CHAPTER 6

Lecture 6 (March 13)

- We have recalled the notion of normal spaces, Urysohn's lemma and Urysohn's metrization theorem, and then we proved the metrization theorem (the proof is written down in the previouis paragraph).
- Then we started looking at compact spaces. See below.

1. Compact spaces

Probably many of you have seen the notion of compact space in the context of subsets of $\mathbb{R}^n$, as sets which are closed and bounded. Although not obviously at all, this is a topological property (it can be defined using open sets only).

DEFINITION 6.1. *Given a topological space $(X, \mathcal{T})$ and $A \subset X$, an open cover of A in X is a family $\mathcal{U} = \{U_i : i \in I\}$ (I-some index set) consisting of open sets $U_i \subset X$ such that*

$$A \subset \cup_{i \in I} U_i.$$

A subcover is any cover $\mathcal{V}$ with the property that $\mathcal{V} \subset \mathcal{U}$.

When $A = X$ (so that $X = \cup_i U_i$), we talk about an open cover of X.

DEFINITION 6.2. *We say that a topological space $(X, \mathcal{T})$ is compact if from any open cover $\mathcal{U}$ of X one can extract a finite open subcover.*

We say that a subset A of X is compact in $(X, \mathcal{T})$ if from any open cover of A in X one can extract a finite open subcover.

EXERCISE 6.1. *Show that the set*

$$\{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$$

is compact in $\mathbb{R}$.

REMARK 6.3. Although this is not completely clear from the defintion, the notion of compactness of a subspace $A \subset X$ is not a relative notion, i.e. it does not really depend on X, but only on the induced topology $\mathcal{T}_A = \mathcal{T}|_A$ on A. More precisely, what happens is that A is compact in $(X, \mathcal{T})$ if and only the topological space $(A, \mathcal{T}_A)$ is compact.

PROOF. Recall that $\mathcal{T}_A$ consists of sets of type $A \cap U$ with $U \in \mathcal{T}$. Also, for $U_i \in \mathcal{T}$ we have the equivalence

$$\bigcup_i (A \cap U_i) = A \Longleftrightarrow A \subset \bigcup_i U_i,$$

from which the assertion follows. □

EXAMPLE 6.4.

(1) $(X, \mathcal{T}_{\mathrm{discr}})$ is compact if and only if X is finite (use the cover of X by the opne-point opens).

(2) $\mathbb{R}$ is not compact. Indeed,

$$\mathbb{R} = \bigcup_{n \in \mathbb{Z}} (-n, n)$$

is an open cover from which we cannot extract a finite open subcover. Generalizing a bit this argument, we see that if a set $A \subset \mathbb{R}$ is compact, then A must be bounded (write $A \subset \cup_n (-n, n)$).

(3) $[0, 1)$ is not compact. Indeed,

$$[0, 1) \subset \bigcup_n (-\infty, 1 - \frac{1}{n})$$

defines an open cover of $[0, 1)$ in $\mathbb{R}$, from which we cannot extract a finite open subcover. Again, this argument can be extended to show that if a set $A \subset \mathbb{R}$ is compact, then sup$A$ belongs to A (and similarly inf$A$). Indeed, if sup$A$ was not in A, for each $b \in A$ we would find a biger element in A, i.e. there would exist $a \in A$ such that $b \in (-\infty, a)$. Hence

$$A \subset \bigcup_{a \in A} (-\infty, a),$$

and this is an open cover of A in $\mathbb{R}$ which does not admit a finite open subcover. Indeed, if

$$(-\infty, a_1), \ldots, (-\infty, a_n)$$

was a finite subcover, taking $a = max\{a_1, \ldots, a_n\}$ we would have

$$A \subset \bigcup_{i=1}^{n} (-\infty, a_i) = (-\infty, a)$$

which is impossible because $a \in A$.

(4) As we shall see later, $[0, 1]$ is compact. Actually, we will see that $A \subset \mathbb{R}$ is compact of and only if it is bounded and closed.

Exercise 6.2. *Given a set X, when is $(X, \mathcal{T}_{cf})$ compact?*

Exercise 6.3. *Look at the examples (2)-(4) above.*

(1) *We have seen that any compact $A \subset \mathbb{R}$ must be bounded. Prove the same for subsets of $\mathbb{R}^n$.*
(2) *We have seen that if $A \subset \mathbb{R}$ is compact, then supA and infA must belong to A. Use the same type of argument to prove that if $A \subset \mathbb{R}$ is compact, then A is closed.*
(3) *We have seen that if $A \subset \mathbb{R}$ is compact then supA and infA must belong to A. How would you generalize this to subsets A of $\mathbb{R}^n$?*

Remark 6.5. Related to Remark 6.3, here is another one: if $(X, \mathcal{T})$ is a compact space, then any closed subset $A \subset X$ is itself compact. Indeed, if $\mathcal{U}$ is an open cover of A in X, then adding $X - A$ to $\mathcal{U}$ (which is open since A is closed), we get an open cover of X. Extracting a finite subcover (which may or may not contain $X - A$), denoting by $U_1, \ldots, U_n$ the elements of this finite subcover which are different from $X - A$, this will define a finite subcover of the original cover of A in X.

Exercise 6.4. *Prove that if two spaces are homeomorphic and one of them is compact, then so is the other one. Then show that*

1. *assuming that $[0, 1]$ is compact, show that any closed and bounded subset A of $\mathbb{R}$ is compact.*
2. *assuming also that the product of two compact spaces is compact, prove that any closed and bounded subset A of $\mathbb{R}^n$ is compact.*

1.1. Topological properties of compact spaces: We now turn to some basic properties of compact spaces. First of all, the property pointed out in (2) of Exercise 6.3 holds in much greater generality.

THEOREM 6.6. *In a Haudorff space $(X, \mathcal{T})$, any compact set is closed.*

Before we turn to the proof, here is the relation with the previous lecture.

THEOREM 6.7. *Any compact Hausdorff space is normal.*

These two theorems will follow quite easily from the following

PROPOSITION 6.8. *In a Haudorff space $(X, \mathcal{T})$, any two disjoint compact sets A and B can be separated topologically.*

PROOF. (of Theorem 6.6 and Theorem 6.7, assuming Proposition 6.8) For Theorem 6.6, assume that $A \subset X$ is closed, and we prove that $\overline{A} \subset A$: if $x \in \overline{A}$, then, for any neighborhood U of x, $U \cap A \neq \emptyset$, and this shows that x cannot be separated from A; using the proposition, we conclude that $x \in A$. Theorem 6.7 now: if A, B are closed disjoint in the compact space X, we know that A and B must be compact (see Remark 6.5), hence using the proposition, A and B can be separated. $\square$

PROOF. (of Proposition 6.8): NEXT TIME. $\square$

EXERCISE 6.5. *Find a topological space $(X, \mathcal{T})$ which is compact but is not Hausdorff. In the example that you found, exhibit a compact set $A \subset X$ such that A is not closed.*