CHAPTER 3

Lecture 3 (February 20)

1. Bases: continuation of the previous lecture

3.1. **Product topology** As we mentiond in the previous lecture, One of the best illustration of the usefulness of bases is the construction of product topologies. Given two sets X and Y we consider their cartesian product

$$X \times Y = \{(x, y) : x \in X, y \in Y\}.$$

Given a topology $\mathcal{T}_X$ on X and a topology $\mathcal{T}_Y$ on Y, one would like to construct a "product topology" $\mathcal{T}_X \times \mathcal{T}_Y$ on $X \times Y$. It is natural to require that $U \times V$ is open whenever U and V are open, i.e. that $\mathcal{T}_X \times \mathcal{T}_Y$ contains

$$\mathcal{P} := \{U \times V : U \in \mathcal{T}_X, Y \in \mathcal{T}_Y\}. \tag{1.1}$$

The point is that $\mathcal{P}$ is not a topology. Instead, it satisfies conditions (B1) and (B2), hence it is a basis for a topology on $X \times Y$. The induced topology is denoted by $\mathcal{T}_X \times \mathcal{T}_Y$ and is called the product topology. Hence an open in the product topology is any subset $D \subset X \times Y$ with the property that, for all $(x, y) \in D$, there exists $U \in \mathcal{T}_X$ containing x and $V \in \mathcal{T}_Y$ containing y such that $U \times V \subset D$.

EXERCISE 3.1. *Show that a sequence $\{(x_n, y_n)\}_{n\geq 1}$ converges to (x, y) with respect to the product topology if and only if $(x_n)_{n\geq 1}$ converges to x, and $(y_n)_{n\geq 1}$ converges to y.*

Since in examples $\mathcal{T}_X$ and $\mathcal{T}_Y$ might be defined with the help of bases, it is interesting to point out induced basis for the product topology.

LEMMA 3.2. *If $\mathcal{B}_1$ is a basis for $\mathcal{T}_X$, B_2 is a basis for $\mathcal{T}_Y$, then*

$$\mathcal{B} := \{B_1 \times B_2 : B_1 \in \mathcal{B}_1, B_2 \in \mathcal{B}_2\}$$

is a basis for the product topology.

PROOF. First of all, $\mathcal{B}$ is a basis (check this!), and we have to show that $\mathcal{T}(\mathcal{B}) = \mathcal{T}(\mathcal{P})$. For this one uses Lemma 2.11 twice to show that $\mathcal{T}(\mathcal{B}) \subset \mathcal{T}(\mathcal{P})$ and $\mathcal{T}(\mathcal{P}) \subset \mathcal{T}(\mathcal{B})$. □

EXAMPLE 3.3. Consider $\mathbb{R} \times \mathbb{R}$, where we endow the two factors with one of the various topologies on $\mathbb{R}$ which we have discussed in 2.13. Using the previous lemma, we can discribe bases for the resulting topologies. For instance (see Figure 1):

- for $\mathbb{R}_{\mathrm{eucl}} \times \mathbb{R}_{\mathrm{eucl}}$ the basis is given by rectangles which look like B_1 in the picture (open on all sides).
- for $\mathbb{R}_{\mathrm{l}} \times \mathbb{R}_{\mathrm{eucl}}$ the basis is given by rectangles which look like B_2 in the picture (which contain the left vertical side).
- for $\mathbb{R}_{\mathrm{l}} \times \mathbb{R}_{\mathrm{l}}$ the basis is given by rectangles which look like B_2 in the picture (which contain the left vertical side and the lower horisontal side).
- for $\mathbb{R}_{\mathrm{dis}} \times \mathbb{R}_{\mathrm{eucl}}$ the basis is given by vertical open segments in the plane.

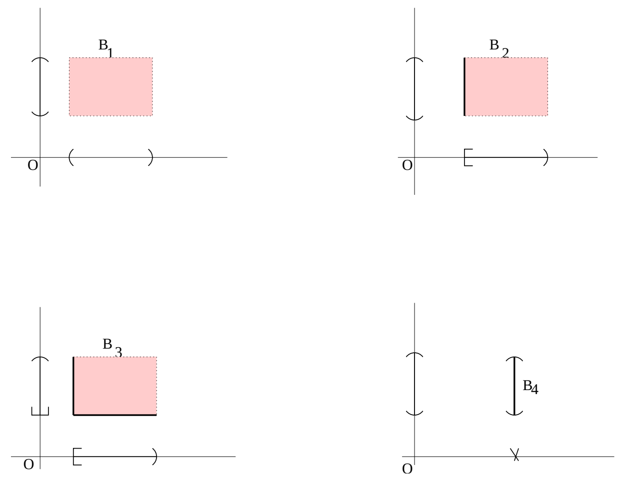

FIGURE 1.

2. Countability axioms

DEFINITION 3.4. *We say that a topological space $(X, \mathcal{T})$ satisfies the second countability axiom, or that it is second-countable if it admits a basis $\mathcal{B}$ which is countable.*

EXAMPLE 3.5. For $\mathbb{R}$ with the usual topology, the basis consisting of all open intervals is not countable. However, it does admit a countable basis: the collection of all open intervals of the form (p, q) with p, q-rational numbers. Similarly, $\mathbb{R}^n$ with the euclidean topology (induced by the euclidean metric) is second countable: a countable basis is the collection of all open balls $\{B(x, \frac{1}{n}) : x \in \mathbb{Q}^n, n \in \mathbb{N}\}$. However, not all metric spaces are second countable.

The first countability axiom refers to basis around points.

DEFINITION 3.6. *Let $(X, \mathcal{T})$ be a topological space and $x \in X$. A basis of neighborhoods of x is a collection $\mathcal{B}_x$ of open sets, all containing x, with the property that*

$$\forall\ V \in \mathcal{V}(x)\ \exists\ B \in \mathcal{B}_x :\ B \subset V.$$

We say that $(X, \mathcal{T})$ satisfies the first countability axiom, or that it is first-countable if for each point $x \in X$ there exists a countable basis of neighborhoods of x.

REMARK 3.7. If there exists a countable basis of neighborhoods of x, then one can also find a decreasing one, i.e. one of type

$$\mathcal{B}_x = \{B_1, B_2, B_3, \ldots\}, \text{with } \ldots \subset B_3 \subset B_2 \subset B_1.$$

(If $\{U_n : n \geq 1\}$ is an arbitrary countable basis of neighnorhoods, put $B_n = V_1 \cap V_2 \cap \ldots \cap V_n$).

EXAMPLE 3.8. Clearly, if $\mathcal{B}$ is a basis of $(X, \mathcal{T})$, then

$$\mathcal{B}(x) = \{B \in \mathcal{B} : x \in B\}$$

is a basis of neighborhoods of x. Hence the second countability axiom implies the first one. In particular, the euclidean space $\mathbb{R}^n$ are first countable. However, unlike the case of the second countability axiom. all metric space are first countable: if (X, d) is a metric space and $x \in X$, then

$$\mathcal{B}_x := \{B(x, \frac{1}{n}) : n \in \mathbb{N}\}$$

is a countable basis of neighborhoods of x.

Hence, regarding the metrizability problem, what we have learned so far is that if a topological space is metrizable, then it is Hausdorff and first-countable.

3.9. **Exercise** *Find examples of spaces which satisfy the first countability axiom and does not satisfy the second one. (Hint: try one of the extreme topologies).*

3.10. **What are the countability axioms good for?** The second countability axiom is one of the fundamental axioms in modern geometry: "geometric spaces" satisfy the axiom, while the axiom itself is of great importance for deriving fundamental properties of such spaces (such as metrizability, existence of partitions of unity, etc). For examples coming from functional analysis however, this axiom is quite restrictive. Hence what we should really have in mind is that this is "an axiom for geometry". We will soon return to it.

On the other hand, the role-in this course- of the first countability axiom is more a pedagogical one: it is under this axiom that the notion of sequence can be used in its full power. For instance, for a function $f : X \longrightarrow \mathbb{R}$, continuity will imply that f maps convergent sequences into convergent sequences, but the converse only holds if X satisfies the first countability axiom (and we will see lots of other examples like this). Of course, since the second countability axiom, or metrizability, they both imply the first countability axiom, this will be an axiom which is "innocent for geometry" and "not very restrictive for functional analysis".

To illustrate the relation between the the first countability axiom and the notion of sequence, here is a good exercise:

Exercise 3.2. *(1-point exercise) We say that two topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ on X have the same convergency of sequences if, for a sequence $(x_n)_{n\geq 1}$ of elements of X, and $x \in X$, one has:*

$$x_n \xrightarrow{\mathcal{T}_1} x \Longleftrightarrow x_n \xrightarrow{\mathcal{T}_2} x.$$

Show that

- *If $\mathcal{T}_1$ and $\mathcal{T}_2$ satisify the first countability axiom and have the same convergence of sequences, then $\mathcal{T}_1 = \mathcal{T}_2$.*
- *This is no longer true if one gives up the first countability axiom.*

(Hint for (ii): try $\mathcal{T}_{discr}$ and $\mathcal{T}_{cf}$. If it does not work, try to change one of them).

3. Topological aspects associated to subspaces

We now discuss several topological aspects that are associated to subspace of a topological space. So, let $(X, \mathcal{T})$ be a topological space, and let $A \subset X$.

3.11. **The induced topology:** We say that $B \subset X$ is <u>open in A</u> or that it is <u>open relative to A</u> if $B \subset A$ and there exists an open U such that $B = A \cap U$. The collection of all opens in A describes a topology on A, called <u>the restriction of $\mathcal{T}$ to A</u>, or <u>the induced topology on A</u>, denoted $\mathcal{T}_A$ (or $\mathcal{T}|_A$ if "$\mathcal{T}_A$" is too ambiguous). Hence:

$$\mathcal{T}_A := \{U \cap A : U \in \mathcal{T}\}.$$

A together with $\mathcal{T}_A$ is called a subspace of $(X, \mathcal{T})$.

Exercise 3.3. *Consider two topological spaces $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$, $A \subset X$ endowed with the induced topology $\mathcal{T}_A$, $B \subset Y$ endowed with the induced topology $\mathcal{T}_B$. Show that*

- *If $\mathcal{B}$ is a basis for the topology $\mathcal{T}_X$, then $\mathcal{B}|_A := \{U \cap A : U \in \mathcal{B}\}$ is a basis for the induced topology on A.*
- *The restriction of the product topology $\mathcal{T}_X \times \mathcal{T}_Y$ to $A \times B$ coincides with the product topology $\mathcal{T}_A \times \mathcal{T}_B$.*

3.12. **Closed sets:** We say that A is closed in X if the complement of A in X, $C_X A = X - A$, is open. Since $C_X C_X(A) = A$, we see that A is closed (open) if and only if $C_X A$ is open (respectively closed). Since for a familly $\{A_i : i \in I\}$ of subsets of X,

$$C_X(\cup_{i\in I} A_i) = \cap_{i\in I} C_X(A_i), C_X(\cap_{i\in I} A_i) = \cup_{i\in I} C_X(A_i),$$

we deduce that the axioms for a topology are equivalent to

- $\emptyset$ and X are closed.
- arbitrary intersections of closed sets is closed.
- unions of a finite number of closed sets is closed.

In conclusion, to define a topology on X it is necessary and sufficient to define the collection of all closed sets of the topology- and this can be an arbitrary collection of subsets of X that satisfies the previous three conditions.

To avoid confusion, here is one more piece of terminology: given a topological space $(X, \mathcal{T})$, $A, Y \subset X$, we say that A is closed in Y if $A \subset Y$ and A is a closed set of the topological space $(Y, \mathcal{T}|_Y)$.

EXERCISE 3.4. *Given $(X, \mathcal{T})$ and a closed subset $Y \subset X$ and $A \subset Y$, show that A is closed in Y if and only if it is closed (... in X).*

EXAMPLE 3.13.

- A closed interval $[r, s]$ $(r, s \in \mathbb{R}, r \leq s)$ is closed in $\mathbb{R}$ (with the euclidean topology) since
$$\mathbb{R} - [r, s] = (-\infty, r) \cup (s, \infty).$$
In particular, all one-element sets, $A = \{r\}$, are closed.
- The half open interval $A = [0, 1)$ is neither closed not open in $\mathbb{R}$.
- Considering $A = [0, 1)$ in the topological space $X = (-1, 1)$ endowed with the topology induced from $\mathbb{R}$, A is closed in X (but not open).
- Considering $A = [0, 1)$ in the topological space $X = \mathbb{R}$ endowed with the lower limit topology, A is both open and closed.

EXAMPLE 3.14. We could have defined the cofinal topology $\mathcal{T}_{\text{cof}}$ on X as the topology whose closed sets are the finite subsets of X. For a general topology $\mathcal{T}$ on X, finite sets may fail to be closed. However, since any finite sets is a finite union of sets containing only one element, we see that the closedness of all finite subsets is equivalent to the closedness of all sets of type $A = \{x\}$ with $x \in X$. This happens e.g. when $(X, \mathcal{T})$ is Hausdorff (prove this!).

EXERCISE 3.5. *(ex. 13, pp. 101) Show that X is Hausdorff if and only if the diagonal*

$$\Delta = \{(x, x) : x \in X\}$$

is closed in $X \times X$ (endowed with the product topology).

3.15. **Interior, closure, boundary:** Given a topological space $(X, \mathcal{T})$ and $A \subset X$, define:

- the interior of A:
$$\overset{\circ}{A} = \bigcup_{U-\text{open contained in } A} U.$$
It is sometimes denoted by $\text{Int}(A)$. Note that $\overset{\circ}{A}$ is open, is contained in A, and its is the largest set with these properties.
- the closure of A:
$$\overline{A} = \bigcap_{F-\text{closed containing } A} F.$$

It is sometimes denoted by $\mathrm{Cl}(A)$. Note that $\overline{A}$ is closed, contains A, and its is the smallest set with these properties.

- <u>the boundary of A</u>:

$$\partial(A) = \overline{A} - \overset{\circ}{A}\ .$$

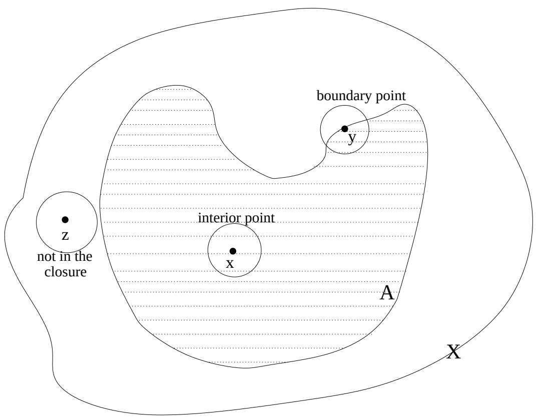

FIGURE 2.

LEMMA 3.16. *Let $(X, \mathcal{T})$ be a topological space, $x \in X$, and assume that $\mathcal{B}_x$ is a basis of neighborhoods around x (e.g. $\mathcal{B}_x = \mathcal{T}(x)$). Then:*

(i) *$x \in \overset{\circ}{A}$ if and only if there exists $U \in \mathcal{B}_x$ such that $U \subset A$.*
(ii) *$x \in \overline{A}$ if and only if, for all $U \in \mathcal{B}_x$, $U \cap A \neq \emptyset$.*
(iii) *If $(X, \mathcal{T})$ satisfies the first axiom of countability, then $x \in \overline{A}$ if and only if there exists a sequence $(a_n)_{n \geq 1}$ of elements of A such that $a_n \to a$.*

See Figure 2.

EXAMPLE 3.17. Take the "open disk" in the plane

$$A = \overset{\circ}{D}{}^2 \ (= \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}).$$

Then the interior of A is A itself (it is open!), the closure is the "closed disk"

$$A = D^2 (= \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}),$$

while the boundary is the unit circle

$$\partial(A) = S^1 (= \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}).$$

See Figure 3. The same is true for balls in any metric space (check this!).

EXAMPLE 3.18. Take $A = [0, 1) \cup \{2\} \cup [3, 4)$ in $X = \mathbb{R}$. Using the lema (and the basis given by open intervals) we find

$$\overset{\circ}{A} = (0, 1) \cup (3, 4),\ \overline{A} = [0, 1] \cup \{2\} \cup [3, 4],\ \partial(A) = \{0, 1, 2, 3, 4\}.$$

However, considering A inside $X' = [0, 4)$ (with the topology induced from $\mathbb{R}$),

$$\overset{\circ}{A} = [0, 1) \cup (3, 4),\ \overline{A} = [0, 1] \cup \{2\} \cup [3, 4), \partial(A) = \{1, 2, 3\}.$$

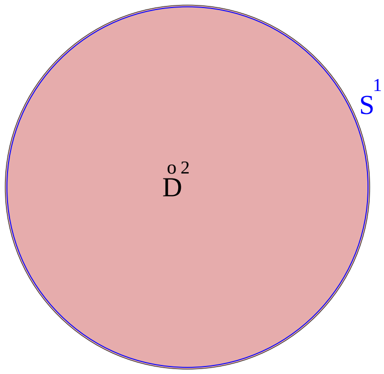

FIGURE 3.

PROOF. (of the lemma) You should first convince yourself that (i) is easy; we prove here (ii) and (iii). To prove the equivalence in (ii), is sufficient to prove the equivalence of the negations, i.e.

$$[x \notin \mathrm{Cl}(A)] \Longleftrightarrow [\exists\, U \in \mathcal{B}_x : U \cap A = \emptyset].$$

From the definition of $\overline{A}$, the left hand side is equivalent to:

$$\exists\, F - \text{closed} : A \subset F, x \notin F.$$

Since closed sets are of those of type $F = X - U$ with U-open, this is equivalent to

$$\exists\, U - \text{open} : A \cap U = \emptyset, x \in U,$$

i.e.: there exists $U \in \mathcal{T}(x)$ such that $U \cap A = \emptyset$. On the other hand, any $U \in \mathcal{T}(x)$ contains at least one $B \in \mathcal{B}_x$, and the condition $U \cap A = \emptyset$ will not be destroyed if we replace U by B. This concludes the proof of (ii). For (iii), first assume that $x = lim a_n$ for some sequence of elements of A. Then, for any $U \in \mathcal{T}(x)$, we find n_U such that $a_n \in U$ for all $n \geq n_U$, which shows that $U \cap A \neq \emptyset$, By (ii), $x \in \overline{A}$. For the converse, assume that $\{V_n : n \geq 1\}$ is a decreasing countable basis of open neighborhoods of x (see Remark 3.7). For each n we choose $a_n \in V_n \cap A$. We have $a_n \to x$: for $U \in \mathcal{T}(x)$ arbitrary, there exists n_0 such that $V_{n_0} \subset U$ and then, for all $n \geq n_0$,

$$a_n \in V_n \subset V_{n_0} \subset U.$$

□

EXERCISE 3.6. *(ex. 6, pp. 101) Let A, B and B_i denote subsets of a topological space X, where i runs in a set of indices I. Prove that*

(a) *If $A \subset B$ then $\overline{A} \subset \overline{B}$.*
(b) $\overline{A \cup B} = \overline{A} \cup \overline{B}$.
(c) $\cup_i \overline{A_i} \subset \overline{\cup_i A_i}$*; give an example where the equality fails. When does the equality hold?*

For the case of metric spaces, let us point out the following corollary. To state it, recall that given a metric space (X, d), $A \subset X$, and $x \in X$, one defines the distance between x and A as

$$d(x, A) = \inf\{d(x, a) : a \in A\}.$$

COROLLARY 3.19. *If A is a subset of a metric space (X, d), $x \in X$, then the following are equivalent:*

(1) $x \in \overline{A}$.
(2) *there exists a sequence (a_n) of elements of A such that $a_n \to x$.*
(3) $d(x, A) = 0$.

PROOF. The equivalence of (1) and (2) follows directly from (iii) of the lemma. Next, the condition (3) means that, for all $\epsilon > 0$, there exists $a \in A$ such that $d(x, a) < \epsilon$. In other words, $A \cap B(x, \epsilon) \neq \emptyset$ for all $\epsilon > 0$. Using (iii) of the lemma (with $\mathcal{B}_x$ beeing the collection of all balls centered at x), we find that (3) is equivalent to (1). $\square$