CHAPTER 4

Lecture 4

- finnish what is left fromthe previous lecture

1. Continuous functions

DEFINITION 4.1. *Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be two topological spaces, $x \in X$, and let $f : X \longrightarrow Y$. We say that*

- *f is continuous if for every $V \in \mathcal{T}_Y$, $f^{-1}(V) \in \mathcal{T}_X$.*
- *f is continuous at x if for every $V \in \mathcal{T}_Y(f(x))$ there exists $U \in \mathcal{T}_X(x)$ such that $f(U) \subset V$.*

When we want to emphasize the topologies (e.g. in order to avoid confusion) we also say that $f : (X, \mathcal{T}_X) \longrightarrow (Y, \mathcal{T}_Y)$ is continuous (or continuous at x).

PROPOSITION 4.2. *For $f : (X, \mathcal{T}_X) \longrightarrow (Y, \mathcal{T}_Y)$, the following are equivalent:*

(c1) *f is continuous.*
(c2) *$f^{-1}(B)$ is closed for all $B \subset Y$ closed.*
(c3) *f is continuous at each point $x \in X$.*

Given $x \in X$, a basis $\mathcal{B}_x$ of neighborhoods of x in X, and a basis $\mathcal{B}_{f(x)}$ of neighborhoods of $f(x)$, the following are equivalent

$(\text{c1})_x$ *f is continuous at x.*
$(\text{c2})_x$ *for every $V \in \mathcal{B}'_{f(x)}$, there exists $U \in \mathcal{B}_x$ such that $f(U) \subset V$.*

If $(X, \mathcal{T}_X)$ satisfies the first axiom of countability, then the last two conditions are also equivalent to:

$(\text{c3})_x$ *If $\lim x_n = x$ in $(X, \mathcal{T}_X)$, then $\lim f(x_n) = f(x)$ in $(Y, \mathcal{T}_Y)$.*

Before we go to the proof, we make a few remarks.

EXAMPLE 4.3.

(1) For any topological space $(X, \mathcal{T})$, the identity map $\text{Id}_X : (X, \mathcal{T}) \longrightarrow (X, \mathcal{T})$ is continuous. The composition of two continuous functions is continuous. However, the inverse of a continuous (bijective) function may fail to be continuous (see below).
(2) If $\mathcal{T}_1$ and $\mathcal{T}_2$ are two topologies on X, then the identity map $\text{Id}_X : (X, \mathcal{T}_1) \longrightarrow (X, \mathcal{T}_2)$ is continuous if and only if $\mathcal{T}_2$ is smaller then $\mathcal{T}_1$.
(3) On one extreme, any function $f : (X, \mathcal{T}_{\text{dis}}) \longrightarrow (Y, \mathcal{T}_Y)$ is continuous. On the other extreme, any function $f : (X, \mathcal{T}_X) \longrightarrow (Y, \mathcal{T}_{\text{triv}})$ is continuous.
(4) Given $(Y, \mathcal{T})$ and $A \subset Y$, the inclusion $i : (A, \mathcal{T}|_A) \longrightarrow (Y, \mathcal{T})$ is continuous.
(5) Given $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$, the first projection $\text{pr}_X : (X \times Y, \mathcal{T}_X \times \mathcal{T}_Y) \longrightarrow (X, \mathcal{T}_X)$ and the second projection $\text{pr}_Y : (X \times Y, \mathcal{T}_X \times \mathcal{T}_Y) \longrightarrow (Y, \mathcal{T}_Y)$ are both continuous.

EXERCISE 4.1. *Use the previous proposition to show that, given two metric spaces (X, d_X) and (Y, d_Y), then a function $f : X \longrightarrow Y$ is continuous (with repsect to the topologies induced by the two metrics) if and only if: for all $x \in X$, and all $\epsilon > 0$, there exists $\delta > 0$ such that*

$$d_Y(f(x'), f(x)) < \epsilon, \text{ for all } x' \text{ with } d_X(x', x) < \delta.$$

(hint: use basis of neighborhoods of x given by open balls centered at x).

EXERCISE 4.2. *Deduce from the properties above that, if $f : (X, \mathcal{T}_X) \longrightarrow (Y, \mathcal{T}_Y)$ is continuous and $A \subset X$ is endowed with the induced topology $\mathcal{T}_A$, then the restriction $f|_A : (A, \mathcal{T}_A) \longrightarrow (Y, \mathcal{T}_Y)$ is continuous.*

EXAMPLE 4.4 (a general principle for constructing topologies). Many topological constructions are what we call "natural", or "canonical" (in any case, not arbitrary). Very often, when one looks for a topology, one wants that certain (given maps) are continuous. This happens e.g. with the induced topologies and product topologies.

A general setting is as follows: assume that the set X is given, and that one has a collection of maps $f_i : X \longrightarrow X_i$ where each X_i is endowed with a topology $\mathcal{T}_i$ (i runs in a set I of indices). An almost interesting question is to find topologies $\mathcal{T}_X$ on X such that $f_i : (X, \mathcal{T}_X) \longrightarrow (X_i, \mathcal{T}_i)$ are all continuous. "Almost" because the question has an obvious answer: $\mathcal{T}_X = \mathcal{T}_{\text{dis}}$- which does not reflect the functions f_i. One should also remark that the smaller $\mathcal{T}_X$ becomes, the smaller are the chances that f_i are continuous. With these, the really interesting question is to find the smallest topology on X for which all the functions f_i are continuous.

An example: given $(Y, \mathcal{T}_Y)$ and $A \subset Y$, the induced topology $\mathcal{T}_A$ is the smallest with the property that the inclusion $i : (A, \mathcal{T}_A) \longrightarrow (Y, \mathcal{T}_Y)$ is continuous.

Another example: given $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$, the product topology is the smalles topology on $X \times Y$ with the property that the projections pr_X and pr_Y are continuous.

PROOF. (of Proposition 4.2) (c1) $\Longleftrightarrow$ (c2): this follows from the fact that a set is closed if and only if its complement is open, and the fact that $C_X(f^{-1}(B)) = f^{-1}(C_Y(B))$.

(c1) $\Longleftrightarrow$ (c3) in the definition of continuity, just rewrite the condition $f^{-1}(V)$-open using the criteria for openess from the second lecture ((T3') there).

$(\text{c1})_x \Longleftrightarrow (\text{c2})_x$: This follows from the definition of basis.

$(\text{c1})_x \Longrightarrow (\text{c3})_x$: Assume that $\lim x_n = x$. Let $V \in \mathcal{T}_Y(f(x))$. We have to show that there exists n_0 such that $f(x_n) \in V$ for all $n \geq n_0$. Choose $U \in \mathcal{T}_X(x)$ such that $f(U) \subset V$. Since $x_n \to x$, we find an n_0 such that $x_n \in U$ for all $n \geq n_0$. Then $f(x_n) \in f(U) \subset V$ for all $n \geq n_0$.

$(\text{c3})_x \Longrightarrow (\text{c1})_x$: It is here that we need the first axiom of countability. Assume that $(\text{c1})_x$ does not hold (and we will produce a counterexample to $(\text{c3})_x$). This means that there exists $V \in \mathcal{T}_Y(f(x))$ such that, for each $U \in \mathcal{T}_X(x)$, $f(U)$ is not contained in V. Let $\{U_n : n \geq 1\}$ be a countable basis of neighborhoods around x which we assume to be decreasing (cf. Remark 3.7), and choose $U = U_n$ in the previous assertion; for each n we will find an element $x_n \in U_n$ such that $f(x_n) \notin V$. Then $\lim x_n = x$ while $\lim f(x_n) \neq f(x)$. $\square$

COROLLARY 4.5. *(glueing continuous functions) Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be two topological spaces and let $\{A_i : i \in I\}$ be a collection of subsets of X which cover X (i.e. $X = \cup_i A_i$). Assume that*

$$f_i : A_i \longrightarrow Y$$

are continuous functions (where each A_i is endowed with the topology induced from X), and assume that

$$f_i|_{A_i \cap A_j} = f_j|_{A_i \cap A_j}$$

for all i and j. Consider the function $f : X \longrightarrow \mathbb{R}$ defined by $f|_{A_i} = f_i$. Then f is continuous in each of the following cases:

1. *If all A_i are open.*
2. *If all A_i are closed and I is finite.*

PROOF. For an arbitrary subset F of Y, $f_i^{-1}(F) = f^{-1}(F) \cap A_i$, hence

$$f^{-1}(F) = \cup_{i \in I} f_i^{-1}(F).$$

For the first part we prove that $f^{-1}(V)$ is open in X for all opens $V \subset Y$. Since f_i is continuous, $f_i^{-1}(V)$ is open in A_i; since A_i is open, it follows that $f_i^{-1}(V)$ is open in X. Applying the previous equality to $F = V$, we deduce that $f^{-1}(F)$ is open.

The second part is proven similarly, by showing that $f^{-1}(B)$ is closed in X for all closed sets $B \subset Y$. □

PROPOSITION 4.6. *(closed sets as level sets) If (X, d) is a metric space and $A \subset X$, then the function*

$$d_A : X \longrightarrow \mathbb{R}, \quad d_A(x) = d(x, A)$$

is continuous.

In particular, any closed set of a metric space X can be realized as the level set $f^{-1}(0)$ of some continuous function $f : X \longrightarrow \mathbb{R}$.

PROOF. In general,

$$d(x, A) \leq d(x, y) + d(y, A)$$

for all x and y. Indeed, from the definition of $d(y, A)$, we find a sequence $(a_n)_{n \geq 1}$ of points of A such that $d(y, a_n) \to d(y, A)$. Then

$$d(x, A) \leq d(x, a_n) \leq d(x, y) + d(y, a_n)$$

for all n, hence making $n \to \infty$, we find the desired inequality. Next, remark that this inequality implies that d_A satisfies

$$|d_A(x) - d_A(y)| \leq d(x, y)$$

for all x and y. The (ϵ, δ)- description of continuity is now easy to check. □

4.7. **Homeomorphisms, embeddings:** Continuous maps do not satisfy properties that one might expect at first, such as preservation of open/closed sets. In particular, if $f : (X, \mathcal{T}_X) \longrightarrow (Y, \mathcal{T}_Y)$ is continuous and bijective, it may happen that f^{-1} is not continuous. For f^{-1} to be continuous, one would need that for each $U \subset X$ open, $f(U) \subset Y$ is open (this is the definition of continuity together with the fact that the preimage of U by f^{-1} is $f(U)$). In other words, f would need to be open.

EXAMPLE 4.8. Consider $X = [0, 2\pi)$ with the topology induced from $\mathbb{R}$, and $Y = S^1$- the unit circle in the plane with the topology induced from the euclidean topology of $\mathbb{R}^2$. The function

$$f : [0, 2\pi) \longrightarrow S^1, f(t) = (cos(t), sin(t)),$$

is continuous and bijective. However, it is not open: $[0, \pi)$ is open in X, while $f([0, 2\pi))$ is a half circle closed at one end and open at the other- hence not open (see the picture).

EXERCISE 4.3. *Show that $f([0, 2\pi))$ is not open in S^1.*

DEFINITION 4.9. *We say that a map $f : (X, \mathcal{T}_X) \longrightarrow (Y, \mathcal{T}_Y)$ is a* <u>*homeomorphism*</u> *if f is bijective and both f and f^{-1} are continuous.*

We say that two topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are homeomorphic if there exists an homeomorphism between them.

An <u>*embedding*</u> *of a topological space $(X, \mathcal{T}_X)$ into a topological space $(Y, \mathcal{T}_Y)$ is any injective continuous map $f : X \longrightarrow Y$ with the property that it defines a homeomorphism between X (with the topology $\mathcal{T}_X$) and $f(X)$ (with the topology induced from Y).*

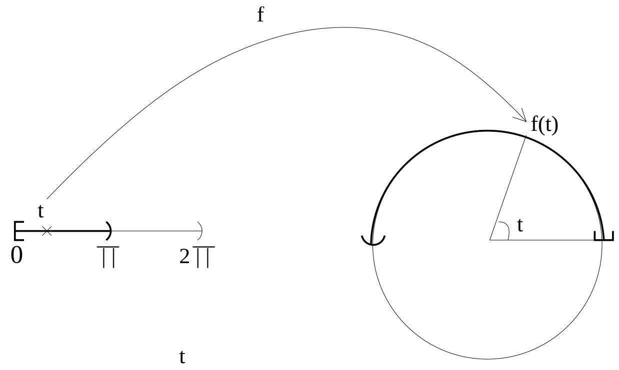

FIGURE 1.

EXAMPLE 4.10. There are injective continuous maps that are not embeddings. For instance, $f : [0, 2\pi) \longrightarrow \mathbb{R}^2$, $f(\alpha) = (cos(\alpha), sin(\alpha))$ (see Example 4.8).

EXERCISE 4.4. *Show that any two open intervals in $\mathbb{R}$ are homeomorphic (by open interval we mean an interval of type (r, s), with $-\infty \leq r \leq s \leq \infty$).*

4.11. **Important remark:** Note that two homeomorphic spaces have exactely the same topological properties: a homeomorhism $f : (X, \mathcal{T}_X) \longrightarrow (Y, \mathcal{T}_Y)$ can be used to move from X to Y and backwards, without loosing any topological information. For this reason, in topology, homeomorphic spaces are not viewed as beeing different from each other.

Actually, many notions are motivated precisely by this question:

<u>*how do we decide if two spaces are homeomorphic or not ?*</u>

One possible way to attack the problem is to understand the possible topological properties of a space (a topological property is a property that can be characterized using open sets only). Given a topological property P, if X satisfies P and Y does not, we can conclude that X and Y are not homeomorphic. All the topological properties we discuss in this course (the countability axioms, Hausdorfness, connectedness, compactness, etc) could be motivated by this problem. For instance, try to prove now that $(0, 2)$ and $(0, 1) \cup (1, 2)$ are not homeomorphich (if you mannage, you have probably discovered the notion of connectedness). Then try to prove that $\mathbb{R}$ and $\mathbb{R}^2$ are not homeomorphic. Next, try to prove that $\mathbb{R}^2$ and $\mathbb{R}^3$ are not homeomorphic (this is much more difficult). Or try something a bit different: $(0, 1)$ and $[0, 1]$ are not homeomorphic. Later on we will see how to prove such results.

EXERCISE 4.5. *Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be two homeomorphic topological space. Show that*

(1) *If X is Hausdorff, then so is Y.*
(2) *If X satisfies the first countability axiom, then so does Y.*
(3) *If X satisfies the second countability axiom, then so does Y.*
(4) *If X is metrizable, then so is Y.*

EXERCISE 4.6. *Show that if a topological space $(X, \mathcal{T})$ can be embedded into a metrizable space, then $(X, \mathcal{T})$ is metrizable.*