned}$$

□

Another useful proposition which we will need is the following:

Proposition 10. *If $\langle \cdot, \cdot \rangle$ is a bilinear form on a vector space V an U is a subspace of V, then the following are equivalent:*

- *U is non-singular*
- *$U \cap U^{\perp} = \{0\}$*

Proof. If U is non-singular then $\langle \cdot, \cdot \rangle_{|U \times U}$ is non-degenerate. Which is the same as $U^{\perp_U} = \{0\}$. On the other hand $U^{\perp_U} = \{x \in U | x \perp U\} = U \cap U^{\perp}$. Hence $U \cap U^{\perp} = \{0\}$.
Let $U \cap U^{\perp} = 0$. Take $x \in U$ such that $\langle x, y \rangle = 0$ for all $y \in U$. This property is the same as $x \perp U$, hence $x \in U^{\perp}$. This means that $x \in U \cap U^{\perp} = \{0\}$, hence $x = 0$. □

The following statement states that for finite-dimensional vector space their dual space is isomorphic to the space itself.

Lemma 11. *let V be a finite-dimensional vector space and $\langle \cdot, \cdot \rangle$ a non-degenerate bilinear form on V. Then the linear map $\lambda : V \to V^*, v \mapsto \lambda_v = \langle v, \cdot \rangle$ is an isomorphism.*

Proof. Let V be a vector space over K and $dim \quad V = n$ for some $n \in \mathbb{N}$. Let $\langle \cdot, \cdot \rangle : V \times V \to K$ be a non-degenerate bilinear form. Define $\lambda : V \to V^*$ by $v \mapsto \lambda_v \langle v, \cdot \rangle$. We see that λ is linear since $\lambda_{cv+w} = \langle cv + w, \cdot \rangle = \langle cv, \cdot \rangle + \langle w, \cdot \rangle$ for all $v, w \in V$ and $c \in K$. The kernel of λ is $\{0\}$ hence λ is injective. It is also surjective since $dim \quad V^* = dim \quad V$, and finite-dimensional. proof of kernel λ being $\{0\}$

$$\begin{aligned} \ker \lambda &= \{v \in V | \lambda_v = 0\} \\ &= \{v \in V | \lambda_v(w) = 0 \quad \forall w \in V\} \\ &= \{v \in V | \langle v, w \rangle = 0 \quad \forall w \in V\} \\ &= \{0\} \end{aligned}$$

□

This statement allows us to prove the next statement:

Theorem 12. *If V is a vector space over K, $\langle \cdot, \cdot \rangle$ is a bilinear form on V and $U \subset V$ a finite-dimensional non-singular subspace then*

$$V = U \oplus U^{\perp}$$

If we also have that $\langle \cdot, \cdot \rangle$ is non-degenerate then $U^{\perp}$ is non-singular.

Proof. Assume that $U \subset V$ is non-singular and that the dimension of U is finite. Then $U \cap U^{\perp} = \{0\}$. Thus we have the direct sum $U \oplus U^{\perp}$. Now we have to show that this equals V. We see that $U \oplus U^{\perp}$ is a subset of V since U and $U^{\perp}$ are both subsets of V who only have $\{0\}$ as a common element. Now we need to show that V is a subset of $U \oplus U^{\perp}$.
Given a vector $w \in V$ we are going to split it in two parts, one in U and one in $U^{\perp}$. Since U is finite-dimensional and the restriction $\langle \cdot, \cdot \rangle_{|U \times U}$ is non-degenerate, then $\mu : U \to U^*, v \mapsto \mu_v = \langle v, \cdot \rangle_{|U \times U}$ is an isomorphism by the previous lemma. Now let $\lambda V \to V^*, v \mapsto \lambda_v = \langle v, \cdot \rangle$. The form $\lambda_{w|U}$ belongs to U^* as we just ahve shown. Therefore there is a vector $u \in U$, such that $\mu_u = \lambda_{w|U}$. This is the same as:

$$\langle u, v \rangle = \langle w, v \rangle \quad \forall v \in U \tag{3.12}$$

If we subtract the right hand side from the equation we get

$$\langle u - w, v \rangle = 0 \quad \forall v \in U \tag{3.13}$$

We now define $u' = u - w$. Then $u' - w \perp U$ hence $u' \in U^{\perp}$ and $w = u + u'$. Thus we have proven the first part of the statement, namely $V = U \oplus U^{\perp}$.
Now assume that $\langle \cdot, \cdot \rangle$ is non-degenerate, that $U^{\perp}$ is singular and that we have $x \in U^{\perp} \cap (U^{\perp})^{\perp}$. If we can prove that $x = 0$ we are done. We start by taking an element y in V. By the previous part of the proof we know we can write it as $u + u'$, with $u \in U$ and $u' \in U^{\perp}$. By our assumption $x \in U^{\perp}$ and $x \in (U^{\perp})^{\perp}$. This means that $\langle x, u \rangle = \langle x, u' \rangle = 0$. We can add these together and get $\langle x, u \rangle + \langle x, u \rangle = \langle x, u + u \rangle = \langle x, y \rangle = 0$. This is true for all y. Since $\langle \cdot, \cdot \rangle$ is non-degenerate this means that $x = 0$. □

This theorem has 2 corollaries:

Corollary 13. *If $\langle \cdot, \cdot \rangle : V \times V \to K$ is a bilinear form on a vector space V and $U \subset V$ is a finite-dimensional non-singular subspace, then the dimension of V is the sum of the dimensions of U and $U^{\perp}$.*

Proof. We just use:

$$\begin{aligned} dim \quad V &= dim \quad (U \oplus U^{\perp}) \\ &= dim \quad U + dim \quad U^{\perp} \end{aligned}$$

and the theorem is proven. □

Corollary 14. *If $\langle \cdot, \cdot \rangle : V \times V \to K$ is a non-degenerate bilinear form and U is a finite-dimensional non-singular subspace of V then $U = (U^{\perp})^{\perp}$.*

Proof. We already know that U is a subset of $(U^{\perp})^{\perp}$ so all we need to do is prove the other inlcusion. Let $x \in (U^{\perp})^{\perp}$. We know that $V = U \oplus U^{\perp}$. Hence we can write $x = u + u'$ with $u \in U$ and $u' \in U^{\perp}$. But then $u \in (U^{\perp})^{\perp}$ since $u \in U$. Hence $x - u \in (U^{\perp})^{\perp}$. We also know that $x - u = u' \in U^{\perp}$. This implies that $x - u \in U^{\perp} \cap (U^{\perp})^{\perp}$ and hence $x - u = 0$. If we now add u on both sides we get $x = u$ hence $x \in U$. □

This concludes our discussion on quadratic forms.

3.2 Composition algebras

In this section we are going to reveal the structure which a composition algebra naturally has. We define a composition algebra:

Definition 15. *A composition algebra over a field K is a pair (C, N) where C is a nonzero unital algebra and $N : C \to K$ a non-degenerate quadratic form that satisfies*

$$N(xy) = N(x)N(y) \quad \forall x, y \in C \tag{3.14}$$

The quadratic form N is somethimes called a norm and the associated bilinear form $\langle \cdot, \cdot \rangle$ an inner product. However the inner product need not be an inner product in the usual sence since it need not be positive semidefinite.

Definition 16. *A composition subalgebra of an composition algebra (C, N) is a pair (D, N) with D a unital subalgebra of C.*

There are several important examples of composition algebras. The real numbers are an example, $(\mathbb{R}, |.|)$ is a composition algebra. In the same fashion are the complex numbers, the quaternions and the octonions composition algebras with the euclidean norm. We will use these algebras as examples of composition algebras.

Proposition 15. *Let (C, N) be any composition algebra, then the identity e satisfies*

$$N(e) = 1 \tag{3.15}$$

Proof.

$$N(e) = N(ee) = N(e)N(e) \tag{3.16}$$

Hence $N(e) = 0$ or $N(e) = 1$, since in a field these are the only two elements that obey the property that they are equal to their squares.
Now suppose $N(e) = 0$. Then this implies that $N(x) = N(x)N(e) = 0$ for all $x \in C$. Hence $\langle e, x \rangle = 0 \quad \forall x \in C$. However, since $\langle \cdot, \cdot \rangle$ is non-degenerate this means that $e = 0$. But e cannot be the zero element, hence $N(e) = 1$. □

Proposition 16. *If x is an invertible element of a composition algebra (C, N), then*

$$N(x^{-1}) = N(x)^{-1} \tag{3.17}$$

In particular we know that $N(x)$ and $N(x^{-1})$ cannot be zero.

Proof. Let x be an invertible element of a composition algebra (C, N). Then the following holds:

$$1 = N(e) = N(xx^{-1}) = N(x)N(x^{-1}) \tag{3.18}$$

Hence, $N(x^{-1}) = N(x)^{-1}$. □

We will later show that $N(x) \neq 0$ is enough to have an inverse, and that that inverse is unique. However, to show that we need to construct extra machinery.

Proposition 17. *In every composition algebra (C, N) the following identities hold for all $x, x_1, x_2, y, y_1, y_2 \in C$*

$$\langle x_1 y, x_2 y \rangle = \langle x_1, x_2 \rangle N(y)$$
$$\langle x y_1, x y_2 \rangle = N(x) \langle y_1, y_2 \rangle$$
$$\langle x_1 y_1, x_2 y_2 \rangle + \langle x_1 y_2, x_2 y_1 \rangle = \langle x_1, x_2 \rangle \langle y_1, y_2 \rangle$$

Proof. We start by proving the first equality. The second equality follows from the exact argument, but then mirrored.

$$\begin{aligned}\langle x_1 y, x_2 y\rangle &= N(x_1 y + x_2 y) - N(x_1 y) - N(x_2 y)\\ &= N((x_1 + x_2)y) - N(x_1 y) - N(x_2 y)\\ &= N(x_1 + x_2)N(y) - N(x_1)N(y) - N(x_2)N(y)\\ &= (N(x_1 + x_2) - N(x_1) - N(x_2))N(y)\\ &= \langle x_1, x_2\rangle N(y)\end{aligned}$$

The third identity we have

$$\begin{aligned}\langle x_1, x_2\rangle N(y_1 + y_2) &= \langle x_1(y_1 + y_2, x_2(y_1 + y_2)\rangle\\ &= \langle x_1 y_1 + x_2 y_2, x_2 y_1 + x_2 y_2\rangle\\ &= \langle x_1 y_1, x_2 y_1 x_2 y_2\rangle + \langle x_1 y_2, x_2 y_1 + x_2 y_1 + x_2 y_2\rangle\\ &= \langle x_1 y_1, x_2 y_1\rangle + \langle x_1 y_1, x_2 y_2\rangle + \langle x_1 y_2, x_2 y_1\rangle + \langle x_1 y_2, x_2 y_2\rangle\\ &= \langle x_1, x_2\rangle N(y_1) + \langle x_1 y_1, x_2 y_2\rangle + \langle x_1 y_2, x_2 y_1\rangle + \langle x_1, x_2\rangle N(y_2)\end{aligned}$$

But we also have

$$\begin{aligned}\langle x_1, x_2\rangle N(y_1 + y_2) &= \langle x_1, x_2\rangle(\langle y_1, y_2\rangle + N(y_1) + N(y_2))\\ &= \langle x_1, x_2\rangle\langle y_1, y_2\rangle + \langle x_1, x_2\rangle(N(y_1) + N(y_2))\end{aligned}$$

If we subtract $\langle x_1, x_2\rangle(N(y_1) + N(y_2))$ from the last terms of both equations we get the last equation. ☐

Corollary 18. *Let (C, N) be a composition algebra and let $x, y \in C$ with $\langle x, y\rangle = 0$. Then for all $x_1, y_1 \in C$ we have*

$$\langle x_1 x, y_1 y\rangle = -\langle x_1 y, y_1 x\rangle \tag{3.19}$$

Proof. This statement follows directly from the last equality we have just shown. ☐

Proposition 19. *Let (C, N) be a composition algebra and x an element in C. Then the following identity holds*

$$x^2 - \langle x, e\rangle x + N(x)e = 0 \tag{3.20}$$

Proof. We take $x \in C$. Then for abritrary y we are going to recreate the inner product of $x^2 - \langle x, e\rangle x + N(x)e$ with y. This yields

$$\langle x^2 - \langle x, e\rangle x + N(x)e, y\rangle = \langle x^2, y\rangle - \langle x, e\rangle\langle x, y\rangle + N(x)\langle e, y\rangle$$

Then we can take in the $N(x)$ into the inner product. This yields

$$\langle x^2 - \langle x, e\rangle x + N(x)e, y\rangle = \langle x^2, y\rangle - \langle x, e\rangle\langle x, y\rangle + \langle xe, xy\rangle$$

Then, since $\langle x^2, y\rangle + \langle xe, xy\rangle = \langle x, e\rangle\langle x, y\rangle$ by the last equation of the previous proposition, we see that the first and the last term cancel versus the second term. This implies

$$\langle x^2 - \langle x, e\rangle x + N(x)e, y\rangle = 0$$

This holds for all y, and since the quadratic form is not degenerate we have $x^2 - \langle x, e\rangle x + N(x)e = 0$ ☐

Corollary 20. *Every composition algebra is a quadratic algebra*

Proof. Let (C, N) be a composition algebra. Then for all $x \in C$, $x^2 = \langle x, e\rangle x - N(x)e$. Hence $x^2 \in span\{e, x\}$. ☐

3.2.1 Imaginary elements and conjugation

On the real algebras we have a concept of complex numbers. For example in the complex field $\mathbb{C}$ there are the numbers so that they squared are an real number, but the number themselves are not a member of $\mathbb{R} - \{0\}$.

Definition 17. *The set of all purely imaginary numbers of a composition algebra (C, N) over K is the set*

$$Im \quad C = \{x \in C | x^2 \in Ke, x \notin Ke - \{0\}\} \tag{3.21}$$

Proposition 21. *Let (C, N) be a composition algebra over a field K with char K unequal to 2. Then $Im \quad C$ is a non-singular subspace of C, $Im \quad C = (Ke)^{\perp}$ and $C = Im \quad C \oplus Ke$*

Proof. It is enough to prove that Ke is a non-singular subspace and that Im $\quad C = (Ke)^{\perp}$. The rest will follow from the theorem we proved in the previous section since the dimension of Ke is 1.
Asume that Ke is singular. Then $x \in (Ke)^{\perp} \cap Ke$ for some non-zero $x \in C$. Hence $x = \lambda e, \lambda \neq 0$ and $\langle x, e \rangle = 0$. However, $0 = \langle e, e \rangle = \lambda \langle e, e \rangle = 2\lambda$. Hence $\lambda = 0$, and thus $x = 0$. Thus this leads to a contradiction, which implies that (Ke) is singular.
Now we are going to show Im $\quad C = (Ke)^{\perp}$. As a first step we will show Im $\quad C \subset (Ke)^{\perp}$. Let $x \in$ Im $\quad (C) - \{0\}$. Thus x and e are linearly independent and $x^2 = \lambda e$ for some λ. We also have $N(x) = -\lambda$ and $\langle x, e \rangle = 0$. Hence $x \in (Ke)^{\perp}$.
Now the other inclusion. Assume $x \in (Ke)^{\perp}$. Then $\langle x, e \rangle = 0$ and we again obtain $x^2 = \langle x, e \rangle x - N(x)e = -N(x)e$. Since $x \perp Ke$ we have $x \neq Ke - \{0\}$ since Ke is non-singular. Hence $x \in$ Im $\quad C$. Thus $C = (Ke)^{\perp}$. This concludes the proof. □

This proposition has an intresting corollary, which states that for a given algebra there is just one quadratic form.

Corollary 22. *If (C, M) and (C, N) are composition algebras, then $M = N$.*

Proof. For every $x \in C$ we have by the previous proposition

$$x^2 = \langle x, e \rangle_M x - M(x)e$$
$$x^2 = \langle x, e \rangle_N x - N(x)e$$

Hence

$$\langle x, e \rangle_M x - M(x)e = \langle x, e \rangle_N x - N(x)e \tag{3.22}$$

It suffices to show that $\langle x, e \rangle_M x = \langle x, e \rangle_N x$. If we take this to the other side we see that this is equivalent with $(\langle x, e \rangle_M - \langle x, e \rangle_N)x = 0$, and this is equivalent with $\langle x, e \rangle_M = \langle x, e \rangle_N$.
First suppose e and x are linearly dependent. Then $x = \lambda_0 e + \sum_{i=1}^{n} \lambda_i e_i$ for some $\lambda \in K^n$. Then we obtain

$$M(x) = \lambda^2 M(e) = \lambda^2 = \lambda^2 N(e) = N(x)$$

This proves the claim in this case.
Now suppose they are linearly independent. Consider the splitting of C into $Ke \oplus$ Im $\quad C$. Then Im $\quad C$ is perpendicular Ke with respect to both inner products. If n is the dimension of C, we can find a basis for Im $\quad C$, $(e_2, .., e_n)$. If we take $e_1 = e$ we can write $x = sum_{i=1}^{n} \lambda_i e_i$. This gives $\langle x, e \rangle_M = \langle sum_{i=1}^{n} \lambda_i e_i, e \rangle$ and since Im $\quad C$ is perpendicular to Ke with respect to $\langle , \rangle_M$ we get $\langle x, e \rangle_M = \langle \lambda_1 e, e \rangle = \lambda_1 \langle e, e \rangle_M$. In the same way we get $\langle x, e \rangle_N = \lambda_1 \langle e, e \rangle_n$. Now there are two cases, the characteristic is other than two or not. Suppose the characteristic is other than two, then we get that $\langle e, e \rangle_M = \frac{1}{2} = \langle e, e \rangle_N$. On the other hand if the characteristic is 2 then $\langle e, e \rangle_M = 0 = \langle e, e \rangle_N$. Hence in both cases $\langle e, e \rangle_N = \langle e, e \rangle_M$. This yields $-M(x) = -N(x)$. Hence both cases lead to $M(x) = N(x)$. □

For the complex numbers we have a complex conjugation map. In general there also exists a conjugation map which behaves a lot like the conjugation map for complex numbers.

Definition 18. *Let (C, N) be a composition K-algebra. The conjugation map $\bar{\cdot} : C \to C, x \mapsto \overline{x}$ is defined by*

$$\overline{x} = \langle x, e\rangle e - x \tag{3.23}$$

Geometrically speaking, this is minus the reflection of x in $(Ke)^{\perp}$. For the complex numbers the definition is the same with the usual definition of complex conjugates.

Proposition 23. *Let (C, N) be a composition algebra. Then conjugation is a linear map and $\overline{\overline{x}} = x$.*

Proof. Linearity follows straightforward of the definition since $\overline{x}$ is the sum of two linear maps ($\langle x, e\rangle e$ and minus the identity map). To show that $\overline{\overline{x}} = x$ we directly compute it:

$$\begin{aligned}
\overline{\overline{x}} &= \langle \overline{x}, e\rangle e - \overline{x} \\
&= \langle \langle x, e\rangle e - x, e\rangle e - \langle x, e\rangle e + x \\
&= \langle x, e\rangle \langle e, e\rangle e - \langle x, e\rangle e - \langle x, e\rangle e + x \\
&= 2\langle x, e\rangle e - 2\langle x, e\rangle e + x \\
&= x
\end{aligned}$$

Here we used that $\langle e, e\rangle = 2N(e) = 2$. ∎

Other important properties of conjugation hold too. They have to be slightly modified.

Proposition 24. *In every composition algebra (C, N) the following holds for all $x, y \in C$.*

- $x\overline{x} = \overline{x}x = N(x)e$
- $\overline{xy} = \overline{yx}$
- $N(\overline{x}) = N(x)$
- $\langle \overline{x}, \overline{y}\rangle = \langle x, y\rangle$

Proof. The first property we will show one side, and the other side is symmetric:

$$\begin{aligned}
N(x)e &= \langle x, e\rangle x - x^2 \\
&= (\langle x, e\rangle e - x)x \\
&= \overline{x}x
\end{aligned}$$

For the second property observe

$$xy = \langle x, e\rangle e + \langle x, e\rangle y - \langle x, y\rangle e - yx \tag{3.24}$$

This holds since

$$\begin{aligned}
xy &= (x + y)^2 - x^2 - y^2 - yx \\
&- \langle x + y, e\rangle (x + y) - N(x + y)e - \langle x, e\rangle x + N(x)e0\langle y, e\rangle y + N(y)e - yx \\
&= \langle x, e\rangle y + \langle y, e\rangle x - (N(x + y) - N(x) - N(y))e - yx \\
&= \langle y, e\rangle x + \langle x, e\rangle y - \langle x, y\rangle e - yx
\end{aligned}$$

By the definition of conjugation we have

$$\begin{aligned}
\overline{y}\,\overline{x} &= (\langle y, e\rangle e - y)(\langle x, e\rangle e - x) \\
&= \langle x, e\rangle \langle y, e\rangle e - \langle x, e\rangle - \langle y, e\rangle x + yx \\
&= \langle xy, e\rangle e - xy \\
&\overline{xy}
\end{aligned}$$

Hence we have shown the second property.

For the third property we use the two previous properties. This yields

$$N(\overline{x})e = \overline{\overline{x}}\,\overline{x} = x\overline{x} = N(x)e \tag{3.25}$$

Hence $N(\overline{x}) = N(x)$

Now if we use the third property we can show the fourth property

$$\begin{aligned}
\langle \overline{x}, \overline{y}\rangle &= N(\overline{x} + \overline{y}) - N(\overline{x}) - N(\overline{y}) \qquad\qquad = N(\overline{x + y}) - N(\overline{x} - N(\overline{y}) \\
&= N(x + y) - N(x) - N(y) \\
&= \langle x, y\rangle
\end{aligned}$$

□

As we said earlier about existence of inverses, we can show the following proposition now

Proposition 25. *In every composition algebra (C, N) and for every $x \in C$ the following two statements are equivalent:*

- *x has an inverse*
- *$N(x) \neq 0$*

In the case $N(x)^{-1}\overline{x}$ is the inverse of x

Proof. We have shown in proposition 16 on page 16 that if x has in inverse then $N(x) \neq 0$.

Now suppose we know $N(x) \neq 0$. Define $y = N(x)^{-1}\overline{x}$. This yields

$$xy = x(N(x)^{-1}\overline{x}) = N(x)^{-1}(x\overline{x}) = N(x)^{-1}N(x)e = e \tag{3.26}$$

The other equality $yx = e$ is just the same statement but then mirrored hence $xy = yx = e$. □

3.2.2 Associativity properties

So far we have not looked at any form of associativity properties of composition algebras. We will start investigating these now. It will turn out that although composition algebras need not be associative they obey a lot of other properties that are almost the same. We will start by showing not every composition algebra must be associative. We know the octonions are a composition algebra with the usual euclidean norm. However, the octonions are not associative. What we will see in the end is that every non associative composition algebra still obeys a lot of weaker associativity properties such as alternativity, and that every non associative composition K-algebra is an 8 dimensional K algebra and there are no K algebras with higher dimension than 8.

Proposition 26. *Let (C, N) be a composition algebra and let $x, y, z \in C$. Then the following three equalities hold:*

$$\langle xy, z\rangle = \langle y, \overline{x}z\rangle \tag{3.27}$$

$$\langle xy, z\rangle = \langle x, z\overline{y}\rangle \tag{3.28}$$

$$\langle xy, \overline{z}\rangle = \langle yz, \overline{x}\rangle \tag{3.29}$$

$$\tag{3.30}$$

Proof.

$$\begin{aligned}
\langle y, \overline{x}z\rangle &= \langle y, (\langle x, e\rangle e - x)z\rangle \\
&= \langle y, \langle x, e\rangle z\rangle - \langle y, xz\rangle \\
&= \langle x, e\rangle\langle y, z\rangle - \langle y, xz\rangle \\
&= \langle xy, z\rangle + \langle xz, y\rangle - \langle y, xz\rangle \\
&= \langle xy, z\rangle
\end{aligned}$$

To prove the second equality we use the following

$$\begin{aligned}
\langle xy, z\rangle &= \langle y, \overline{x}z\rangle \\
&= \langle \overline{y}, \overline{z}x\rangle \\
&= \langle z\overline{y}, x\rangle \\
&= \langle x, z\overline{y}\rangle
\end{aligned}$$

$$\begin{aligned}
\langle xy, \overline{z}\rangle &= \langle \overline{z}xy\rangle \\
&= \langle z, \overline{xy}\rangle \\
&= \langle z, \overline{xy}\rangle \\
&= \langle yz, \overline{x}\rangle
\end{aligned}$$

□

With this proposition we can now define adjoint operators on a composition algebra. First recall that the adjoint of an linear operator f on an inner product space V is the unique linear operator such that $\langle f(x), y\rangle = \langle x, f^*(y)\rangle$ for all $x, y \in V$. Using this definition the adjoints in composition algebras are given by

$$L_x^* = L_{\overline{x}}$$

$$R_y^* = R_{\overline{y}}$$

They are adjoints since $\langle L_x(y), z\rangle = xy, z\rangle = \langle y, \overline{x}z\rangle = \langle y, L_{\overline{x}}(z)\rangle$, and similarly for R_y.
The next proposition is a generalization of proposition 24 of page 19.

Proposition 27. *If (C, N) is a composition algebra and $x, y \in C$. Then the following holds:*

$$x(\overline{x}y) = N(x)y \tag{3.31}$$

$$(x\overline{y})y = N(y)x \tag{3.32}$$

Proof. We will show that $\langle x(\overline{x}y), z\rangle = \langle N(x)y, z\rangle$ for all z. Then we have equality due to proposition 9 on page 14

$$\begin{aligned}\langle x(\overline{x}y), z\rangle &= \langle \overline{x}y, \overline{x}z\rangle \\ &= \langle x\overline{x}y, z\rangle \\ &= N(x)\langle y, z\rangle \\ &= \langle N(x)y, z\rangle\end{aligned}$$

This holds for all z hence $x(\overline{x}y) = N(x)y$. If we now use conjugation we get the following equation:

$$\overline{x(\overline{x}y)} = (\overline{\overline{x}y})\overline{x} = (\overline{y}x)\overline{x} \tag{3.33}$$

and we also have

$$\overline{N(x)y} = N(x)\overline{y} \tag{3.34}$$

These two combined prove the second equation. □

The previous proposition has the first result on associativity:

Corollary 28. *Let (C, N) be a composition algebra. Then for all x and y in C the following holds*

$$x(\overline{x}y) = (x\overline{x})y \tag{3.35}$$

$$x(\overline{y}y) = (x\overline{y})y \tag{3.36}$$

$$\tag{3.37}$$

Proof. This result follows directly from the previous proposition and using $x\overline{x} = N(x)e$ □

Another corollary states the uniqueness of inverses:

Corollary 29. *Let (C, N) be a composition algebra. Then every element $x \in C$ that satisfies $N(x) \neq 0$ has an unique inverse $N(x)^{-1}\overline{x}$*

Proof. Let $x \in C$ and $N(x) \neq 0$. Then x has an inverse $y = N(x)^{-1}\overline{x}$. Assume that there is another element z such that $xz = zx = e$. Then

$$\begin{aligned}y &= y(xz) \\ &= N(x)^{-1}\overline{x}(xz) \\ &= N(x)^{-1}N(x)z \\ &= z\end{aligned}$$

□

There is another form of associativity rules which composition algberas obey. They are the Moufang identities:

Proposition 30. *In every composition algebra (C, N) the Moufang identities hold:*

$$(ax)(ya) = a((xy)a) \tag{3.38}$$

$$a(x(ay)) = (a(xa))y \tag{3.39}$$

$$x(a(ya)) = ((xa)y)a \tag{3.40}$$

for all $a, x, y \in C$.

Proof. We will take the inner product of $(ax)(ya)$ with an arbitrary element $z \in C$. Then we will show that that inner product equals the inner product of $a((xy)a)$ with z and hence these two elements are equal.

$$\begin{aligned}
\langle (ax)(ya), z \rangle &= \langle ya, (\overline{a}\overline{x})z \rangle \\
&= \langle ya, (\overline{xa})z \rangle \\
&= \langle y, \overline{xa} \rangle \langle a, z \rangle - \langle yz, (\overline{xa})a \rangle \\
&= \langle xy, \overline{a} \rangle \langle a, z \rangle - \langle yz, N(a)\overline{x} \rangle \\
&= \langle xy, \overline{a} \rangle \langle a, z \rangle - N(a) \langle yz, \overline{x} \rangle \\
&= \langle xy, \overline{a} \rangle \langle a, z \rangle - N(a) \langle xy, \overline{z} \rangle \\
&= \langle xy, \overline{a} \rangle \langle a, z \rangle - N(a) \langle (xy)z, e \rangle \\
&= \langle xy, \overline{a} \rangle \langle a, z \rangle - \langle (xy)z, \overline{a}a \rangle \\
&= \langle (xy)a, \overline{a}z \rangle \\
&= \langle a((xy)a), z \rangle
\end{aligned}$$

This proves the first equality. The second equality is again proven by a long chain of equalities.

$$\begin{aligned}
\langle a(x(ay)), z \rangle &= \langle x(ay), \overline{a}z \rangle \\
&= \langle x, (\overline{a}z)(\overline{ay}) \rangle \\
&= \langle \overline{x}, \overline{(\overline{a}z)(\overline{ay})} \rangle \\
&= \langle \overline{x}, (ay)(\overline{z}a) \rangle \\
&= \langle x, \overline{a((y\overline{z})a)} \rangle \\
&= \langle x, ((\overline{(y\overline{z})a}))\overline{a} \rangle \\
&= \langle x, (\overline{a}(z\overline{y}))\overline{a} \rangle \\
&= \langle xa, \overline{a}(z\overline{y}) \rangle \\
&= \langle a(xa), x\overline{y} \rangle \\
&= \langle (a(xa))y, z \rangle
\end{aligned}$$

This proves the second equality. To prove the third we start with the second and we conjugate it. This yields the equality we want. □

Proposition 31. *Let (C, N) be a composition algebra. Then C is alternative.*

This proposition can be generalised into this proposition, which has the same proof:

Proposition 32. *Let A be an unital Algebra that obeys the Moufang identities, then A is alternative (see definition 8 on page 7.*

Proof. We need to prove the following equalities:

$$x(yx) = (xy)x \tag{3.41}$$

$$(xx)y = x(xy) \tag{3.42}$$

$$(xy)y = x(yy) \tag{3.43}$$

$$\tag{3.44}$$

The first equality follows directly if we multiply this by e on both sides. This yields

$$(xy)x = (xy)(ex) = x((ye)x) = x(yx) \tag{3.45}$$

The second equation follows from

$$x(xy) = x(e(xy)) = (x(ex))y = (xx)y \tag{3.46}$$

The third equality follows from

$$x(yy) = x(y(ey)) = ((xy)e)y = (xy)y \tag{3.47}$$

This concludes our proof. □

If we summarize our results into this theorem:

Theorem 33. *Let (C, N) be a composition algebra. Then C is a quadratic, alternative algebra that satisfies the Moufang identities.*

3.2.3 Doubling

We will now start our work on the doubling construction. This will allow us to construct all the possible composition algebras over a field.

Lemma 34. *Let (C, N) be a composition algebra. If $D \subset C$ is a finite-dimensional proper non-singular subspace of C, then there is an element $a \in D^{\perp} - \{0\}$ such that $N(a) \neq 0$.*

Proof. Asumme that $N(x) = 0$ for all $x \in D^{\perp}$. We know that $C = D \oplus D^{\perp}$ where $D^{\perp}$ is non-singular. We also know that $D \neq C$, hence $D^{\perp} \neq \{0\}$. Now take any element a in D which is nonzero. Then

$$\langle a, x \rangle = N(a + x) - N(a) - N(x) = 0 - 0 - 0 = 0 \quad \forall x \in D^{\perp} \tag{3.48}$$

Hence since $D^{\perp}$ is non-singular we have $a = 0$. This contradicts our assumption, hence not all elements of $D^{\perp} - \{0\}$ can have nonzero norm. □

If we take D a proper subset of C and D is also a finite-dimensional proper composition subalgebra, then this lemma tells us that we can create another subalgebra of C

$$D_2 = D_1 \oplus D_1 a \tag{3.49}$$

where a is an nonzero element of $D^{\perp}$ such that $N(a) \neq 0$. We first start with an example of the doubling construction. Consider the complex field $\mathbb{C}$. In there we can identify the real numbers by $\mathbb{R}1$. Then we show that there exists an number in $\mathbb{R}1^{\perp}$. For example take $i \in \mathbb{C}$. Then we can see $\mathbb{C} = \mathbb{R} + \mathbb{R}i$. Then we can define addition of $(x, y) + (w, z) = (x + w, y + z)$ and multiplication as $(xw - yz, xz + yw)$. We will now start the doubling construction:

Proposition 35. *Let (C, N) be a composition algebra. If D_1 is a finite-dimensional proper composition subalgebra of C, $a \in D_1^{\perp} - \{0\}$ and $N(a) \neq 0$ then the subspace*

$$D_2 = D_1 \oplus D_1 a \tag{3.50}$$

of C is a composition subalgebra of C with dimension of D_2 twice the dimension of D_1. Multiplication of D_2 then is defined in the following way:

$$(x + ya)(z + wa) = (xz - N(a)\overline{w}y) + (wx + y\overline{z})z \tag{3.51}$$

Proof. We will split the proof into 2 parts. The first part will proof that the sum is direct. The second part will work on the proof of the multiplication. If we show that the claimed multiplication rule is the correct rule, we have shown that D_2 is closed under multiplication and non-singular.
We start by showing that the sum is a direct sum. We will show that $D_1 a \subset D_1^\perp$. This proves that D_2 is a direct sum. Hence, what we need to do is $xa \perp D_1$. Hence what we need to show is

$$\langle xa, y\rangle = 0 \quad \forall x, y \in D_1 \tag{3.52}$$

We know that D_1 is closed under conjugation and multiplication since it is a composition subalgebra. We know that $\langle xa, y\rangle = \langle a, \overline{x}y\rangle$. But $\overline{x}y$ is an element of D_1 and $a \in D_1^\perp$. Hence this inner product is zero. Thus $xa \perp y$. Thus $D_1 a \subset D_1^\perp$.
The second part of the proof is to work out the multiplication. The left hand side of the multiplication is the multiplication what it should be in C. If we work this multiplication out we get

$$(x + ya)(z + wa) = xz + x(wa) + (ya)z + (ya)(wa) \tag{3.53}$$

If we show the following equalities we are done:

$$x(wa) = (wx)a \tag{3.54}$$

$$(ya)z = (y\overline{z})a \tag{3.55}$$

$$(ya)(wa) = -N(a)\overline{w}y \tag{3.56}$$

At first we note that $\overline{v} = -v$ for all $v \in C$ with $v \perp e$. In particular we get

$$\overline{va} = -va \tag{3.57}$$

for all $v \in D_1^\perp$
The next step is to prove the first equality. We take an arbitrary $v \in D_1$. Then we do the following

$$\begin{aligned}
\langle x(wa), v\rangle &= \langle wa, \overline{x}v\rangle \\
&= \langle \overline{wa}, \overline{\overline{x}v}\rangle \\
&= \langle -wa, \overline{v}x\rangle \\
&= -\langle wa, \overline{v}x\rangle \\
&= \langle wx, \overline{v}a\rangle \\
&= \langle wx, (\langle v, e\rangle e - v)a\rangle \\
&= \langle v, e\rangle\langle wx, a\rangle - \langle wx, va\rangle \\
&= \langle v, e\rangle 0 - \langle wx, va\rangle \\
&= -\langle (wx)\overline{a}, v\rangle \\
&= \langle (wx)a, v\rangle
\end{aligned}$$

Since this holds for arbitrary v it holds that $x(wa) = (wx)a$ for all $x, w, a \in D_2$. The second equality follows since

$$\begin{aligned}
\langle (ya)u, z\rangle &= \langle ya, z\overline{u}\rangle \\
&= -\langle y\overline{u}, za\rangle \\
&= -\langle (y\overline{u})\overline{a}, z\rangle \\
&= \langle (y\overline{u})a, z\rangle
\end{aligned}$$

The third equality follows from the equality

$$va = \overline{\overline{va}} = \overline{-va} = -\overline{a}\overline{v} = a\overline{v} \tag{3.58}$$

This allows us to compute

$$\begin{aligned}
(ya)(wa) &= (a\overline{y})(wa) \\
&= a((\overline{y}w)a) \\
&= a(a(bar\overline{y}w)) \\
&= a(a(\overline{w}a)) \\
&= (aa)(\overline{w}y) \\
&= -(a\overline{a})(\overline{w}y) \\
&= -N(a)(\overline{w}y)
\end{aligned}$$

Hence D_2 is closed under multiplication. What is left to do is to show D_2 is non-singular. We look at the norm N_{D_2}. We get

$$\begin{aligned}
N(x + ya) &= \frac{1}{2}\langle x + ya, x + ya\rangle \\
&= \frac{\langle x, x\rangle + \langle ya, ya\rangle + \langle x, ya\rangle + \langle ya, x\rangle}{2} \\
&= N(x) + N(ya) + \langle x, ya\rangle \qquad = N(x) + N(y)N(a)
\end{aligned}$$

The last step holds since $ay \perp x$.
Now we take a look on the inner product, we get for $x, yv, w \in D_1$

$$\begin{aligned}
\langle x + ya, v + wa\rangle &= \langle x, v\rangle + \langle x, wa\rangle + \langle ya, v\rangle + \langle ya, wa\rangle \\
&= \langle x, v\rangle + \langle ya, wa\rangle \\
&= \langle x, v\rangle + \langle y, w\rangle N(a)
\end{aligned}$$

fix $x, y \in D_1$. Now assume that the inner product stated above is zero for all $v, y \in D_1$. Then $\langle x, v\rangle + \langle y, w\rangle N(a) = 0$ for all $v, w \in D$. Suppose $w = 0$. Then $\langle x, v\rangle = 0$ for all v and hence $x = 0$. Suppose $v = 0$. Then we can divide by $N(a)$ since $N(a)$ is a non-zero field element. Hence yield $\langle y, w\rangle = 0$ for all w. This gives $y = 0$. Thus the only element for which this inner product with all other elements of D_2 is zero is 0. Thus D_2 is non-singular.
The last step is to prove the claim that the dimension of D_2 is twice the dimension of D_1. What we need to show is that the dimension of D_1a is the dimension of D_1. We define R_a to be the map $x \mapsto xa$. This is a linear map since multiplication is bilinear. It is also invertible since a has an inverse. Hence $R_a^{-1} = R_{a^{-1}}$. This gives a bijective map and hence it is an isomorphism. This implies that the dimension of D_1a is the dimension of D_1. Since the dimension of D_2 is the dimension of D_1 plus the dimension of D_1a, we get that the dimension of D_2 is twice the dimension of D_1. □

This yields us a powerfull tool to create new algebras. But we have not shown much properties of this new algebra.

Proposition 36. *Let (C, N) be a composition algebra and D a finite-dimensional proper composition subalgebra. Then D is associative.*

Proof. Let $a \in D^{\perp} - \{0\}$ with $N(a) \neq 0$. We already have shown that such an element must exist. Then we use the following equality $N((x+ya)(z+wa)) = N(x+ya)N(z+wa)$ for $x, y, zw \in D$. This yields

$$\begin{aligned}N((x+ya)(z+wa)) &= N((xz - N(a)\overline{w}y) + (wx + y\overline{z})a) \\ &= N(xz - N(a)\overline{w}y) + N(a)N(wx + y\overline{z}) \\ &= \langle xz, -N(a)\overline{w}y\rangle + N(xz) + N(-N(a)\overline{w}y) + N(a)(\langle wx, y\overline{z}\rangle + N(wx) + N(y\overline{z}))\end{aligned}$$

And we also get

$$\begin{aligned}N(x+ya)N(z+wa) &= (N(x) + N(y)N(a))(N(z) + N(w)N(a)) \\ &= N(x)N(z) + N(x)N(w)N(a) + N(y)N(z)N(a) + N(y)N(w)N(a)N(a) \\ &= N(xz) + N(wx)N(a) + N(y\overline{z})N(a) + N(a)^2N(\overline{w}y)\end{aligned}$$

These two statements are equal, hence we can equate them. We will first subtract the statements which are the same expression. This yields

$$\langle xz, -N(a)\overline{w}y\rangle + N(a)\langle wx, y\overline{z}\rangle = 0 \tag{3.59}$$

Hence we can take $-N(a)$ outside of the inner product and move that term to the other side. This yields

$$N(a)\langle xz, \overline{w}y\rangle = N(a)\langle wx, y\overline{z}\rangle \tag{3.60}$$

Since $N(a)$ is invertible we can divide by this and get

$$\langle xz, \overline{w}y\rangle = \langle wx, y\overline{z}\rangle \tag{3.61}$$

We will now transform these equations into a more workable form

$$\begin{aligned}\langle wx, y\overline{z}\rangle &= \langle y\overline{z}, wx\rangle \\ &= \langle (y\overline{z})\overline{x}, w\rangle \\ &= \langle x(z\overline{y}), \overline{w}\rangle\end{aligned}$$

And we get

$$\langle xz, \overline{w}y\rangle = \langle (xz)\overline{y}, \overline{w}\rangle$$

Thus we have

$$\langle x(z\overline{y}), \overline{w}\rangle = \langle (xz)\overline{y}, \overline{w}\rangle \tag{3.62}$$

This last equation holds for all w, therefore $x(z\overline{y}) = (xz)\overline{y}$ for all $x, y, z \in D$. Since we can transform any multiplication into this form we get associativity. $\square$

We will now show an proposition that states that every finite proper composition subalgebra of an associative composition algebra is associative and commutative.

Proposition 37. *Let (C, N) be a composition algebra and D_1 a finite-dimensional proper composition subalgebra. Take $a \in D_1^{\perp}$ with $N(a) \neq 0$ and let $D_2 = D_1 \oplus D_1 a$. Then the two statements are equivalent:*

- *D_2 is associative*
- *D_1 is associative and commutative.*

Proof. D_2 is the composition subalgebra of the doubling proposition. We will show first that if D_2 is associative then D_1 is associative and commutative.

If D_2 is associative then every subalgebra of D_2 is also associative. Hence D_1 is associative. Suppose $x, y \in D_1$ then $(xy)a = x(ya)$ and we know $x(ya) = (yx)a$. Combining this gives

$$xy = xyaa^{-1} = ((xy)a)a^{-1} = ((xy)a)a^{-1} = yxaa^{-1} = yx \tag{3.63}$$

This shows the first direction. For the other direction assume that D_1 is both commutative and associative. Then we want to show that multiplication in D_2 is associative. Pick 3 elements of D_2. These are $z_i = x_i + y_i a$ for $i = 1, 2, 3$. Then using the multiplication in D_2 we get

$$(z_1z_2)z_3 = (x_1x_2)x_3 - N(a)((\overline{y_2}y_1)x_3 + \overline{y_3}(y_2x_1) + \overline{y_3}(y_1\overline{x_1})) + (y_3(x_1x_2) - N(a)(y_3(\overline{y_y}y_1) + (y_2x_1)\overline{x_3} + (y_1\overline{x_2})\overline{x_3})a$$
$$z_1(z_2z_3) = x_1(x_2x_3) - N(a)(x_1(\overline{y_3}y_2) + (y_3x_2)y_1 + (y_2\overline{x_3})y_1) + ((y_3x_2)x_1 + (y_2\overline{x_3})x_1 + y_1(\overline{x_2}\overline{x_2}) - N(a)y_1(\overline{y_2}y_3))a$$

Since D_1 is both commutative and associative we have two observations. First $(x_1x_2)x_3 = x_1(x_2x_3)$ and second all the terms containing an term a (not $N(a)$) are equal since we can just move the terms around. To show that the statements are equal we need to show that the 3 terms with an $N(a)$ in are equal. Hence

$$N(a)(\overline{y_2}y_1)x_3 = N(a)x_1(\overline{y_3}y_2)$$
$$N(a)\overline{y_3}(y_2x_1) = N(a)\overline{y_3}(y_1\overline{x_1})$$
$$N(a)(y_3x_2)y_1 = N(a)(y_2\overline{x_3})y_1$$

If we use that $(ya)z = (y\overline{z})a$ then we arrive at the following:

$$N(a)xyz = a\overline{a}xyz = a\overline{z}((ya)x) = \overline{a}z((y\overline{x})a) = N(a)\overline{x}yz \tag{3.64}$$

Since we have commutativity we can do the same for y and z. Hence we see

$$N(a)xyz = N(a)\overline{x}yz = N(a)x\overline{y}z = N(a)xy\overline{z} \tag{3.65}$$

This proves the equalities above since we can just move the terms around until they are in the correct form. □

3.3 The 1,2,4,8 theorem and the classification of all Composition algebras

We have now done all the work to do the mayor goal this section. We can now classify all composition algebras.

Theorem 38. *Let (C, N) be a composition algebra over a field K. Then C has dimension $1, 2, 4$ or 8 and C is a quadratic alternative algebra that satisfies the Moufang identities. Moreover, it must also satisfy the following:*

dim C	*K*	*commutative*	*associative*	*alternative*
1	*char* $\neq 2$	*yes*	*yes*	*yes*
2	*any*	*yes*	*yes*	*yes*
4	*any*	*no*	*yes*	*yes*
8	*any*	*no*	*no*	*yes*

Proof. Let C be a division algebra. We have already shown it is a quadratic alternative algebra that satisfies the Moufang identities. Now we need to show the rest.

We start with any composition algebra C. Then if the characteristic of K is unequal to 2, we can find a one-dimensional composition subalgebra $D_1 = Ke$. It is non-singular since $\langle \lambda e, \mu e\rangle = \lambda\mu\langle e, e\rangle = 2\lambda\mu \neq 0$ if $\lambda, \mu \neq 0$. However if the characteristic of K is 2 then $\langle \lambda e, \mu e\rangle$ is equal to zero, thus Ke must be singular. Therefore the dimension of C must be bigger than 2 in that case.

If the dimension of C is bigger than 1 then in the case the characteristic is not 2 we get a two dimensional subalgebra $D_2 = d_1 \oplus D_1 a$ for some $a \in C$. Since D_1 is both associative and commutative (it is isomorphic to the underlying field) we get that D_2 is associative. We need to show it is commutative. Let $x, y, z, w \in K$, i.e. $xe, ye, ze, we \in D_1$. Then we have

$$\begin{aligned}(xe + ya)(ze + wa) &= xze + x(wa) + (ya)ze + (ya)(wa) \\ &= zxe + wxa + zya + (wa)(ya) \\ &= (ze + wa)(xe + ya)\end{aligned}$$

Hence D_2 is commutative. If the characteristic of K is two, we will prove it as a lemma after this theorem. Hence no matter the characteristic we have a commutative associative composition subalgebra of C of dimension 2, if the dimension of C is bigger than 1.

Now suppose that the dimension of C is greater than 2. Then we can perform the doubling another time and get a four-dimensional subalgebra $D_3 = d_2 \oplus d_2 b$. It is associative since D_2 is associative and commutative. However it is not commutative itself.

We start by defining an element y which is perpendicular on e. If the characteristic of K is not 2 then take $y = a$. This yields $\langle y, e\rangle = 0$. Hence $\overline{y} = -y \neq y$¿ if the characteristic of the underlying field is two, then assume that $\overline{y} = y$. We then have $\overline{y} = \langle y, e\rangle e - y$. This implies $2y = 0 = \langle y, e\rangle e$. However this contradicts our choice of y, which had to be perpendicular on e. Thus $y \neq \overline{y}$. We will show that $D_2 b$ is non-singular. Assume that there is some $xb \in D_2 b$ for which we have $\langle xb, zb\rangle = 0$ for all $zb \in D_2 b$. Then $\langle xb, zb\rangle = \langle x, z\rangle N(b) = 0$. Hence $\langle x, z\rangle = 0$ for all $z \in D_2$. Since D_2 is non-singular this means that $x = 0$, and thus $xb = 0$. This shows that $D_2 b$ is non-singular. We can now find an element $x \in D_2 b$ such that $N(x) \neq 0$. Since $x \in D_2 b$ we know that $x \in D_2^{\perp}$. Hence $\overline{xy} = -xy$. However $\overline{xy} - \overline{yx} = -\overline{y}x$ since $x \perp e$. If we combine these results with $y \neq \overline{y}$ we get

$$xy = \overline{y}x \neq yx \tag{3.66}$$

So we have shown that D_3 is not commutative.

There is one final doubling we can make. If we assume that the dimension of C is bigger than 4 we can double once more to get $D_4 = d_3 \oplus D_3 c$. It is not associative since D_3 is not commutative. It is not commutative since we can identify D_3 as a subalgebra of D_4, and D_3 itself is not commutative. Moreover, it cannot be a proper subalgebra of C, thus $C = D_4$. ☐

Now we prove the lemma which we used in the proof of the first doubling.

Lemma 39. *If (C, N) is a composition K algebra with char $K = 2$, then there exist an $a \in C$ such that $\langle e, a\rangle \neq 0$. Then $D = Ke \oplus Ka$ is a two dimensional associative and commutative composition subalgebra.*

Proof. Assume that $\langle e, a\rangle = 0$ for all a. Then $e = 0$ since $\langle , \rangle$ is non-degenerate on C. So $\langle e, a\rangle \neq 0$ for some $a \in C$. Note that $a \notin Ke$, since in that case $\langle e, a\rangle = \langle e, \lambda e\rangle = 0$. Hence D is two dimensional. It is non singular, since if $\langle \lambda e + \mu a, x\rangle = 0$ for all $x \in D$ then in particular $\langle \lambda e + \mu a, e\rangle = 0$ and $\langle \lambda e + \mu a, a\rangle$ yield $\mu = 0$ and$\lambda = 0$, respectively. This would imply $\lambda e + \mu a = 0$. Hence D is non-singular. It is closed under multiplication follows from

$$\begin{aligned}xy &= (\alpha e + \beta a)(\gamma e + \delta a) \\ &= \alpha\gamma e + (\alpha\delta + \beta\gamma)a + \beta\delta(\langle a, e\rangle a - N(a)e) \\ &= (\alpha\gamma - \beta\delta N(a))e + (\alpha\delta + \beta\gamma + \beta\delta\langle a, e\rangle)a\end{aligned}$$

Associativity follows from the same proof as the last proposition of the previous section. Commuativity follows from this equation, since α, β, γ and δ are elements of the field they commute. This allows us to change the order of all the elements in the previous equation and see that $xy = yx$. □

3.3.1 Hurwitz problem

Hurwitz theorem is a theorem about the (non)existance of solutions for the Hurwitz problem.

Theorem 40. *Let K be a field with characteristic other than 2. The only values of $n \in \mathbb{N} - \{0\}$ for which the following statement is true are $1, 2, 4, 8$: there exists n bilinear functions $z_i : K^n \times K^n \to K$ such that for all $x, y \in K^n$ this equation holds:*

$$(\sum_{i=1}^{n} x_i^2)(\sum_{i=1}^{n} y_i^2) = \sum_{i}^{n} z_i(x,y)^2 \tag{3.67}$$

Note that if we allowed for fields of characteristic two, this statement is trivially true, since the sum of squares is the square of the sums.

Proof. We will first show existence. For this we look at a few examples. We have seen that the following real algebras are composition algebras: the real numbers, the complex numbers, the quaternions and the octonions. If we take a closer look at the square of the norm, we see that it involves sums of squares, and multiplication. For example $|(x+iy)(w+zi)|^2 = |(x+iy)|^2|(w+zi)|^2 = (x^2+y^2)(w^2+z^2)$. But also we see $(xw-yz)^2+(xzwy)^2$. This also holds for the quaternions and the octonions. This gives an indication what formulas the use when we want to show existence of solutions. As will turn out, these formulas hold in general:
In the case of $n = 1$ the statement is trivial.
In the case $n = 2$ we take $z_1 = x_1y_1 + x_2y_2$ and $z_2 = x_1y_2 - x_2y_1$
In the case $n = 4$ we have the following z_i

$$z_1 = x_1y_1 - x_2y_2 - x_3y_3 - x_4y_4 \tag{3.68}$$
$$z_2 = x_1y_2 + x_2y_1 + x_3y_4 - x_4y_3 \tag{3.69}$$
$$z_3 = x_1y_3 - x_2y_4 + x_3y_1 + x_4y_2 \tag{3.70}$$
$$z_4 = x_1y_4 + x_2y_3 - x_3y_2 + x_4y_1 \tag{3.71}$$
$$\tag{3.72}$$

And in the last case when $n = 8$ we take

$$z_1 = x_1y_1 - x_2y_2 - x_3y_3 - x_4y_4 - x_5y_5 - x_6y_6 - x_7y_7 - x_8y_8 \tag{3.73}$$
$$z_2 = x_1y_2 + x_2y_1 + x_3y_4 - x_4y_3 + x_5y_6 + x_6y_5 + x_7y_8 - x_8y_7 \tag{3.74}$$
$$z_3 = x_1y_3 + x_2y_4 + x_3y_1 + x_4y_2 + x_5y_7 - x_6y_8 + x_7y_5 + x_8y_6 \tag{3.75}$$
$$z_4 = x_1y_4 + x_2y_3 - x_3y_2 + x_4y_1 + x_5y_8 + x_6y_7 - x_7y_6 + x_8y_5 \tag{3.76}$$
$$z_5 = x_1y_5 - x_2y_6 - x_3y_7 - x_4y_8 + x_5y_1 + x_6y_2 + x_7y_3 + x_8y_4 \tag{3.77}$$
$$z_6 = x_1y_6 + x_2y_5 + x_3y_8 - x_4y_7 - x_5y_2 + x_6y_1 - x_7y_4 + x_8y_3 \tag{3.78}$$
$$z_7 = x_1y_7 - x_2y_8 + x_3y_5 + x_4y_6 - x_5y_3 + x_6y_4 + x_7y_1 - x_8y_2 \tag{3.79}$$
$$z_8 = x_1y_8 + x_2y_7 - x_3y_6 + x_4y_5 - x_5y_4 - x_6y_3 + x_7y_2 + x_8y_1 \tag{3.80}$$
$$\tag{3.81}$$

Now suppose we have another n for which there exists a solution for Hurwitz problem. Then we can define $N(x) = \sum_{i=1}^{n} x_i^2$. This yields a non-degenerate quadratic form. Moreover $N(x)N(y) = N(z(x, y))$. The only thing missing before (K^n, N) becomes a composition algebra is an unit element. We will prove there exist an unit element in the next lemma. Now since we have a quadratic form on an algebra which is non-singular, we have a composition algebra. Hence the dimension must be $1, 2, 4, 8$ □

Lemma 41. *Let $N : A \to K$ be a non-degenerate quadratic form and $A \times A \to A, (x, y) \mapsto xy$ a bilinear map such that*

$$N(xy) = N(x)N(y) \quad \forall x, y \in A \tag{3.82}$$

Then there is a map $* : A \times A \to A, (x, y) \mapsto x * y$ such that $N(x * y) = N(x)N(y)$ and there is an element $e \in A$ such that

$$e * x = x * e = x \quad \forall x \in A \tag{3.83}$$

Proof. Let $v \in A$ such that $N(v) \neq 0$. It exists since N is non-degenerate. Let $u = N(v)^{-1}v^2$, then $N(u) = 1$ and $N(ux) = N(xu) = N(x)$ for all $x \in A$. We also have $\langle ux, uy\rangle = N(ux+uy)-N(ux)-N(uy) = N(x+y)-N(x)-N(y) = \langle x, y\rangle$. This yields

$$\langle L_u(x), L_u(y)\rangle = \langle x, y\rangle \tag{3.84}$$

and

$$\langle R_u(x), R_u y\rangle = \langle x, y\rangle \tag{3.85}$$

Let L_u^* be the adjoint of L_u. Then for all $x, y \in A$ we have

$$\langle x, y\rangle = \langle L_u(x), L_u(y)\rangle = \langle x, L_u^*(L_u(y))\rangle \tag{3.86}$$

This means that $y = L_u^* L_u(y))$ for all $y \in A$. Hence $L_u^* = L_u^{-1}$. Notice that $n(L_u^{-1}x = N(uL_u^{-1}(x)) = N(x)$. We can do the same argument for R_u. Now we define a map $* : A \times A \to A$ as

$$x * y = R_u^{-1}(x)L_u^{-1}(y) \tag{3.87}$$

Then $N(x * y) = N(R_u^{-1}(x))N(L_u^{-1}(y)) = N(x)N(y)$. The only thing we need to check is that we now have an unit.

$$u^2 * x = R_u^{-1}(u^2)L_u^{-1}(x) = UL_u^{-1}(x) = x$$
$$x * u^2 = R_u^{-1}(x)L_u^{-1}(u^2) = R_u^{-1}(x)u = x$$

Hence u^2 is the unit element. □

3.3.2 Real composition division algebras

We can also classify all real composition division algebras. These are all the normed real algebras where one can define division in. If we have a real division composition algebra, there are the following options. We either have an associative division algebra. Then it is isomorphic to either $\mathbb{R}, \mathbb{C}, \mathbb{H}$. Now there is also the option there is an non-associative real division composition algebra. Then it must have dimension 8. What turns out is that there is up to isomorphism just one way to construct a division algebra from the Quaternion. These are the octonions. The idea is that the a we chose behaves either like a positive or a negative real number. If it is a positive real number we get idempotent elements, if a behaves like a negative real number we get a division algebra.

Chapter 4

Topological groups

In order to answer our second question, for which n can S^n be a topological group, we need to define what it means to be a topological group. We will show a couple of examples of topological groups.

Definition: Topological groups

A topological group is a group G, together with a topological structure on G such that the groups product $m : G \times G \to G$ is a continuous function, and the inverse function $x \mapsto x^{-1}$ is also continuous.

4.1 Examples of topological groups

4.1.1 $\mathbb{R}$ is a topological group

We take as group multiplication the additive structure of $\mathbb{R}$ and denote it with $+$. First note that $+$ is a continuous function from $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$, since it is just translation by a fixed amount in either argument, and translation is continuous in $\mathbb{R}$. The inverse operation $-$ is also continuous, since if the distance between $-x$ and $-y$ is d, then so is the distance between x and y. Hence both operations are continuous, and thus $\mathbb{R}$ is a topological space.

4.1.2 $\mathbb{R}^n$ is a topological group

We look at $\mathbb{R}^n$ as a vector space, and take addition of vectors as multiplication. This addition is continuous in both arguments, and hence it is continuous. The inverse operation $-$ is continuous by the same argument as before, namely, the distance between x and y is the same as the distance between $-x$ and $-y$.

4.1.3 Hilbert spaces form a topological group

First notice that addition is continuous and that taking x to $-x$ is also continuous. Furthermore addition is associative and closed. Hence any Hilbert space H is a topological group.

4.2 The spheres

Here we will show that the $0, 1$ and 3 sphere form a topological group. It will turn out that these spheres are the only ones which can become a topological group.

4.2.1 S^0 is a topological group

First note that S^0 is a space consisting of 2 elements, namely 1 and -1. The topology on this space is the discrete topology, namely every subset is open: $\phi, \{1\}, \{-1\}$ and $\{1, -1\}$ are all open sets in $\S^0$. Hence every map is continuous. The group operations are m where 1 is the identity and -1 is the self inverse element. Hence m is continuous and so is the inversion map. This makes S^0 into a topological group.

4.2.2 S^1 is a topological group

First note that S^1 is the set of all the elements of distance 1 in $\mathbb{R}^2$, or, equivalently, the set of all elements of norm 1 in the complex numbers. These can be represented by $e^{\theta i}$. Group inversion becomes $e^{\theta i} \mapsto e^{-\theta i}$ and multiplication becomes $e^{\theta_1 i} e^{\theta_2 i} = e^{(\theta_1+\theta_2)i}$, which are both continuous functions. Hence S^1 is a topological group. One can also look at this topological group as follows. Take S^1 as a subset of $\mathbb{C}$, namely $\{z| \quad ||z|| = 1, z \in \mathbb{C}\}$. Then multiplication of complex numbers is the group operation on S^1, this again produces numbers of norm 1 in S^1. We can also invert the numbers on S^1 since $0 \notin S^1$ and $\mathbb{C}$ is a division algebra. If we take the inverse of an element of norm 1 we have seen that this yields a number of norm 1 again. Both these operations are continuous maps, thus S^1 forms a topological group.

4.2.3 S^3 is a topological group

Again notice that S^3 is the set of all elements of distance 1 in $\mathbb{R}^4$, or equivalently, the set of all elements of norm 1 in the quaternions. We can define the multiplication on S^3 as multiplication of quaternionic numbers of norm 1. Again we have shown that this yields another number of norm 1 thus it is on the sphere. We can also take inverses, which will also have norm 1. Both these operations are continuous maps on $\mathbb{H}$, hence restrict to a continuous map on S^3.

4.3 $\mathbb{R}^n$ real associative division composition algebra implies S^{n-1} is a topological group

We will show that S^{n-1} is a topological group if $\mathbb{R}^n$ is a associative division composition algebra.
Notice that the maps $x \mapsto a \cdot x$ and $x \mapsto x \cdot a$ are linear maps, which are continuous. Furthermore, there is an inversion map $x \mapsto x^{-1} = \frac{1}{N(x)}\overline{x}$ for all $x \neq 0$ which is continuous. Note that S^{n-1} is a subset of $\mathbb{R}^n$. Suppose $x, y \in S^{n-1}$. Then we can define $m(x, y) = \frac{x \cdot y}{|x \cdot y|}$, which is continuous. We can also define the inversion map $x \mapsto \frac{x^{-1}}{|x^{-1}|}$. This is also continuous. Further notice that m is associative, since $\cdot$ is associative and the norm function obeys $|x \cdot y| = |x||y|$. Hence S^{n-1} is a topological group.
We have seen that the only real associative division composition algebra are the real numbers, complex numbers and the quaternions. The natural question arises, is it possible for S^{n-1} to be a group when $\mathbb{R}^n$ cannot be endowed with a real associative division composition algebra? The answer to that question will be no. We will show that S^{n-1} can be something that comes close to a group when $n = 1, 2, 4, 8$. These are the only S^{n-1} which have a chance at being a group. Then it turns out that S^7 is not a group, hence the answer will be no.

Chapter 5

Lie groups

A Lie group is just like a topological group except it is on smooth manifolds and the multiplication and inversion maps should be smooth maps. This makes some questions in this settings a whole lot easier, since smoothness is far stronger than just continuity.

Definition 19. *A Lie group* $(G, \cdot)$ *is a smooth manifold* G *such that* $(G, \cdot)$ *is a group and the multiplication map* $\cdot : G \times G \to G$ *and inversion map* $\cdot^{-1} : G \to G$ *are smooth maps.*

This gives rather strong properties for the geometry which exists on M.

Lemma 42. *Any Lie group* G *is parallelizable.*

Proof. We need to show that TG is isomorphic to $G \times T_eG$. We have an induced map $(L_g)_*$ from the map L_g which is defined by $L_g h = gh$. Now if we have an element at the tangent space at g, call it X_g we can move it over the entire manifold by this transformation. Then we get $TG = G \times T_eG$ by sending X_g to $(g, (L_{g^{-1}})_* X_g)$, this gives an isomorphism between T_gG and T_eG, and hence get the required relation. □

5.1 Spheres which are Lie groups

Theorem 43. *The only* n *for which* S^n *can be made into Lie groups are* $n = 0, 1, 3$.

Proof. We start by noting that for $n = 0$ the statement is trivial since $S^0 = \{\pm 1\}$. Now move to the higher dimensional case. For all $n \geq 1$ the sphere S^n is connected. For connected Lie groups G there is the following theorem on the de Rham cohomology: if $H^1(G) = 0$ then $H^3(G) \neq 0$, see [7]. For the spheres we have $H^n(S^n) = \mathbb{Z}$ and $H^i(S^n) = 0$ if $i \neq n$, for a proof of this see [4]. Suppose S^n is a Lie group. If $n = 1$ then $H^1(S^1) \neq 0$. Hence it can be a Lie group. If $n > 1$ then $H^1(S^n) = 0$. This implies that $H^3(S^n)$ must be nonzero if S^n is a Lie group. We see that the only moment this is true when $n = 3$. Hence the only possible Lie groups are S^0, S^1, S^3. These are Lie groups since we can endow them with the group structure of the elements of norm 1 of $\mathbb{R}, \mathbb{C}, \mathbb{H}$ respectively. The multiplication and inversion maps are smooth maps in $\mathbb{R} - \{0\}, \mathbb{C} - \{0\}$ and $\mathbb{H} - \{0\}$. In particular they are smooth when restricted to S^0, S^1 or S^3. This proves the theorem. □

Chapter 6

Vector Bundles

This is a short chapter which contains the theory about complex vector bundles we need, and some general theory about vector bundles like their definition. From now on we will assume that map means a continuous function. Vector bundles are topological objects that locally look like a Cartesian product of a topological space with a vector space.

Definition 20. *Let B be a topological space. An n dimensional real vector bundle over B is a topological space E and a map $p : E \to B$ together with a real vector space structure of $p^{-1}(b)$ for each $b \in B$, such that the following condition holds: There exists an open cover $\mathcal{U}$ with opens U_α such that for each U_α there exists a homeomorphism $h_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^n$ taking $p^{-1}(b)$ to $\{b\} \times \mathbb{R}^n$ by a vector space isomorphism for each $b \in U_\alpha$. Such an h_α is called a local trivialization of the vector bundle. The space B is called the base space, E is the total space and the vector spaces $p^{-1}(b)$ are the fibers. If we take $\mathbb{C}$ instead of $\mathbb{R}$ we get complex vector bundles, which we will use later on. For now, we write $\mathbb{K}$ instead of $\mathbb{R}$ or $\mathbb{C}$ when results hold for both fields.*

Some examples of vector bundles:

1. A trivial vector bundle is the bundle $E = B \times \mathbb{K}^n$ with p the projection onto B.

2. Tangent spaces of smooth manifolds form vector bundles.

3. The normal bundle to S^n in $\mathbb{R}^{n+1}$, consisting of pairs (x, v) such that V is perpendicular to the tangent plane to S^n at x. This makes $E \subset S \times \mathbb{R}^{n+1}$.

4. The open Möbius band is a vector bundle over S^1.

5. The infinite cylinder is also a vector bundle over S^1. These two vector bundles locally look both like a surface, however they are not homeomorphic.

Definition 21. *We call two vector bundles $p_1 : E_1 \to B, p_2 : E_2 \to B$ over the same base space B isomorphic if there is a homeomorphism $h : E_1 \to E_2$ taking each fiber $p_1^{-1}(b)$ to the corresponding fiber $p_2^{-1}(b)$ by a linear isomorphism. We denote isomorphism of vector bundles by $E_1 \approx E_2$.*

Lemma 44. *A continuous map $h : E_1 \to E_2$ between vector bundles over the same base space B is an isomorphism if it takes each fiber $p_1^{-1}(b)$ to the corresponding fiber $p_2^{-1}(b)$ by a linear isomorphism.*

Proof. Notice that h must be bijective by the hypothesis. Thus all we have to show is that h^{-1} is continuous. Since this is a local question we can restrict to an open subset $U \subset B$ over which E_1 and E_2 are trivial. We can now compose with the local trivialization since this is a homeomorphism. Thus we have to show that h is locally a continuous map $U \times \mathbb{K}^n \to U \times \mathbb{K}^n$

of the form $h(x, v) = (x, g_x(v))$. Here g_x is the composition from the linear isomorphisms from $x \times \mathbb{K}^n$ to $p_1^{-1}(b)$, then again from $p_1^{-1}(b)$ to $p_2(p_1^{-1}(b))$ which are both linear isomorphisms. Thus here g_x is an element of the group $GL_n(\mathbb{K})$ of invertible linear transformations of $\mathbb{K}^n$, and thus g_x depends continuously on x. This means that g_x can be regarded as a $n \times n$ matrix, and its n^2 entries depend continuously on x, since $g_x^{-1} = \frac{1}{\det g_x}$ times the adjoint matrix of g_x. Therefore $h^{-1}(x, v) = (x, g_x^{-1}(v))$ is continuous. □

Definition 22. *We define the direct sum of two vector bundles E_1 and E_2 as $E_1 \oplus E_2 = \{(v_1, v_2) | p_1(v_1) = p_2(v_2)\}$*

Lemma 45. *The direct sum of two vector bundles is again a vector bundle.*

Proof. First note that if we have a vector bundle $p : E \to B$ and a subspace $A \subset B$ then $p|_A : p^{-1}(A) \to A$ is a vector bundle over A.

Also note that if we have base spaces B_1, B_2 and vector bundles $p_i : E_i \to B_i$ then the product

$$p_1 \times p_2 : E_1 \times E_2 \to B_1 \times B_2 \tag{6.1}$$

is again a vector bundle, since we can make a trivialization

$$h_{\alpha,\beta} : p_1^{-1}(U_\alpha) \times p_2^{-1}(U_\beta) \to U_\alpha \times \mathbb{K}^n \times U_\beta \times \mathbb{K}^m \tag{6.2}$$

by setting $h_{\alpha,\beta} = h_\alpha \times h_\beta$.

Then notice that $E_1 \oplus E_2$ is the restriction of $E_1 \times E_2$ over the diagonal (b, b) for $b \in B$. Hence this is also a vector bundle by our observations. □

We will now work on showing that every vector bundle has a counterpart such that their direct sum is a trivial vector bundle. Our first observation is that we can define an inner product on vector bundles. The idea is that we pull back the inner product on $\mathbb{K}^n$ to $p^{-1}(U_\alpha)$ via the trivialization. Since our space B is compact Hausdorff it has a partition of unity subordinate ϕ_β to the open cover $U_\alpha(\beta)$, such that the support of ϕ_β is contained in $U_\alpha(\beta)$. We then define our inner product

$$\langle v, w\rangle = \sum_\beta \phi_\beta p(v) \langle v, w\rangle_{\alpha(\beta)} \tag{6.3}$$

Lemma 46. *Let $E \to B$ is a vector bundle and B be paracompact. Suppose E_0 is a vector subbundle, then there exists a vector subbundle $E_0^\perp \subset E$ such that $E_0 \oplus E_0^\perp = E$.*

Proof. We have an inner product on E. Define $E_0^\perp$ to be the subspace of E such that each fiber consists of all vectors orthogonal to vectors in E_0. Then $E_0 \oplus E_0^\perp$ is isomorphic to E by sending (v, w) to $v + w$. Now we need to show that $E_0^\perp$ is a vector bundle. All we have to show is that it admits local trivializations.

Since trivializations are local, we can restrict ourselves to U_α and suppose $E = B \times \mathbb{K}^n$. E_0 is a vector bundle. It has dimension $m \leq n$. Thus it has m independent local sections $b \mapsto (b, s_i(B))$ near each point $b_0 \in B$.

If $m < n$ then we can enlarge this set of m independent local sections to a set of n independent local sections $b \mapsto (b, s_i(b))$ of E by choosing $s_{m+1}, .., s_n$ in the fiber $p^{-1}(b_0)$. We can now take that vector for all nearby fibers $p^{-1}(b)$ since the determinant function is continuous, and vectors are linearly independent iff $det(s_1, ..., s_n) \neq 0$. Thus they will remain independent. Now we can apply the Gramm-Schmidt orthonormalization process. This is a continuous function, hence the s'_i will remain orthogonal in a neighborhood. Also notice that the first m will remain a basis for E_0 since the first m s'_i values are the same if we just did the Gramm-Schmidt process on E_0. The sections s'_i allow us to define a local trivialization $h : p^{-1}(U) \to U \times \mathbb{K}^n$ with $h(b, s'_i(b))$ equal to the i-th standard basis vector of $\mathbb{K}^n$. Notice that h splits, such that h carries E_0 to $U \times \mathbb{K}^m$ and $E_0^\perp$ to $U \times \mathbb{R}^{n-m}$. □

We will now use this result to show the following lemma

Lemma 47. *Every vector bundle has a counterpart such that their sum is a trivial vector bundle.*

Suppose the result holds. Then E is a subbundle of a trivial bundle, i.e. E is a subbundle of $B \times \mathbb{R}^n$. Then by the previous lemma we have a vector bundle $E^{\perp}$ such that the direct sum is the trivial bundle. If we include E in the trivial bundle and then project onto $\mathbb{R}^n$ we have a map $E \to \mathbb{R}^n$ that is a linear injection in each fiber. We can now reverse this. We build a map $E \to \mathbb{R}^n$ that is a linear injection. Then we will show that this map gives an embedding of E in $B \times \mathbb{R}^n$.

Proof. First notice that for each element x in B there is a open neighborhood U_x of x such that E over U_x is trivial. By Urysohns Lemma there is a map $\phi_x : B \to [0,1]$ that is 0 outside U_x and non-zero at x. Letting x vary, the sets $\phi_x^{-1}(0,1]$ form an open cover of B. Since B is compact, we can extract a finite subcover. Let the U_x and ϕ_x corresponding to this subcover be labeled U_i and ϕ_i. Define a map $g_i : E \to \mathbb{R}^n$ by $g_i(v) = \phi_i(p(v)) \cdot (\pi_i h_i(v))$, where p is the project $E \to B$, π_i the projection from $U_i \times \mathbb{R}^n$ onto $\mathbb{R}^n$, and $h_i : p^{-1}(U_i) \to U_i \times \mathbb{R}^n$. the local trivialization. This makes g_i a linear injection on each fiber over $\phi_i^{-1}(0,1]$. If we make the various g_i the coordinates of a map $g : E \to \mathbb{R}^N$, where $\mathbb{R}^N$ is a product of copies of $\mathbb{R}^n$, then g in a linear injection on each fiber.
The map g is the second coordinate of a map $f : E \to B \times \mathbb{R}^N$, with $f = (p, g)$. The image of f is a subbundle of the product $B \times \mathbb{R}^N$. This is since if we project $\mathbb{R}^N$ onto the $i-th$ $\mathbb{R}^n$ factor, we have the second coordinate of a local trivialization over $\phi_i^{-1}(0,1]$. This shows that E is isomorphic to a subbundle of $B \times \mathbb{R}^N$, hence by preceding lemma we find a complementary subbundle $E^{\perp}$ such that their direct sum is isomorphic to $B \times \mathbb{R}^N$. □

We will now define tensors of vector bundles.

Definition 23. *Let E_1, E_2 be vector bundles with projection p_1, p_2 and trivialization h_1, h_2. The tensor product of E_1, E_2 is denoted $E_1 \otimes E_2$. This is a set, formed by the disjoint union of the vector spaces $p_1(x)^{-1} \otimes p_2^{-1}(x)$. Then we will need a topology on this set.*
Choose isomorphisms $h_i : p_i^{-1}(U) \to U \times \mathbb{R}^N$ for each open $U \subset B$ over which E_1 and E_2 are trivial. We define τ_U on the set $p_1^{-1}(U) \otimes p_2^{-1}(U)$ is defined by letting the fiber wise tensor product map $h_1 \otimes h_2 : p_1^{-1}(U) \otimes p_2^{-1}(U) \to U \times (\mathbb{R}^{n_1} \otimes \mathbb{R}^{n_2})$. Note that this topology is independent of h_i since h_1 and h_2 agree on overlaps, and hence the other choices are obtained by composing with isomorphisms of $U \times \mathbb{R}^{n_1}$ of the form $(x, v) \mapsto (x, g_i(x)(v))$ for continuous maps $g_i : U \to GL_{n_i}(\mathbb{R})$. Hence $h_1 \otimes h_2$ changes by composing these morphims with the tensors, and the maps $g_1 \otimes g_2$ are continuous maps $U \to U \times (\mathbb{K}^{n_1} \otimes \mathbb{K}^{n_2})$. When we replace U by an open subset V, the topology on $p_1^{-1}(V) \otimes p_2^{-1}(V)$ induced by τ_U is the same as the topology $\tau(V)$, since the local trivializations agree on the overlap between U and V. Hence we get a well defined topology on $E_1 \times E_2$, which makes this a vector bundle over B.

We have now shown the properties we need to define the functor $K(\cdot)$. If you wish you can go forward to the next chapter on Topological K-theory. The last part of this chapter proves additional statements which will be used to show the last property of functors: if we have a map $f : X \to Y$ we get a map between $K(Y)$ and $K(X)$. In this chapter we will be doing the vector bundle side of the statements. In the next chapter we will use these theorems to show that the result hold for K.

Proposition 48. *Given a map $f : A \to B$ and a vector bundle $p : E \to B$, then there exists a vector bundle $p' : E' \to E$ with a map $f' : E' \to E$ taking the fiber of E' over each point $a \in \alpha$ isomorphically onto the fiber of E over $f(A)$, and such a vector bundle E' is unique up to isomorphism.*

We call E' the pullback of E by f.

Proof. We will first show the existence of a pullback of E by f and then show this pullback is unique up to isomorphism. We set $E = \{(a, v) \in A \times E| \quad f(a) = p(v)\}$. Then we set $p'(a, v) = a$ and $f'(a, v) = v$. Then $fp' = pf'$, since they both send (a, v) to $f(a) = p(v)$. Thus we have a commutative diagram:

$$\begin{array}{ccc} E & \xleftarrow{\;f^*\;} & E' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p^*} \\ B & \xleftarrow[\;f\;]{} & A \end{array}$$

Now we need to show that E' is a vector bundle. We need to show it has a trivialization functions. We will show that E' is homeomorphic to a vector bundle. First we look to the graph of f: $\Delta_f = \{(a, f(a))\}$. Then p' factors as maps from $E' \to \Delta_f \to A$ □

Proposition 49. *This induced map f^* commutes with direct sum and tensoring.*

Proof. Notice that if $E = E_1 \oplus E_2$ we can find E_1' and E_2', and then $E' = E_1' \oplus E_2'$. The same holds for tensoring. Thus f^* commutes with direct sum and tensoring. □

Theorem 50. *Given a vector bundle $p : E \to B$ and homotopic maps $f_0, f_1 : A \to B$, then the induced vector bundles $f_0^*(E)$ and $f_1^*(E)$ are isomorphic if A is paracompact.*

Proof. This theorem is a special case of the next proposition. □

Proposition 51. *The restrictions of a vector bundle $E :\to X \times I$ over $X \times \{0\}$ and over $X \times \{1\}$ are isomorphic if X is compact Hausdorff.*

Lemma 52. *A vector bundle $p : E \to X \times [a, b]$ is trivial if it restrictions over $X \times [a, c]$ and $X \times [c, b]$ are both trivial for some $c \in (a, b)$.*

Proof. We define the restrictions $E_1 = p^{-1}(X \times [a, c])$ and $E_2 = p^{-1}(X \times [c, b])$ and we have isomorphisms h_1 and h_2 the restrictions to $[a, c]$ and $[c, b]$. These isomorphisms need not agree on $p^{-1}(X \times \{c\})$, but if we replace h_2 by the isomorphism $X \times [c, b] \times \mathbb{K}^n \to X \times [c, b] \times \mathbb{K}^n$ on which each slice $X \times \{x\} \times \mathbb{K}^n$ is given by $h_1 h_2^{-1} : X \times \{c\} \times \mathbb{K}^n$. Once h_1 and h_2 agree on $E_1 \cap E_2$ they define a trivialization of E. □

Lemma 53. *For a vector bundle $p : E \to X \times I$ there exists an open cover U_α of X so that each restriction $p^{-1}(U_\alpha \times I \to U_\alpha \times I$ is trivial.*

Proof. This is because for each $x \in X$ we can find open neighborhoods $U_{x,1}, ...U_{x,k}$ in X and a partition $0 = t_0 < t_1 < ... < t_k = 1$ of $[0, 1]$ such that the bundle is trivial over $U_{x,i} \times [t_{i-1}, t_i]$ using compactness of $[0, 1]$. Then by previous lemma the bundle is trivial over $U_\alpha \times I$ where $U_\alpha = U_{x,1} \cap ... \cap U_{x,k}$. □

Proof. We will now prove the proposition. We can take an open cover $\{U_\alpha\}$ so that E is trivial over each $U_\alpha \times I$. We extract a finite subcover of X. We relabel them as $U_1, ..., U_m$. As shown before there is a corresponding partition of unity by functions ϕ_i with the support of ϕ_i contained in U_i. Define $\psi_i = \sum_{j=0}^{i} \phi_j$. Let X_i be the graph of ψ_i. Then we can restrict $p_i : E_i \to X_i$ be the restriction of the bundle E over X_i. Since E is trivial over $U_i \times I$ the natural projection homeomorphism $X_i \to X_{i-1}$ lifts to a homeomorphism $h_i : E_i \to E_{i-1}$ which is the identity outside $p^{-1}(U_i \times I)$ and which takes each fiber of E_i isomorphically onto the corresponding fiber of E_{i-1}. Explicitly, on points in $p^{-1}(U_i \times I) = U_i \times I \times \mathbb{K}^n$ we let $h_i(x, \psi_i(x), v) = (x, \psi_{i-1})(x), v)$. The composition $h = h_1 h_2 ... h_m$ is then a isomorphism from the restriction of E over $X \times \{1\}$ to the restriction over $X \times \{0\}$.

□

Corollary 54. *A homotopy equivalence $f : A \to B$ of paracompact spaces induces a bijection f^* between the sets of all $\mathbb{K}$ vector bundles over A and B.*

Proof. If g is a homotopy inverse of f then $f^*g^* = 1^* = 1$ and $g^*f^* = 1^* = 1$, hence we have that f^* is a bijection with inverse g^*. □

Proposition 55. *If $p : E \to B$ is a fiber bundle whose fiber F and base B are both finite cell complexes, then E is also a finite cell complex, whose cells are products of cells in B with cells in F.*

Proof. We will use induction. Suppose B is a cell complex with only 1 cell. Then $p^{-1}(B) = e^n \times F$, which is a finite cell complex, since F is a finite cell complex, and we can make $e^n \times F$ by taking the e^m cells in F and replacing them by e^{n+m} cells, and the attaching maps are (id, p) where p is the attaching map in F.
Now suppose B is obtained from a subcomplex B' by attaching an n-cell e^n. We assume that $p^{-1}(B')$ is a finite cell complex. If $\Phi : D^n \to B$ is a characteristic map for e^n then the pullback bundle $\Phi^*(E) \to D^n$ is a product since D^n is contractable. Since F is a finite cell complex, this means that we may obtain $\Phi^*(E)$ from its restriction of S^{n-1} by attaching cells. Hence we may obtain E from $p^{-1}(B')$ by attaching cells. □

The next goal is to define exterior powers for vector bundles. An exterior power $\lambda^k(V)$ for vector spaces V is constructed as follows:
First we take the k-fold tensor product. This yields $V \otimes \cdots \otimes V$.
In this space we have a subspace generated by $v_1 \otimes \cdots \otimes v_k - sign(\sigma)v_{\sigma_1} \otimes \cdots \otimes v_{\sigma_v}$ where σ is a permutation of the elements $\{1, .., k\}$.
Note that if V has dimension n then $\lambda^k(V)$ has dimension $\binom{n}{k}$.
We will locally do the same for the exterior products of vector bundles. This will yield the set E which we will then endow with a topology via local trivialisations in the same way as for tensor products.

Definition 24. *The exterior product of a vector bundle is the set E formed by first taking the disjoint union of all $\lambda^k(p^{-1}(x))$, i.e. point-wise exterior product. Then we use the same construction as for tensors products to create a trivialization for this set E, and via this route we also are able to endow E with a topology.*

In order to show that tensor product had a well defined topology we used that the tensor product $\phi \otimes \psi$ depended continuously on ϕ and ψ for ϕ, ψ linear maps. We will need to show this also holds for λ

Lemma 56. *Given a linear map $\phi : \mathbb{R}^n \to \mathbb{R}^n$, we have a linear map $\lambda^k(\phi) : \lambda^k(\mathbb{R}^n) \to \lambda^k(\mathbb{R}^n)$ which depends continuously on ϕ.*

Proof. Since ϕ is a linear map it induces a map on the k-fold tensor of $\mathbb{R}^n$. Then we take the quotient. Thus our map $\tilde{\psi}$ is the quotient map of a k fold tensor of a map ψ by itself, which is continuous. □

We will end this chapter with an example:

6.1 Bundles over S^N

The n-sphere S^n may be covered by two open discs U_1 and U_2. The intersection is homotopy-equivalent to $S^n - 1$. Any bundle $E \to S^n$ with fiber $V = \mathbb{C}^k$ is trivial when restricted to U_1 and U_2, since open disks are contractable. If we look from a gluing point of view, E is fully determined by the homotopy class of the transition function $g : S^{n-1} \to GL(V)$.
If $n = 1$, this set contains a single element, since $GL(V)$ is path connected. Thus every complex vector bundle over S^1 is trivial.
When $n = 2$, the set of all homotopy classes of maps from $S^{n-1} \to GL(V)$ is the same as the fundamental group, which is $\mathbb{Z}$ for any V. This is due to $GL(k, \mathbb{C})$ being connected.
In general, if $n > 1$ the set of homotopy classes from $S^{n-1} \to GL(V)$ is the $n-1$-th homotopy group of $GL(V)$.

Chapter 7

Topological K-theory

In topological K-theory we introduce an Functor K which maps compact Hausdorff topological spaces to rings. So given a topological space X we are going to introduce a ring $K(X)$. A ring has to obey the following axioms:

- $K(X)$ is an Abelian group under addition.
- $K(X)$ is a monoid under multiplication, eg $\cdot$ is associative and there is an identity.
- multiplication is distributive with respect to addition: $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ and $(b + c) \cdot a = (b \cdot a) + (c \cdot a)$

So first we need a Abelian group for addition. After that, we need an multiplication operation which distributes over the addition. We shall show that the direct sum of vector bundles on X can be made into an Abelian group, and that tensoring vector bundles over X defines a multiplication.
Furthermore, to be a functor, morphisms from one topological space into the other should induce morphisms from one ring to the other. In this case, the morphisms from one topological space to the other is a continuous function, and the morphisms from the one ring to the other are ring homomorphisms. In other words, the following diagram commutes:

$$\begin{array}{ccc} top & \xrightarrow{F} & top \\ \Big\downarrow K & & \Big\downarrow K \\ rings & \xrightarrow[F_*]{} & rings \end{array}$$

In this diagram *top* stands for compact Hausdorff topological spaces.

7.1 Direct sum of vector bundles induces an Abelian group

We start with a few basic constructions. First we fix a compact Hausdorff space X. It is convenient to take a slightly broader definition of vector bundles in this chapter. We allow fibers of a vector bundle $p : E \to X$ to be vector spaces of different dimensions. There still should be local trivializations $h : p^{-1}(U) \to U \times \mathbb{C}^n$.

Lemma 57. *Dimensions of fibers of vector bundles are locally constant.*

Proof. Suppose that we have an open cover $\mathcal{U} = \{U_\alpha\}$ of X. Then we can split every open U_α in $\mathcal{U}$ into connected components $V_{\alpha,\beta}$. This gives another open cover $\mathcal{V}$.

Then note that for any $V_{\alpha,\beta}$ the space $V_{\alpha,\beta} \times \mathbb{K}^n$ has constant dimension, due to dimension being locally constant in $V_{\alpha,\beta}$ and $\mathbb{K}^n$ having dimension n.
Now suppose that we have two opens V_1, V_2 such that $p^{-1}(V_1) \cap p^{-1}(V_2) = W$ is non empty. Then we have two homeomorphisms $p : W \to p(W) \subset V_1$ and $p : W \to p(W) \subset V_2$. Hence W has the same dimension as V_1 and V_2. Thus they are equal. This shows that dimension is locally constant. □

We now define a equivalence relation on the set of all vector bundles.

Definition 25. *We call two vector bundles E_1 and E_2 equivalent, $E_1 \sim E_2$ iff there exist trivial vector bundles ϵ^n, ϵ^m of dimension n and m such that $E_1 \oplus \epsilon^n \approx E_2 \oplus \epsilon^m$*

We shall show that the equivalence classes of $\sim$ form an Abelian group.

Theorem 58. *The equivalence classes of $\sim$ form an Abelian group with the operation $\oplus$.*

We define $[E_1] \oplus [E_2] = [E_1 \oplus E_2]$. This is clearly closed since any direct sum of vector bundles is again a vector bundle. It is also associative and the class of the trivial bundle $[\epsilon^0]$ forms the identity $[E_1] + [\epsilon^0] = [E_1 \oplus \epsilon^0] = [E_1]$. Thus the only thing we need to show is that there exist inverses.
We have shown that if for a vector bundle E all the fibers have the same dimension n, there exist an vector bundle E' such that their direct sum is a trivial vector bundle.
Now suppose that we have a vector bundle $p : E \to X$. We define $X_i = \{x \in X | dim \quad p^{-1}(x) = i\}$. This is a disjoint set in X, is open and hence forms an open cover. By compactness we can extract a finite subcover, but we cannot leave any non open cover out due to every open being disjoint from the rest. Hence there are only a finite amount of such X_i. We can find for each such X_i a vector bundle E'_i such that the direct sum of $p^{-1}X_i \oplus E'_i$ is trivial. These E'_i together can then be made the fibers of a vector bundle over X and we have found our inverse. This forms an Abelian group $\tilde{K}(X)$. We shall construct $K(X)$ by using a stronger equivalence relation $\approx_s$.

Definition 26. *We call two vector bundles E_1, E_2 over X stably isomorphic, $E_1 \approx_s E_2$ if $E_1 \oplus \epsilon^n \approx E_2 \oplus \epsilon^n$*

Note that if $E_1 \approx_s E_2$ then $E_1 \sim E_2$. Now we construct the Abelian group $K(X)$ using $\approx_s$. We cannot have inverses in the same way as in $\tilde{K}(X)$ as two positive vector bundles cannot be added to form the zero dimensional trivial vector bundle. Thus we seek another property. Observe the following:

Lemma 59. *If $E_1 \oplus E_2 \approx_s E_1 \oplus E_3$ then $E_2 \approx_s E_3$.*

Proof. We know that $E_1 \oplus E_2 \approx_s E_1 \oplus E_3$ and we know that for E_1 there exists an E'_1 such that $E_1 \oplus E'_1 \approx \epsilon^n$ for some n. Now add this E'_1 to both sides of the first equation. This yields $E'_1 \oplus E_1 \oplus E_2 \approx \epsilon^n \oplus E_2 \approx_s E_3 \oplus \epsilon^n$. And thus $E_2 \approx_s E_3$. □

We will use formal differences of vector bundles to form the Abelian group $K(X)$. The addition in $K(X)$ is defined in the following way:

$$(E_1 - E'_1) \oplus (E_2 - E'_2) = (E_1 \oplus E_2) - (E'_1 \oplus E'_2) \tag{7.1}$$

for formal differences $E_i - E'_i$, where E_i and E_i are vector bundles. The zero element defined to be $E - E$ for any vector bundle E. Notice that the inverse of an element $E - E'$ is $E' - E$. Also notice that every element $E - E'$ can be represented by $\tilde{E} - \epsilon^n$ for some n and $\tilde{E}$. We can add a bundle E'' to both E and E' such that $E' \oplus E'' \approx \epsilon^n$. We then add $E'' - E''$ to $E - E'$. This gives $(E \oplus E'') - (E' \oplus E'') = \tilde{E} - \epsilon^n$

Theorem 60. $K(X) = \tilde{K}(X) \oplus \mathbb{Z}$

Proof. Since $\approx_s$ was a stronger equivalence relation, we can find a natural homomorphism $K(X) \to \tilde{K}(X)$. Notice that the homomorphism is surjective, and the kernel consist of elements of the form $\epsilon^m - \epsilon^n$. This subgroup of $K(X)$ is isomorphic to $\mathbb{Z}$. □

7.2 Tensoring vector bundles induces a multiplication

We have just defined the additive structure of $K(X)$ and are going to work on the multiplicative structure of $K(X)$. If we have two elements of $K(X)$, namely E_1, E_2, we define their product as follows $E_1 \otimes E_2$. Since the elements of $K(X)$ are in general represented by differences of vector bundles $E - E'$ we can define their product as follows:

$$(E_1 - E_1')(E_2 - E_2') = E_1 \otimes E_2 - E_1 \otimes E_2' - E_1' \otimes E_2 + E_1' \otimes E_2' \tag{7.2}$$

Notice that tensoring is associative. distributively follows from the definition. All we have to show is that there exists an identity. ϵ^1 is the identity by $\epsilon^1 \otimes (E - E') = \epsilon^1 \otimes E - \epsilon \otimes E' = E - E'$. By commutativity ϵ^1 is the multiplicative identity. Also notice that $\epsilon^n \otimes E$ gives n copies of E hence we can abbreviate ϵ^n by n and let nE stand of n copies of E.
Now we want to show that this multiplication is well defined. Suppose that E_1, E_2 come from the same equivalence class, and we have another element E_3. Then consider $E_1 \otimes E_3$ and $E_2 \otimes E_3$. We can both add them up $(E_1 + \epsilon^n) \otimes E_3$ and $(E_2 + \epsilon^m) \otimes E_3$. These are equivalent to the first statement, however, this also yields $E_1 \otimes E_3 + \epsilon^n \otimes E_3$. And also $E_2 \otimes E_3 + \epsilon^m \otimes E_3$. These terms are equal since we multiplied the same element of $K(x)$ by E_3. However, the last part is equivalent to zero, hence $E_1 \otimes E_3$ is equivalent to $E_2 \otimes E_3$. We can exactly mirror this argument for the other side of multiplication. This shows that multiplication is well defined. As we will see in the next section, continuous maps induce ring homomorphisms. If we take the map which maps X to $x_0 \in X$ this comes down to restricting vector bundles over the fibers to x_0. Its kernel is $\tilde{K}(X)$. Hence it is an ideal. Then $\tilde{K}(X)$ obeys all the ring axioms except it might not have an identity element.

7.3 Continuous maps induce ring homomorphisms

Given two compact Hausdorff spaces X, Y and a continuous map between them. As we have seen, a map $f : X \to Y$ induces a map in vector bundles $f^* : E_Y \to E_X$. We have seen that $f^*(E_1 \oplus E_2) = f^*(E_1) \oplus f^*(E_2)$ and $f^*(E_1 \otimes E_2) = f^*(E_1) \otimes f^*(E_2)$ hence f^* is a homomorphism. Further more $(fg)^* = g^* f^*$, $1^* = 1$. Thus K obeys all the required relations for a functor.

7.4 Extra properties of $K(X)$

These properties will be proved in later chapters

7.4.1 External product

We define the external product $\mu : K(X) \otimes K(Y) \to K(X \times Y)$ to be $\mu(a \otimes b) = p_1^*(a) p_2^*(b)$ where p_1 and p_2 are the projects of $X \times Y$ onto X and Y. Notice that we tensor rings. The product of $a \otimes b \cdot c \otimes d$ is defined to be $a \cdot c \otimes b \cdot d$.

7.4.2 The fundamental Product theorem

Suppose H is the canonical line bundle over $S^2 = \mathbb{C}P^1$. In [5] example 1.13 is shown that $(H \otimes H) \oplus 1 \approx H \oplus H$. In $K(S^2)$ this yields $H^2 + 1 = 2h$, thus $H^2 - 2H + 1 = 0$ and otherwise stated: $(H - 1)^2 = 0$. Thus we have a ring homomorphism $\mathbb{Z}[h]/(H-1)^2 \to K(S^2)$. We will define a homomorphism μ as the composition

$$\mu : K(X) \otimes \mathbb{Z}[H]/(H-1)^2 \to K(X) \otimes K(S^2) \to K(X \times S^2) \tag{7.3}$$

Theorem 61. *The homomorphism $\mu : K(X) \otimes \mathbb{Z}[H]/(H-1)^2 \to K(X \times S^2)$ is an isomorphism of rings for all compact Hausdorff X.*

Proof. For a proof we refer to [5], there it is theorem 2.2 on page 41. Here we will only introduce the theorem, and later we will use it. □

This theorem yields directly the following corollary by taking $X = \{0\}$:
$\mathbb{Z}[H]/(H-1)^2 \to K(S^2)$ is an isomorphism of rings. This implies directly that $K(S^2) = \mathbb{Z}$

7.4.3 Complex Bott periodicity

The Bott periodicity theorem for complex vector bundles states that there exist a homomorphism β such that $\beta : \tilde{K}(X) \to \tilde{K}(S^2X)$ with $\beta(a) = (H-1) * a \quad \forall a \in \tilde{K}(X)$ is an isomorphism for all compact Hausdorff X, more details are given in the next chapter.
As a corollary we have that $\tilde{K}(S^{2n+1}) = 0$ and $\tilde{K}(S^{2n}) = \mathbb{Z}$.

7.4.4 K-theory as a Cohomology

We can make K-theory into a cohomology theory, by defining $\tilde{K}^{-n}(X) = \tilde{K}(S^n X)$. This obeys the axioms of cohomology. This will be shown in the chapter Extending to a cohomology theory. We then set $K^{-n}(X) = K(\Sigma^n X)$, where Σ is the reduced suspension, i.e. where we send the north and south pole to the same point.

7.4.5 The Splitting Principle

Given a vector bundle $E \to X$ with X compact Hausdorff space, there is a compact Hausdorff space $F(E)$ and a map $: F(E) \to X$ such that induced map $p^* : K^*(X) \to K^*(F(E))$ is injective and $P^*(E)$ splits as a sum of line bundles. For a proof of the splitting principle we refer to [5]

Chapter 8

Complex Bott periodicity

We start of by introducing exact sequences in $\tilde{K}(X)$.

Proposition 62. *If X is compact Hausdorff and $A \subset X$ is a closed subspace, then the inclusion and quotient maps $A \to^i X \to^q X/A$ induce homomorphisms $\tilde{K}(X/A) \to^{q^*} \tilde{K}(X) \to^{i^*} \tilde{K}(A)$ for which the kernel of i^* equals the image of q^*.*

Proof. The inclusion $Im \quad q^* \subset \ker i^*$ is equivalent to $i^*q^* = 0$. Since qi is equal to the composition $A \to A/A \to X/A$ and $\tilde{K}(A/A) = 0$ we see that $i^*q^* = 0$.
The other inclusion is more work. The general idea is that the only things that get mapped to zero come from the set A. Now suppose that E is in the kernel of i^*, i.e. an element of $\tilde{K}(X)$ such that $i^*(E)$ is 0. Then we are going to show it is an element in the image of q^*. We know that the restriction of E over A is trivial up to adding a trivial vector space. Hence we can add a trivial vector bundle to E, thus staying in the same equivalence class, and get E trivial over A. We have a trivialization $h : p^{-1}(A) \to A \times \mathbb{C}^n$. We now define E/h to be the quotient of E under the relation $h^{-1}(x, v) \sim h^{-1}(y, v)$ for all $x, y \in A$. Then we have a projection E/h to X/A. To see that this is a vector bundle on X/A we need to show that there exists a trivialization over a neighborhood of the point A/A.
To find such a neighborhood, we will make an open cover $\{U_\alpha\}$ of A in X. On this open cover we have sections $s_i : A \cap U_\alpha \to E$ to the fiber of E. These sections can be extended by the Tietze extension theorem. We also have a partition of unity $\{\psi_\alpha, \psi\}$ subordinate to the open cover $\{U_\alpha, A\}$. We then define

$$\sum_\alpha \psi_\alpha s_{i\alpha} \tag{8.1}$$

This is an extension of the section s_i on U_α to a section on X. Since these sections form a basis on A and can be seen as an invertible linear function, they are a basis in a neighborhood of A.
Now we have a trivialization h of E which extends to a neighborhood U of A. Thus we have a trivialization of E/h over U/A. Thus E/h is a vector bundle. Note that we have the following commutative diagram:

$$\begin{array}{ccc} E & \longrightarrow & E/H \\ \Big\downarrow{\scriptstyle p} & & \Big\downarrow \\ X & \underset{q}{\longrightarrow} & X/A \end{array}$$

hence we have an isomorphism $E \approx q^*(E/h)$, and thus we have found an image. □

We will extend our exact sequence. First we begin with a sequence of inclusions. Each space in this sequence is created by the cone of the space two steps back in the sequence. The vertical maps are quotient maps if we collapse that cone to a point.

$$\begin{array}{ccccccccc} A & \longrightarrow & X & \longrightarrow & X \cup CA & \longrightarrow & (X \cup CA) \cup CX & \longrightarrow & ((X \cup CA) \cup CX) \cup C(X \cup CA) \\ & & & & \downarrow & & \downarrow & & \downarrow \\ & & & & X/A & & SA & & SX \end{array}$$

Corollary 63. *If A is contractible, the quotient map $Q : X \to X/A$ induces a bijection $q^* : Vect^n(X/a) \to Vect^n(X)$ for all n.*

Proof. Since A is contractible we notice that the vector bundle $E \to X$ must be trivial. Hence we have a trivialization h. This gives a vector bundle $E/h \to X/h$ as in the previous proof. We will now show that the equivalence class of e/h does not depend on h.
Given two trivializations h_0 and h_1, we can write $h_1 = (h_1 h_0^{-1})h_0$, we see that h_0 and h_1 differ by an element g_x of $GL(n, \mathbb{C})$, i.e. an invertible matrix, for each point x in A. The resulting map $g : A \to GL(n, \mathbb{C})$ is homotopic to a constant map $x \to \alpha \in GL(n, \mathbb{C})$ since A is contractible. Write now $h_1 = (h_1 h_0^{-1} \alpha^{-1})(\alpha h_0)$, we see that composing h_0 with $alpha$ does not change E/h_0. Assume that α is the identity. Then the homotopy from g to the identity gives a homotopy H from h_0 to h_1. We now build a new vector bundle

$$(E \times I)/H \to (X/A) \times I \tag{8.2}$$

This vector bundle restricts to E/h_0 on one end of I, and E/h_1 on the other end of I. Therefore $E/h_0 \approx E/h_1$.
From this we conclude that we have a well defined map $Vect^n(X) \to Vect^n(X/A)$, namely $E \mapsto E/h$. Is is an inverse of q^* since $q^*(E/h) \approx E$, by the preceding proof, and for a bundle $E \to X/A$ we have $q^*(E)/h \approx E$ for the trivialization h of $q^*(E)$ over A. Hence we have shown that q^* is bijective. $\square$

From the previous two statements we have an exact sequence of $\tilde{K}$ groups

$$... \to \tilde{K}(SX) \to \tilde{K}(SA) \to \tilde{K}(X/A) \to \tilde{K}(X) \to \tilde{K}(A) \tag{8.3}$$

Now, suppose that X is the wedge sum $A \vee B$ then $X/A = B$ and the sequence breaks into split short exact sequences. Hence $\tilde{K}(X) \to \tilde{K}(A) \oplus \tilde{K}(B)$ is an isomorphism.
Our next goal is to use the tools we have developed to create a new long exact sequence, from which we can deduce the Bott Periodicity. The first step is to obtain a reduced version of the external product. This will be a ring homomorphism $\tilde{K}(X) \otimes \tilde{K}(Y) \to \tilde{K}(X \wedge Y)$. We will start with the long exact sequence for the pair $(X \times Y, X \vee Y)$:

$$\begin{array}{ccccccccc} \tilde{K}(S(X \times Y)) & \longrightarrow & \tilde{K}(S(X \vee Y)) & \longrightarrow & \tilde{K}(X \wedge Y) & \longrightarrow & \tilde{K}(X \times Y) & \longrightarrow & \tilde{K}(X \vee Y) \\ & & \approx \downarrow & & & & & & \approx \downarrow \\ & & \tilde{K}(SX) \oplus \tilde{K}(SY) & & & & & & \tilde{K}(X) \oplus \tilde{k}(Y) \end{array}$$

This exact sequence can be obtained directly from the previous results. The first vertical isomorphism is due to the quotient map of $SZ \to \Sigma Z$ inducing an isomorphism by the previous lemma. We also use that $\Sigma(X \vee Y) = \Sigma X \vee \Sigma Y$. This then gives the isomorphism which is a split surjection as we can do it per coordinate by projections.
The last horizontal map is also a split surjective, as we can take $(a, b) \in \tilde{K}(X) \oplus \tilde{K}(Y)$, then use the isomorphism to find corresponding points in $\tilde{K}(X \vee Y)$ and use the projection map to pull the back to $\tilde{K}(X \times Y)$. Explicitly we can send $(a, b) \mapsto p_1^*(A) + p_2^*(B)$ where p_1 and p_2 are the projections of $X \times Y$ onto X and Y. In the same fashion, the first horizontal

map splits by $(Sp_1)^* + (Sp_2)^*$.

Thus we have a splitting $\tilde{K}(X \times Y) \approx \tilde{K}(X \wedge Y) \oplus \tilde{K}(X) \oplus \tilde{K}(Y)$.

Recall that $\tilde{K}(X) = \ker(K(X) \to K(x_0))$. The same holds for $\tilde{K}(Y)$. Thus if we have $(a, b) \in \tilde{K}(X) \oplus \tilde{K}(Y)$, the external product $a * b = p_1^*(a)p_2^*(B) \in \tilde{K}(X \times Y)$ has $p_1^*(A)$ restricting to zero in $\tilde{K}(Y)$ and $p_2^*(b)$ restricting to zero in $\tilde{K}(X)$. Thus $p_1^*(a)p_2^*(b)$ restricts to zero in both $K(X)$ and $K(Y)$, and therefore also in $K(X \vee Y)$. This implies that $a*b$ lies in $\tilde{K}(X \times Y)$, and the short exact sequence then implies that $a * b$ pulls back to an unique element of $\tilde{K}(X \wedge Y)$. This defines the reduced external product. We will now show that it is a restriction of the unreduced external product:

$$\begin{array}{ccc}
\tilde{K}(X) \otimes \tilde{K}(Y) & \xrightarrow{\approx} & (\tilde{K}(X) \otimes \tilde{K})Y)) \oplus \tilde{K}(X) \oplus \tilde{K}(Y) \oplus \mathbb{Z} \\
\downarrow & & \downarrow \\
K(X \times Y) & \xrightarrow{\approx} & \tilde{K}(X \wedge Y) \oplus \tilde{K}(X) \oplus \tilde{K}(Y) \oplus \mathbb{Z}
\end{array}$$

Notice that the last part of both lines are equal, thus the only first two parts change. From this we conclude that the reduced external product is also a ring homomorphism.

Now we will set up the last step before we have the Bott periodicity theorem. We know that $S^n \wedge X$ is the n-fold iterated reduced suspension $\Sigma^n X$. This in turn is a quotient of $S^n(X)$ by collapsing an n-disk in $S^n X$ to a point. Hence by the previous lemma the quotient map $S^n X \to S^n \wedge X$ induces an isomorphism on $\tilde{K}$.

Then the reduced external product gives rise to a homomorphism

$$\beta : \tilde{K}(X) \to \tilde{K}(S^2 X) \tag{8.4}$$

where

$$\beta(a) = (H - 1) * a \tag{8.5}$$

where H is the canonical line bundle over $S^2 = \mathbb{C}P^1$.

Theorem 64. *The homomorphism $\beta : \tilde{K}(X) \to \tilde{K}(S^2 X)$ with $\beta(\alpha) = (H - 1) * \alpha$ is an isomorphism for all compact Hausdorff spaces X.*

Proof. The map β is the composition

$$\tilde{K}(X) \to \tilde{K}(S^2) \otimes \tilde{K}(X) \to \tilde{K}(S^2 X) \tag{8.6}$$

where the first map $a \mapsto (H - 1) \otimes A$ is an isomorphism since $\tilde{K}(S^2)$ is infinite cyclic generated by $H - 1$ and the second map is the reduced external product. Then by the diagram above this is equivalent to the product theorem. □

Corollary 65. *$\tilde{K}(S^{2n+1}) = 0$ and $\tilde{K}(S^{2n}) \approx \mathbb{Z}$, generated by the n-fold reduced external product $*^n(H - 1)$.*

Chapter 9

Extending to a cohomology theory

In the previous chapter we shown that

$$\tilde{K}(S^2X) \to \tilde{K}(S^2A) \to \tilde{K}S(X/A) \to \tilde{K}(SX) \to \tilde{K}(SA) \to \tilde{K}(X/A) \to \tilde{K}(X) \to \tilde{K}(A) \tag{9.1}$$

is an exact sequence of groups. By definition of the graded topological K-theory we have the exact same sequence:

$$\tilde{K}^{-2}(X) \to \tilde{K}^{-2}(A) \to \tilde{K}^{-1}(X,A) \to \tilde{K}^{-1}(X) \to \tilde{K}^{-1}(A) \to \tilde{K}^{0}(X,A) \to \tilde{K}^{0}(X) \to \tilde{K}^{0}(A) \tag{9.2}$$

Now by the product theorem we have that $\tilde{K}^{-2}(X)$ is isomorphic to $\tilde{K}^{0}(X)$ by multiplying every element in $\tilde{K}^{0}(X)$ by β. This makes it reasonable by setting $\tilde{K}^{2i}(X) = \tilde{K}(X)$ and $\tilde{K}^{2i+1}(X) = \tilde{K}(SX)$. Then we have the following exact sequence of groups:

$$\begin{array}{ccccc} \tilde{K}^0(X,A) & \longrightarrow & \tilde{K}^0(X) & \longrightarrow & \tilde{K}^0(A) \\ \downarrow & & & & \uparrow \\ \tilde{K}^1(X,A) & \longrightarrow & \tilde{K}^1(X) & \longrightarrow & \tilde{K}^1(A) \end{array}$$

Now we can define a product between $\tilde{K}^i(X) \otimes \tilde{K}^j(Y) \to \tilde{K}^{i+j}(X \wedge Y)$. We have an external product on $\tilde{K}(X) \otimes \tilde{K}(Y)$ which goes to $\tilde{K}(X \wedge Y)$. Then if we replace X by S^iX and Y by S^jY we have our product. By our above statement we can define $\tilde{K}^*(X)$ to be $\tilde{K}^0(X) \oplus \tilde{K}^{-1}(X)$ without losing any information. Hence we have a product of the form

$$\tilde{K}^*(X) \otimes \tilde{K}^*(Y) \to \tilde{K}^*(X \wedge Y)$$

We can also define a relative form of this product:

$$\tilde{K}^*(X,A) \otimes \tilde{K}^*(Y,B) \to \tilde{K}^*(X \times Y, X \times B \cup A \times Y)$$

We define this product by replacing $\tilde{K}^i(X/A)$ by $\tilde{K}(\Sigma^i(X/A)$. Since $X/A \wedge Y/B = (X \times Y)/(X \times B \cup A \times Y)$ we can replace this in the definition of $\tilde{K}^{i+j}(X/A \wedge Y/B)$ and get $\tilde{K}^{i+j}(X \times Y, X \times B \cup A \times Y)$

We can now compose the product $\tilde{K}^*(X) \otimes \tilde{K}^*(X) \to \tilde{K}^*(X \wedge X)$ with the map $\tilde{K}^*(X \wedge X)\tilde{K}^*(X)$ which comes from the map $x \mapsto (x,x)$. This gives a multiplication on $\tilde{K}^*(X)$. This makes $\tilde{K}(X)^*$ into a ring, which extends the ring on $\tilde{K}^0(X)$. Again notice that there need not be an identity, and all other axioms are satisfied.

We can also give a relative form of this product in the same way as before, $\tilde{K}^*(X,a) \otimes \tilde{K}^*(X,B) \to \tilde{K}^*(X, A \cup B)$ which comes from the diagonal map $X/(A \cup B) \to X/A \wedge X/B$.

Proposition 66. *Multiplication in $\tilde{K}^*(X)$ is commutative up to a sign: $\alpha\beta = (-1)^{ij}\beta\alpha$ for all $\alpha \in \tilde{K}^i(X)$ and $\beta \in \tilde{K}^j(X)$.*

Proof. The product is as we have just seen the composition

$$\tilde{K}(S^i \wedge X) \otimes \tilde{K}(S^j \wedge X) \to \tilde{K}(S^i \wedge S^j \wedge X \wedge X) \to \tilde{K}(S^i \wedge S^j \wedge X) \tag{9.3}$$

where the first map is the external product and the second map is induced by the diagonal map. If we replace $\alpha\beta$ by $\beta\alpha$ the factors in the first term $\tilde{K}(S^i \wedge X) \otimes \tilde{K}(S^j \wedge X)$ switch. This corresponds to switching the S^i and the S^j factors in the second term, and also in the third therm. If we view $S^i \wedge S^j$ as $i + j$ wedges of S^1, then switching S^i and S^j is the same as ij times switching two adjacent circles. So all we need to show is that if we switch two adjacent factors we get a -1 sign and then we have proven that $\alpha\beta = (-1)^{ij}\beta\alpha$.
If we transpose the two factors of $S^1 \wedge S^1$. it is equivalent to reflecting S^2 across an equator, since the smash product of two S^1 is homotopic to S^2, and if we switch them the homotopy reflects S^2 across its equator. Since reflecting across the equator of S^2 is the same as reversing the direction in which we do a suspension, since $S^2 = SS^1$. Hence all we need to do is show that reversing the two ends of a suspension SY induces multiplication by -1 in $\tilde{K}(SY)$. We can view $\tilde{K}(SY)$ as $[Y, U]$, where U is the infinite unitary group, we see that switching ends of SY corresponds to the map $U \to U$ sending a matrix to its inverse. The group operation induced by this map is the same as the operation induced by the product in U. Hence we see that reversing the end points of a suspension gives rise to a factor -1 □

We will state the following result without prove for completeness:

Proposition 67. *The following sequence is an exact sequence of $\tilde{K}^*$ modules with maps homomorphisms of $\tilde{K}^*$ modules:*

$$\begin{array}{ccc} \tilde{K}^*(X,A) & \longrightarrow & \tilde{K}^0(X) \\ \uparrow & \swarrow & \\ \tilde{K}^*(A) & & \end{array}$$

We will now show how to create unreduced versions of the group $\tilde{K}(X)$. We define $K^n(X)$ to be $\tilde{K}^n(X_+$, where X is just X with a disjoint extra point called $+$ adjoined. For $n = 0$ this is consistent with the established relations between $\tilde{K}(X)$ and $K(X)$ since $K^0(X) = \tilde{K}^0(X_+) = \tilde{K}(X_+) = \ker(K(X_+) \to K(+)) = K(X)$. When $n = 1$ this yields $K^1(X) = \tilde{K}^1(X)$ since $S(X_+) = SX \vee S^1$ and $\tilde{K}(SX \vee S^1) \approx \tilde{K}(SX) \oplus \tilde{K}(S^1) = \tilde{K}(SX)$. Here we used $\tilde{K}(S^1) = 0$. For a pair (X, A) with A nonempty we define $K^n(X, A) = \tilde{K}^n(X, A)$ and the six term exact sequence which we derived at the beginning of this chapter is again valid. If A is empty then the statement remains true if we consider $X/\emptyset = X_+$.
We now work on creating an product, the idea is again the same. Since $X_+ \wedge Y_+ = (X \times Y)_+$ the external product $\tilde{K}^*(X) \otimes \tilde{K}^*(Y) \to \tilde{K}^*(X \wedge Y)$ gives a product $K^*(X) \otimes K^*(Y) \to K^*(X \times Y)$. If we then again take $Y = X$ and compose with the diagonal map we get a product which makes $K^*(X)$ into a ring.
We can do the same for the relative product $K^i(X, A) \otimes K^j(Y, B) \to K^{i+j}(X \times Y, X \times B \cup A \times Y)$ defined as the external product $\tilde{K}(\Sigma^i(X/A)) \otimes \tilde{K}(\Sigma^j(Y/B)) \to \tilde{K}(\Sigma^{i+j}(X/A \wedge Y/B))$ using the same identification as before: $(X \times Y)/(X \times B \cup A \times Y) = X/A \wedge Y/B$. Again this works when $A = \emptyset$ since we interpret $X/\emptyset$ as X_+ and similarly for (Y, B). Via the diagonal map we also obtain a product $K^i(X, A) \otimes K^j(X, B) \to K^{i+j}(X, A \cup B)$. With these definitions the preceding propositions also become true for K groups.

Chapter 10

S^n is an H-space iff $n = 0, 1, 3, 7$

Definition 27. *A Hopf space, H-space in short, is a topological space X with a multiplication map $m : X \times X \to X$ which is continuous and there exists an identity e such that $m(e, x) = m(x, e) = x \forall x \in X$.*

Lemma 68. *If $\mathbb{R}^n$ is a division algebra, or S^{n-1} is parallelizable, then S^{n-1} is an H-space.*

Proof. If we have a division algebra structure on $\mathbb{R}^n$ with two sides identity, an H-space structure on S^{n-1} is given by $(x, y) \mapsto \frac{xy}{|xy|}$, which is well defined since there are no zero divisors in a division algebra, hence $xy \neq 0$ and $|.|$ is a norm thus $|xy| \neq 0$.
Now suppose S^{n-1} is parallelizable, with tangent vector fields $v_1, .., v_{n-1}$ which are linearly independent at each point of S^{n-1}. Then at each x we can apply the gram Schmidt process to make the vectors $x, v_1(x), ..., v_{n-1}(x)$ orthonormal. Since these vectors at e_1 are non zero, we can deform them to become the basis vectors $e_1, ..e_n$ by deforming vector fields near e_1 and possibly changing the orientation of the vector field by changing sign of e_{n-1}. Then we find for each x an α_x which is a matrix which obeys the following relations: $\alpha_x \alpha_x^T = \alpha_x^T \alpha_x = 1$ and having determinant 1. We define α_x to be the map which sends the standard basis vectors to $x, v_1(x), .., v_{n-1}(x)$. This is an isometry on $\mathbb{R}^n$ and hence the map $(x, y) \to \alpha_x(y)$ defines an H space structure on S^{n-1}. We need to check that $(e, y) = y$ and $(x, e) = x$. We note that on the point e α_e is the identity map, and $\alpha_x(e) = x$, since e is the first basis vector. □

10.1 H-spaces have Hopf invariant ± 1

We begin with a list of easy consequences from earlier statements which we will need later on.

Corollary 69. *$\tilde{K}(S^n)$ is $\mathbb{Z}$ if n is even and 0 if n is odd, and the generator for $\tilde{K}(S^{2k})$ is the $k-fold$ external product $(H-1) * ... * (H-1)$. Multiplication in $\tilde{K}(S^{2k})$ is trivial, since it is the $k-fold$ tensor product of $\tilde{K}(S^2)$, which has trivial multiplication.*

Corollary 70. *The external product $\tilde{K}(S^{2k}) \otimes \tilde{K}(X) \to \tilde{K}(S^{2k} \wedge X)$ is an isomorphism since it is an iterate of the periodicity isomorphism.*

Corollary 71. *The external product $K(S^{2k}) \otimes K(X) \to K(S^{2k} \times X)$ is an isomorphism. This follows from the previous corollary and proof that there is a equivalence between the reduced and unreduced form of the Bott periodicity. Since the external product is a ring isomorphism, the isomorphism $\tilde{K}(S^{2k} \wedge X) \approx \tilde{K}(S^{2k}) \otimes \tilde{K}(X)$ is a ring isomorphism as well. Then because $K(S^{2k})$ can be described by $\mathbb{Z}(\alpha)/(\alpha^2)$ we can deduce that $K(S^{2k} \times K^{2l})$ is $\mathbb{Z}(\alpha, \beta)/(\alpha^2, \beta^2)$ where α and β are the pullbacks of generators of $\tilde{K}(S^{2k})$ and $\tilde{K}(S^{2l})$ under projections of $S^{2k} \times S^{2l}$ onto its factors. Thus we find a basis for $K(S^{2k} \times S^{2l})$ namely $\{1, \alpha, \beta, \alpha\beta\}$.*

Notice that the smash product of any topological space X with S^n is homotopic to the n-reduced suspension of $X : \Sigma^n X$ which in turn is homotopic to $S^n X$.

Suppose S^{2k} is a H-space for $k > 0$. Then $\mu : S^{2k} \times S^{2k} \to S^{2k}$ is its H-space multiplication. The induced homomorphism of rings then has the form $\mathbb{Z}(\gamma)/(\gamma^2) \to \mathbb{Z}[\alpha, \beta]/(\alpha^2, \beta^2)$. Then what is $\mu(\gamma)$? If we take i the inclusion onto the subspace $S^{2k} \times \{e\}$ then i sends α to γ, hence the coefficient of α of $\mu^*(\gamma)$ must be one. The same holds for β if we take i the projection onto the subspace $\{e\} \times S^{2k}$.

Then what is $\mu^*(\gamma)^2$. Since μ^* is a homomorphism $\mu^*(\gamma)^2 = \mu^*(\gamma^2) = \mu^*(0) = 0$. But $\mu^*(\gamma) = \alpha + \beta + m\alpha\beta$ for some $m \in \mathbb{Z}$. If we compute this we get $\mu^*(\gamma)^2 = (\alpha + \beta + m\alpha\beta)^2 = 2\alpha\beta$. This is not zero since α and β are not zero, nor is their product. This leads to a contradiction. Hence S^{2k} cannot be an H-space.

Now we need to show that S^{n-1} cannot be an H space for n even and unequal to $2, 4, 8$. We start by constructing a map $\hat{g} : S^{2n-1} \to S^n$ for every map $g : S^{n-1} \times S^{n-1} \to S^{n-1}$.

We regard S^{2n-1} as the union of two opens: $\partial D^n \times D^n \cup D^n \times \partial D^n = \partial(D^n \times D^n)$.

We take S^n as the union of two discs D^n with their boundary identified. Then we define $\hat{g}$ on $D^n_+ = |y|g(x, y/|y|)$ and on D^n_- $\hat{g}(x, y) = |x|g(x/|x|, y)$. Note that $\hat{g}$ is defined on the boundary as $|x|g(x/|x|, y) = g(x, y) = |y|g(x, y/|y|)$ since $|x| = |y| = 1$. When $|y|$ or $|x|$ go to zero, we notice that since g is bounded, $\hat{g}$ goes to zero. $\hat{g}$ agrees with g on $S^{n-1} \times S^{n-1}$.

When we have a map $f : S^{4n-1} \to S^{2n}$, we define C_f be $S^2 n$ with a cell e^{4n} attached by f. The quotient C_f/S^{2n} is S^{4n}, and $\tilde{K}^1(S^{4n}) = \tilde{K}^1(S^{2n}) = \tilde{K}(S^{2n+1}) = 0$. Therefore the exact sequence of the pair $(C_f, S^{2n}$ becomes

$$0 \to \tilde{K}(S^{4n}) \to \tilde{K}(C_f) \to \tilde{K}(S^{2n}) \to 0 \tag{10.1}$$

Let $\alpha \in \tilde{K}(C_f)$ be the image of the generator of $2n$ times the exterior product of $H - 1$. Thus $*^{2n}(H-1)$. Let β map to $*^n(H-1)$, the generator of $\tilde{K}(S^{2n})$. Then β^2 maps to 0 in $\tilde{K}(S^{2n})$ since every square in $\tilde{K}(S^{2n})$ is zero. By exactness we have $\beta^2 = h\alpha$ for some integer h. This h is called the Hopf invariant of f.

h is well defined since β is unique up to addition of a factor of α, and $(\beta + m\alpha)^2 = \beta^2 + 2m\alpha\beta$ since α^2. And $\alpha\beta = 0$ since α maps to 0 in $\tilde{K}(S^{2n})$, therefore $\alpha\beta$ maps to 0. Therefore $\alpha\beta = k\alpha$ for some integer k. Multiply the equation $k\alpha = \alpha\beta$ on the right by β we have $k\alpha\beta = \alpha\beta^2 = \alpha h\alpha = h\alpha^2$ and this is zero since $\alpha^2 = 0$. Thus $k\alpha\beta = 0$. After we divide by k we see that $\alpha\beta = 0$.

Lemma 72. *If $g : S^{2n-1} \times S^{2n-1} \to S^{2n-1}$ is a H-space multiplication, then the associated map $\tilde{g} : S^{4n-1} \to S^{2n}$ has Hopf invariant ± 1.*

Proof. We begin by constructing a commutative diagram. Let $f = \hat{g}$. By the way we defined f it is natural to see the characteristic map Φ of the $4n$-cell of C_f as a map of the pair $(D^{2n} \times D^{2n}, \partial(D^{2n} \times D^{2n}))$ to the pair (C_f, S^{2n}). This induces a map Φ^* in K-theory. Also notice the isomorphism between $\tilde{K}(D^{2n} \times \{e\}, \partial D^{2n} \times \{e\}) \otimes \tilde{K}(\{e\} \times D^{2n}, \{e\} \times \partial D^{2n})$ and $\tilde{K}(D^{2n} \times D^{2n}, \partial D^{2n} \times D^{2n}) \otimes \tilde{K}(D^{2n} \times D^{2n}, D^{2n} \times \partial D^{2n})$, since everything in the second respectively first D^{2n} factor gets collapsed to zero.

$$
\begin{array}{ccc}
\tilde{K}(C_f)\otimes\tilde{K}(C_f) & \longrightarrow & \tilde{K}(C_f) \\
\uparrow \approx & & \uparrow \\
\tilde{K}(C_f, D^{2n}_-)\otimes\tilde{K}(C_f, D^{2n}_- & \longrightarrow & \tilde{K}(C_f, S^{2n}) \\
\Phi^*\otimes\phi^* \downarrow & & \Phi^* \downarrow \approx \\
\tilde{K}(D^{2n}\times D^{2n}, \partial D^{2n}\times D^{2n})\otimes\tilde{K}(D^{2n}\times D^{2n}, D^{2n}\times\partial D^{2n}) & \longrightarrow & \tilde{K}(D^{2n}\times D^{2n}, \partial(D^{2n}\times D^{2n})) \\
\downarrow \approx & \nearrow \approx & \\
\tilde{K}(D^{2n}\times\{e\}, \partial D^{2n}\times\{e\})\otimes\tilde{K}(\{e\}\times D^{2n}, \{e\}\times\partial D^{2n}) & &
\end{array}
$$

The horizontal maps are the product maps. The diagonal map is the external product $\tilde{K}(S^{2n})\otimes\tilde{K}(S^{2n})\to\tilde{K}(S^{4n})$ which is an isomorphism by iteration of the Bott periodicity isomorphism.
Since f is a H space multiplication, Φ restricts to a homeomorphism from $D^{2n}\{e\}$ onto D^{2n}_- and from $\{e\}\times D^{2n}$ onto D^{2n}_+. Now we take an element $\beta\otimes\beta$ in the upper left group in the diagram. This gets mapped to a generator in the bottom row of the diagram, since β restricts to a generator of $\tilde{K}(S^{2n})$ by definition of β. Thus $\beta\otimes\beta$ gets send to a generator of $\tilde{K}(C_f, S^{2n})$. If we look at the horizontal map in the first row, this is equal to the vertical map from $\tilde{K}(C_f, S^{2n})$ after going through the diagram. Since this diagram commutes, $\beta\otimes\beta$ gets send to $\pm\alpha$ where α is the image of a generator of $\tilde{K}(C_f, S^{2n})$. Thus we have $\beta^2=\pm\alpha$. Thus the Hopf invariant of f is ± 1 since this is the coefficient in front of α when we look at the image of β^2. □

Theorem 73. *There exists a map $f: S^{4n-1}\to S^{2n}$ of Hopf invariant ± 1 only when $n=1,2,4$.*

In the next section we will show the exists of so called Adams operations. They have the following properties

- $\phi^k f^* = f^*\phi^k$ for all maps $f: X\to Y$.
- $\phi^K(L)=L^k$ if L is a line bundle
- $\phi^k\circ\phi^l=\phi^{kl}$
- $\phi^p(x)=x^p \quad mod \quad p$ for p prime.

Then there is an extra property which states that that $\phi^k:\tilde{K}(S^{2n})\to\tilde{K}(S^{2n})$ is multiplication by k^n. This is a proposition we will prove in the next chapter.

Proof. We have $\alpha,\beta\in\tilde{K}(C_f)$. By our previous prop $\phi^k(\alpha)=k^{2n}\alpha$ since α is the image of a generator of $\tilde{K}(S^{4n})$. Similarly, $\phi^k(\beta)=k^n\beta+\mu_k\alpha$ for some $\mu_k\in\mathbb{Z}$. Now we use commutativity of $\phi^k\phi^l=\phi^l\phi^k$. Therefore we have

$$\phi^k\phi^l=\phi^k(l^n\beta+\mu_l\alpha)=k^nl^n\beta+(k^{2n}\mu_l+l^n\mu_k)\alpha \tag{10.2}$$

and

$$\phi^l\phi^k=\phi^l(k^n\beta+\mu_k\alpha)=l^nk^n\beta+(l^{2n}\mu_k+k^n\mu_l)\alpha \tag{10.3}$$

Therefore we have the following relation

$$k^{2n}\mu_l+l^n\mu_k=l^{2n}\mu_k+k^n\mu_l \tag{10.4}$$

This is equal to

$$k^{2n}-k^n)\mu_l=(l^{2n}-l^n)\mu_k \tag{10.5}$$

By property 4 of ϕ^2 we have $\phi^2(\beta) = \beta^2 \mod 2$. Since $\beta^2 = h\alpha$ with the Hopf invariant of f. This gives $\phi^2(\beta) = 2^n\beta + \mu_2\alpha$. Thus $\mu_2 = h \mod 2$. If we assume $h = \pm 1$ then μ_2 is odd. If f is H-space multiplication μ_2 must be odd. Then we have $(2^{2n} - 2^n)\mu_3 = (3^{2n} - 3^n)\mu_2 = 2^n(2^n - 1)\mu_3 = 3^n(3^n - 1)\mu_2$. Thus 2^n divides $3^n(3^n - 1)\mu_2$. Since 3^n and μ_2 are odd, 2^n must divide $3^n - 1$. By the next proposition 2^n divides $3^n - 1$ iff $n = 1, 2, 4$. □

Proposition 74. 2^n *divides* $3^n - 1$ *iff* $n = 1, 2, 4$.

Proof. Write $n = 2^l m$ with m odd. We will find the highest power of 2 dividing $3^n - 1$ by induction on l.
If $l = 0$ we have modulo 4 $3^n - 1 \equiv (-1)^m - 1 \equiv 2$ since m is odd.
if $l = 1$ we have $3^{2m} - 1 \equiv 1 - 1 \equiv 0$ modulo 4. Thus we have divisors. $3^{2m} - 1 = (3^m - 1)(3^m + 1)$. The highest power of 2 dividing the first factor is 2 as we just showed. The highest power of 2 dividing the second factor is 4 since $3^m + 1 \equiv 4$ if we look modulo 8, since even powers of 3 are equal to 1 mod 8 and then odd powers are equal to 3. We add one and get 4.
Now if we go from l to $l+1$ and using $l \geq 1$, or from n to $2n$ with n even since $n = 2^l m$, we write $3^{2n} - 1 = (3^n - 1)(3^n + 1)$. Then $3^n + 1 \equiv 2$ modulo 4 since n is even and even powers of three are equivalent to 1. Thus the highest power of 2 dividing $3^n + 1$ is 2. Thus the highest power of 2 dividing $3^n - 1$ is twice the highest power of 2 dividing $3^n - 1$.
Notice that the highest power of 2 dividing $3^n - 1$ is bounded by 2^{l+2}.
Now suppose 2^n divides $3^n - 1$ we see that the n is less or equal to $l + 2$ since the highest power of 2 dividing $3^n - 1$ is bounded by $l + 2$. Thus $2^l \leq 2^l m = n \leq l + 2$. Thus $l \leq 2$, and hence $n \leq 4$.
This leaves us to check the cases of $n = 1, 2, 3, 4$.

- $n = 1$: 2 divides 2. Factor is 1.
- $n = 2$: 4 divides 8. Factor is 2.
- $n = 3$: 8 does not divide 26. $26 = 3 * 8 + 2$.
- $n = 4$: 16 divides 80. The factor is 5.

This proves the statement. □

10.2 Adams Operations

Theorem 75. *There exists ring homomorphisms* $\phi^K : K(X) \to K(X)$ *defined for all compact Hausdorff spaces* X *and all integers* $k \geq 0$ *which satisfy*

- $\phi^k f^* = f^* \phi^k$ *for all maps* $f : X \to Y$.
- $\phi^K(L) = L^k$ *if* L *is a line bundle*
- $\phi^k \circ \phi^l = \phi^{kl}$
- $\phi^p(x) = x^p \mod p$ *for* p *prime.*

These are called Adams operations.

Proof. We will use exterior powers $\lambda^k(E)$. Exterior powers have the following properties on vector bundles:

- $\lambda^k(E_1 \oplus E_2) = \oplus_i(\lambda^i(E_1) \otimes \lambda^{k-i}(E_2))$.
- $\lambda^0(E) = 1$
- $\lambda^1(E) = E$

- $\lambda^k(E) = 0$ if k is greater than the maximum dimension of the fibers of E.

We will first characterize what we need to do:

If we use property 1 and 2 of ϕ we see that we want ϕ^k of a direct sum of line bundles $\oplus_i L_i$ results in a sum $\sum_i \phi^K(L_i) = \sum_i L_i^k$, since we can use the projection map.

Now we claim that there exists a polynomial s_k with integer coefficients such that $L_1^K + \cdots + L_n^K = S_k(\lambda^1(E), \cdots, \lambda^k(E))$. We define $\phi^K(E) = s_k(\lambda^1(E), \cdots, \lambda^k(E))$.

The idea for construction these polynomials is using interpolation polynomials.

The first step is to show that exterior powers $\lambda^i(E)$ define a polynomial $\lambda_t(E) = \sum_i (\lambda^i(E)t^i \in K(X)[t]$. Since $\lambda^k(E) = 0$ if K is greater than the dimension of the fibers of E this is a finite sum, and hence a polynomial. By property 1 the direct sum $E_1 \oplus E_2$ yields $\lambda_t(E_1 \oplus E_2) = \lambda_t(E_1)\lambda_t(E_2)$ since $\otimes$ is the multiplication operator in $K(X)$. Now let $E = L_1 \oplus \cdots L_n$. Then $\lambda_t(E) = \prod_i \lambda_t(L_i)$. Then this is $\prod_i (1 + L_i + \sum_{k \geq 2} \lambda^k(L_i)) = \prod_i (1 + L_i)$. The coefficient $\lambda^j(E)$ of t^j in $\lambda_t(E)$ is the j-th elementary symmetric function σ_j of the L_i, which is the sum of all products of j distinct L_i. This yields us

$$\lambda^j(E) = \sigma_j(L_1, \cdots, L_n) \tag{10.6}$$

If we now substitute a L_i for a variable t_i we get $(1+t_1)\cdots(1+t_n) = 1 + \sigma_1 + \cdots + \sigma_n$. Then we can define the newton polynomial $s_k(\sigma_1, \cdots, \sigma_n) = t_1^k + \cdots + t_n^k$. If we define $\phi^k(E)$ as stated we get

$$\phi^K(E) = S_k(\lambda^1(E), \cdots, \lambda^k(E)) \tag{10.7}$$
$$= s_k(\sigma_1(L_1, \cdots, L_n), \cdots, \sigma_k(L_1, \cdots, L_k)) \tag{10.8}$$
$$= L_1^k + \cdots + L_n^K \tag{10.9}$$

Our next goal is verifying that this obeys all our properties for ϕ.

For the first property it is just filling in, it follows since $\lambda(f^*E) = f^*(\lambda(E))$.

Now the splitting principle tells us that given a vector bundle $E \to X$ with X compact Hausdorff, there is a compact Hausdorff space $F(E)$ and a map $p : F(E) \to X$ such that the induced map $p^* : K^*(X) \to K^*(F(E))$ is injective and it splits as a sum of line bundles.

This allows us to prove the additivity: We can do pullbacks on E_1 and then on E_2 and then we have $\phi^k(E_1 \oplus E_2) = \phi^k(L_1 \oplus \cdots \oplus L_n) = L_1^k + \cdots + L_n^k$.

Now we are going to show multiplicativity: If E is the sum of line bundles L_i and E is the sum of line bundles L_j then $E \otimes E$ is the sum of line bundles $L_i \otimes L_j$. Hence $\phi^K(E \otimes E) = \sum_{i,j} \phi^k(L_i, \otimes L_j) = \sum_{i,j} L_i^k \otimes L_j^k = \sum_i L_i^k \sum_j L_j = \phi^k(E)\phi^k(E)$. Thus ϕ^k is multiplicative for vector bundles and thus also for elements on $k(X)$.

Now we want to prove property 3. If we use the splitting principle and additivity we are left in the case of line bundles, where $\phi^K(\phi^l(L)) = L^{kl} = \phi^l\phi^k(L)$.

In case of property 4 we again reduce to line bundles, and for line bundles we have $E = L_1 + \cdots + L_n$ then $\phi^p(E) = L_1^p + \cdots + L_n^p = (L_1 + \cdots + L_n)^p = E^p$ if we reduce everything modulo p, since the binomial coefficients will be cancelled. □

We can restrict ϕ^k to be a homomorphism on $\tilde{K}(X) \to \tilde{K}(X)$ by the naturality property, since $\tilde{K}(X)$ is the kernel of the homomorphism $K(X) \to K(x_0)$ for $x_0 \in X$. For the external product $\tilde{K}(X) \otimes \tilde{K}(Y) \to \tilde{K}(X \wedge Y)$ we have

$\phi^k(\alpha * \beta) = \phi^k(\alpha) * \phi^K(\beta)$ since

$$\phi^k(\alpha * \beta) = \phi^k(p_1^*(\alpha)p_2^*(\beta)) \tag{10.10}$$
$$= \phi^k(p_1^*(\alpha))\phi^k(p_2^*(\beta)) \tag{10.11}$$
$$= p_1^*\phi^k(\alpha)p_2^*\phi^K(\beta)) \tag{10.12}$$
$$= \phi^k(\alpha) * \phi^k(\beta) \tag{10.13}$$

Proposition 76. *$\phi^k : \tilde{K}(S^{2n}) \to \tilde{K}(S^{2n})$ is multiplication by k.*

Proof. We will proof this by induction. Notice that we have an isomorphism by the external product $\tilde{K}(S^2)\tilde{K}(S^{2n-2}) \to \tilde{K}(S^{2n})$.
Consider the case $n = 1$. ϕ^k is additive, thus it will suffice to show $\phi^k(\alpha) = k\alpha$ for α of $\tilde{K}(S^2)$. We take $\alpha = H - 1$ for H the canonical line bundle over $S^2 = \mathbb{C}P^1$. Then $\phi^k(\alpha) = \phi^k(H - 1) = \phi^k(H) - \phi^k(1) = H^k - 1$.
Then we write H as $1 + \alpha$. Since $\alpha^i = 0$ for $i > 1$ we can expand the binomial terms, see that that there are everywhere α^i terms except when $i < 2$, thus they become zero. The coefficient for α is k, and thus we get that $H^k - 1 = 1 + k\alpha - 1 = k\alpha$.
Now the induction step. Suppose the result holds for $\tilde{K}(S^{2n-2})$. We have $\phi^K(\alpha*\beta) = \phi^K(\alpha)*\phi^k(\beta) = k\alpha*k^{n-1}\beta = k^n(\alpha*\beta)$, where $\alpha * \beta$ is the generator of $\tilde{K}(S^{2n})$ and hence we are done. □

If we use that S^7 is not a topological group, then we also know which spheres can be groups:

Corollary 77. *The only n for which S^n is a topological group are $0, 1, 3$*

Proof. We know that S^n is a Hopf space if and only if $n \in \{0, 1, 3, 7\}$. If S^n is a topological group then S^n is a Hopf space. Hence this n also shows the only possible spaces where S^n is a topological group. However S^7 is not a topological group. Hence the only n are $0, 1, 3$. □

Corollary 78. *The only dimensions a real division algebra can have are 1, 2, 4 or 8.*

Proof. This follows directly from the fact that an n-dimensional real division algebra induces a Hopf space structure on S^{n-1}, and that the only n for which S^n is a Hopf space are $0, 1, 3, 7$. □

Bibliography

[1] Frank Adams - Classifying vector fields on spheres, Annals of Mathematics, 1962

[2] Frits Beukers - Elementaire getaltheorie, diktaat

[3] Marius Crainic, Inleiding topologie

[4] Allen Hatcher, Algebraic geometry, ISBN 0-521-79540-0, Cambridge University Press, 2002

[5] Allen Hatcher, Vector bundles and K-theory, Preprint from website http://www.math.cornell.edu/ hatcher/

[6] Planetmath, Spheres that are Lie Groups, http://planetmath.org/spheresthatareliegroups

[7] http://planetmath.org/cohomologyofcompactconnectedliegroups

[8] Rasmus Précenth, The (1,2,4,8)-Theorem for Composition Algebras, June 2013

[9] Jason DeVito, Answer on A question, https://math.stackexchange.com/questions/12453/is-there-an-easy-way-to-show-which-spheres-can-be-lie-groups, Nov-2010

[10] Wikipedia, Frobenius theorem of real division algebras, Wikipedia