4. Covering spaces and the homotopy group of S^1

So far, we know that the homotopy group of the spaces $\mathbb{R}^n$ (or any contractible space) is trivial. In this section we will compute the fundamental group of the circle; more precisely, we will show that it isomorphic to the cyclic group in one generator, i.e. to $(\mathbb{Z}, +)$. The basic idea is to relate S^1 to $\mathbb{R}$, by the map

$$p : \mathbb{R} \longrightarrow S^1, p(t) = (cos(2\pi t), sin(2\pi t)).$$

We will use the base point $1 \in S^1$. Note that $p^{-1} = \mathbb{Z}$. It is useful to picture this map as follows: spiral $\mathbb{R}$ above the circle (on a cylinder), as shown in Figure **??**, and then p is just the projection into the circle. In particular, we see that the pre-image p^{-1} of small opens $U \subset S^1$ will consist of a disjoint family of copies of U. It is precisely this property that allows us to compute $\pi(S^1, 1)$. What we will actually do is to look at more general maps which have this property (which will be called covering maps), and extract the relevant information about fundamental groups. Returning to our example, we will be able to conclude the computation of $\pi(S^1, 1)$.

Definition 12.21. *Let $p : E \longrightarrow B$ be a continuous map. Given $U \subset B$ an open subset, a partition of $p^{-1}(U)$ into slices is a family $\{V_i\}_{i \in I}$ of opens in E such that*

- *$\{V_i\}$ is a partition of $p^{-1}(U)$;*
- *$p|_{V_i} : V_i \longrightarrow U$ is a homeomorphism for each $i \in I$.*

A covering map is a continuous surjective map $p : E \longrightarrow B$ with the property that each point $b \in B$ has an open neighborhood U such that $p^{-1}(U)$ admits a partition into slices.

We also say that p is a covering of B, or that E is a covering space of B with covering projection p.

Exercise 12.6. Show that $p : \mathbb{R} \longrightarrow S^1$ is a covering map.

Proposition 12.22. *Let $p : E \longrightarrow B$ be a covering map, and let $b_0 \in B$. Consider*

- *a path $\gamma : [0, 1] \longrightarrow B$ starting at b_0.*
- *a lift e_0 of b_0, i.e. $e_0 \in p^{-1}(b_0)$.*

Then there exists an unique path $\tilde{\gamma} : [0, 1] \longrightarrow E$ such that

- *$\tilde{\gamma}$ starts at e_0;*
- *$\tilde{\gamma}$ is a lift of γ, i.e. $p \circ \tilde{\gamma} = \gamma$.*

Proof. First we prove the uniqueness. Assume that $\tilde{\gamma}$ and $\tilde{\tilde{\gamma}}$ are two lifts of γ starting at the same point e_0. We consider

$$S = \{s \in [0, 1] : \tilde{\gamma}|_{[0,s]} = \tilde{\tilde{\gamma}}\tilde{\gamma}\},$$

and we prove that S is both closed and open in $[0, 1]$this will imply that $S = [0, 1]$, proving the uniqueness.

We first show that S is open. Let $s_0 \in S$. Choose an open neighborhood U of $\gamma(s_0)$ such that $p^{-1}(U) = \cup V_i$ is a partition into slices. Choose i such that $\tilde{\gamma}(s_0) \in V_i$. Since $\tilde{\gamma}$ and $\tilde{\tilde{\gamma}}$ are continuous, we find a neighborhood $D = (s_0 - \epsilon, s_0 + \epsilon) \cap [0, 1]$ of s_0 in $[0, 1]$ such that

$$\tilde{\gamma}(D) \subset V_i,\ \tilde{\tilde{\gamma}}(D) \subset V_i.$$

Since $p|_{V_i} : V_i \longrightarrow U$ is a homeomorphism and $p \circ \tilde{\gamma} = p \circ \tilde{\tilde{\gamma}}$, we deduce that $\tilde{\gamma}|_D = \tilde{\tilde{\gamma}}|_D$, which implies that $D \subset S$.

We now show that S is closed. Let $s_0 \in \overline{S}$, and we show that $s_0 \in S$. For any $s < s_0$, $(s, s_0) \cap S \neq \emptyset$, which implies that $\tilde{\gamma}(s) = \tilde{\tilde{\gamma}}(s)$ (for all $s < s_0$). We still have to show that this equality also holds for $s = s_0$. Assume it does not, and let $e = \tilde{\gamma}(s_0)$, $e' = \tilde{\tilde{\gamma}}(s_0)$. Let U be as above, with $p^{-1}(U) = \cup V_i$, and choose i and j such that $e \in V_i$, $e' \in V_j$. Note that, due to the assumption $e \neq e'$ and the fact that $p|_{V_i}$ is a homeomorphism, we must have $i \neq j$, hence $V_i \cap V_j = \emptyset$. On the other hand, due to continuity, we find a neighborhood D of s_0 such that

$$\tilde{\gamma}(D) \subset V_i, \ \tilde{\tilde{\gamma}}(D) \subset V_j.$$

But this contradicts the fact that $\tilde{\gamma}(s) = \tilde{\tilde{\gamma}}(s)$ for all $s < s_0$. This concludes the proof of the fact that S is closed.

Before we prove the existence part, let us point out the following consequence.

COROLLARY 12.23. *Let $p : E \longrightarrow B$ be a covering map, and let $f : X \longrightarrow B$ be a continuous function defined on a space X. A lift of f is a continuous map $\tilde{f} : X \longrightarrow E$ such that $p\tilde{f} = f$.*

If X is path connected, then any two lifts of f which coincide in the same point of X must coincide everywhere.

PROOF. Let $x_0 \in X$ such that $\tilde{f}(x_0) = \tilde{\tilde{f}}(x_0)$. For $x \in X$ arbitrary, choose a path α from x_0 to x. Then $\tilde{f} \circ \alpha$ and $\tilde{\tilde{f}} \circ \alpha$ are two lifts of $f \circ \alpha$ which coincide at the initial point. By the uniqueness proven above, they coincide everywhere; in particular, the end points (i.e. $\tilde{f}(x)$ and $\tilde{\tilde{f}}(x)$) must coincide. □

We now return to the proof of the proposition (existence part). Let $S' \subset [0, 1]$ be the set of those s with the property that $\gamma|_{[0,s]}$ admits a lift starting at e_0. One can proceed as above and prove that S' is open and closed in $[0, 1]$. We give here a slightly different argument. Remark first that if $s \in S'$, then all $t < s$ are in S'. This implies that $S' = [0, s_0]$ or $S' = [0, s_0)$ for some $s_0 \in [0, 1]$. Also, we can find a lift $\tilde{\gamma} : [0, s_0) \longrightarrow E$ of $\gamma|_{[0,s_0)}$. To see this, we may assume $s_0 \neq 0$, and then $s_0 - \frac{1}{n} \in S'$ for n large enough. Choose $\tilde{\gamma}_n : D_n \longrightarrow E$ lifts of $\gamma|_{D_n}$ starting at e_0 defined on $D_n = [0, s_0 - \frac{1}{n}]$. The uniqueness proven above implies that $\tilde{\gamma}_n$ and $\tilde{\gamma}_m$ coincide on $D_n \cap D_m$ for all n and m, hence we can define $\tilde{\gamma}$ on $D_0 = [0, s_0)$ so that $\tilde{\gamma}|_{D_n} = \tilde{\gamma}_n$ for all n, and this will be the desired lift.

Hence $S' = [0, s_0]$ or $S' = [0, s_0)$ for some $s_0 \in [0, 1]$, and γ has a lift $\tilde{\gamma}$ on $[0, s_0)$, starting at e. Let U be an open neighborhood of $\gamma(s_0)$ such that $p^{-1}(U) = \cup V_i$ is a partition into slices. Choose ϵ such that $D = (s_0 - \epsilon, s_0 + \epsilon) \cap [0, 1]$ is sent by γ inside U. Consider also $D' = (s_0 - \epsilon, s_0)$. Since continuous maps send connected spaces to connected spaces, $\tilde{\gamma}(D')$ must be connected; but it lies in the disjoint union of the V_i's, hence we find i such that

$$\tilde{\gamma}(D') \subset V_i.$$

Since $p|_{V_i}$ is a homeomorphism, we must have

$$\tilde{\gamma}(s) = (p|_{V_i})^{-1}(\gamma(s))$$

for all $s \in D'$. But we can take this formula as the definition of $\tilde{\gamma}$ also for $s \in D$, and we obtain a lift $\tilde{\gamma}$ defined on D. Hence

$$D = (s_0 - \epsilon, s_0 + \epsilon) \cap [0, 1] \subset S'.$$

Recalling that $S' = [0, s_0]$ or $S' = [0, s_0)$, we see that the only possibility is when $S = [0, 1]$. □

REMARK 12.24. In particular, we obtain a map

$$\phi_{e_0} : P(B, b_0, b_0) \longrightarrow p^{-1}(b_0) \tag{4.1}$$

which associates to a path γ the end-point $\tilde{\gamma}(1)$ of its lift evaluated at the end point.

EXAMPLE 12.25. Apply the previous lemma to the covering map $p : \mathbb{R} \longrightarrow S^1$ and $e_0 = 0$. For any path in S^1 starting and ending at $1 \in S^1$ we find a path $\tilde{\gamma}$ in $\mathbb{R}$ starting at 0 such that

$$\gamma(t) = (cos(2\pi\tilde{\gamma}(t)), sin(2\pi\tilde{\gamma}(t))).$$

Note that $\tilde{\gamma}(1) \in p^{-1}(0) = \mathbb{Z}$.

DEFINITION 12.26. *Given a path γ in S^1 starting and ending in $1 \in S^1$, we define the degree of γ by*

$$deg(\gamma) := \tilde{\gamma}(1) \in \mathbb{Z}.$$

For instance, for each $n \in \mathbb{Z}$, the path which goes around the circle n times:

$$\gamma_n(t) = (cos(2\pi nti), sin(2\pi nti)),$$

has as lift

$$\tilde{\gamma}_n(t) = nt,$$

hence

$$\deg(\gamma_n) = n.$$

Our next aim is to show that the degree (or, more generally, the map (4.1)) only depends on path homotopy classes of paths.

LEMMA 12.27. *Let $p : E \longrightarrow B$ be a covering map, $b_0 \in B$, $e_0 \in E$ a lift of b_0. Let $\gamma, \gamma' \in P(B, b_0, b_0)$, and consider their lifts $\tilde{\gamma}$, $\tilde{\gamma}'$ which start at e_0. If γ and γ' are path homotopic, then $\tilde{\gamma}$ and $\tilde{\gamma}'$ have the same end-point and they are path homotopic.*

PROOF. Let $H : [0,1] \times [0,1]$ be a path homotopy between γ and γ'. Let $\gamma_t(s) = H(s,t)$ (so that $\gamma_0 = \gamma$, $\gamma_1 = \gamma'$). For each t we consider the lift $\tilde{\gamma}_t$ of γ_t starting at e_0 (which, by the previous proposition, exists and is unique). Put $\tilde{H}(s,t) = \tilde{\gamma}_t(s)$. Locally, H is the inverse of the restriction of p to a slice, composed with H, from which one deduces that H is continuous (fill in the details!). On the other hand, $t \mapsto \tilde{H}(1,t)$ is a lift of the constant path $H(1,t) = b_0$, hence will be constant (use the uniqueness part of previous proposition). In conclusion, $\tilde{H}$ is a path homotopy between $\tilde{\gamma}_0 = \tilde{\gamma}$ and $\tilde{\gamma}_1 = \tilde{\gamma}'$. ☐

REMARK 12.28. Continuing the previous remark, the map (4.1) will define a map

$$\phi_{e_0} : \pi(B, b_0) \longrightarrow p^{-1}(b_0).$$

In particular, for the covering map $p : \mathbb{R} \longrightarrow S^1$ we find that the notion of degree induces a map

$$\deg : \pi(S^1, 1) \longrightarrow \mathbb{Z}.$$

THEOREM 12.29. *$deg : \pi(S^1, 1) \longrightarrow \mathbb{Z}$ is an isomorphism of groups.*

PROOF. We have already seen that $\deg(\gamma_n) = n$, where $\gamma_n(t) = p(nt)$ for $t \in [0,1]$. Hence deg is surjective. To show it is injective, assume that

$$\deg(\gamma) = \deg(\gamma').$$

This means that, choosing lifts $\tilde{\gamma}$ and $\tilde{\gamma}'$ as above (starting at 0), $\tilde{\gamma}(1) = \tilde{\gamma}'$. But we know that $\mathbb{R}$ is simply connected- any two paths which start and end at the same point are path homotopic. Hence $\tilde{\gamma}$ and $\tilde{\gamma}'$ are path homotopic, which implies that γ and γ' are path homotopic (in explicit formulas, $\tilde{H}(s,t) = (1-t)\tilde{\gamma}(s) + t\tilde{\gamma}'(s)$ is a path homotopy between $\tilde{\gamma}$ and $\tilde{\gamma}'$, while $H(s,t) = p(H(s,t))$ is a path homotopy between γ and γ'.

Hence deg is bijective. To see it is a group homomorphism it suffices to check that $\deg(\gamma_n * \gamma_n) = \deg(\gamma_n) + \deg(\gamma_m)$ for all $n, m\imath\mathbb{Z}$. By the definition of concatenation, we have

$$(\gamma_n * \gamma_m)(t) = \begin{cases} p(2nt) & \text{if } 0 \leq t \leq \frac{1}{2} \\ p(m(2t-1)) & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases},$$

which has as lift

$$t \mapsto \begin{cases} 2nt & \text{if } 0 \leq t \leq \frac{1}{2} \\ m(2t-1) + n & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases},$$

hence

$$\deg(\gamma_n * \gamma_m) = m(2 \cdot 1 - 1) + n = m + n = \deg(\gamma_n) + \deg(\gamma_m).$$

□

EXERCISE 12.7. *If $p : E \longrightarrow B$ is a covering map defined on a simply connected space E, $b_0 \in B$, $e_0 \in p^{-1}(b_0)$, then*

$$\phi_{e_0} : \pi(B, b_0) \longrightarrow p^{-1}(b_0)$$

is a bijection.

COROLLARY 12.30. *There are no continuous retractions of D^2 into S^1 (i.e. continuous maps $r : D^2 \longrightarrow S^1$ such that $r(x) = x$ for all $x \in S^1$).*

PROOF. Assume such a retraction $r : D^2 \longrightarrow S^1$ exists. Let $i : S^1 \longrightarrow D^2$ be the inclusion. Consider the maps induced in the fundamental groups

$$\pi(S^1, 1) \xrightarrow{i_*} \pi(D^2, 1) \xrightarrow{r_*} \pi(S^1, 1).$$

Since $r \circ i = \mathrm{Id}_{S^1}$, we have $r_* \circ i_* = \mathrm{Id} : \pi(S^1, 1) \longrightarrow \pi(S^1, 1)$. But $\pi(D^2, 1) = 0$ (trivial) because D^2 is contractible, hence $\mathrm{Id} = r_* i_*$ will be constant. But this can only happen if $\pi(S^1, 1)$ was trivial- which is not. Hence r does not exist. □

COROLLARY 12.31. *(Brouwer fixed point theorem) Any continuous function $f : D^2 \longrightarrow D^2$ has at least one fixed point.*

PROOF. Assume that $f : D^2 \longrightarrow D^2$ has no fixed point. For each $x \in D^2$, since $f(x) \neq x$, we can talk about the line through x and $f(x)$. This line intersects S^1 in two points, and let $r(x)$ be the point with the property that x lies between $r(x)$ and $f(x)$. Then $r : D^2 \longrightarrow S^1$ will be a retraction of of D^2 into S^1- contradiction. □