CHAPTER 7

Lecture 7 (March 13)

1. Compact spaces- continuation

1.1. Continuation: Compactness- first topological properties: In the previous lecture we have seen the first properties of compact spaces:

- closed inside compact is compact.
- compact inside Hausdorff is closed.
- compact and Hausdorff implies normal.

The proofs (of the last two properties) depended on the following result whose proof we still have to give:

PROPOSITION 7.1. *In a Haudorff space $(X, \mathcal{T})$, any two disjoint compact sets A and B can be separated topologically.*

PROOF. We introduce the following notation: given $Y, Z \subset X$ we write $Y|Z$ if Y and Z can be separated, i.e. if there exist opens U and V (in X) such that $Y \subset U$, $Z \subset V$ and $U \cap V = \emptyset$. We claim that, if Z is compact and $Y|\{z\}$ for all $z \in Z$, then $Y|Z$.

PROOF. (of the claim) We know that for each $z \in Z$ we find opens U_z and V_z such that $Y \subset U_z$, $z \in V_z$ and $U_z \cap V_z = \emptyset$. Note that $\{V_z : z \in Z\}$ is an open cover of Z in X: indeed, any $z \in Z$ belongs at least to one of the opens in the cover (namely V_z). By compactness of Z, we find a finite number of points $z_1, \ldots, z_n \in Z$ such that

$$Z \subset V_{z_1} \cup \ldots \cup V_{z_n}.$$

Denoting by V the last union, and considering

$$U = U_{z_1} \cup \ldots \cup U_{z_n},$$

V is an open containing Z, U is an open containing Y. Moreover, $U \cap V = emptyset$: indeed, if x is in the intersection, since $x \in V$ we find k such that $x \in V_{z_k}$; but $x \in U$ hence $x \in U_z$, which is impossible since $U_{z_k} \cap V_{z_k} = \emptyset$. In conclusion, U and V show that Y and Z can be separated. □

Back to the proof of the proposition, let $a \in A$ arbitrary. Now, since X is compact, we have $\{a\}|\{b\}$ for all $b \in B$. Since B is compact, the claim above implies that $\{a\}|B$, or, equivalently, $B|\{a\}$. This holds for all $a \in A$, hence using again the claim (and the fact that A is compact) we deduce that $A|B$. □

1.2. Compactness of products: Next, we are interested in the compactness of the product of two compact spaces. To prove the main result, we will need the following lemma which is interesting on its own.

LEMMA 7.2. *(the tube lemma) Let X and Y be two topological spaces, $x_0 \in X$, and $U \subset X \times Y$ open (in the product topology) such that*

$$\{x_0\} \times Y \subset U.$$

If Y is compact, then there exists $W \subset Y$ open containing x_0 such that

$$W \times Y \subset U.$$

PROOF. Recall that the product topology is defined as the topology induced by the basis $\{W \times V : W \subset X, V \subset Y \text{opens}\}$. Hence, for each $y \in Y$, since $(x_0, y) \in U$, there exist opens $W_y \subset X$, $V_y \subset Y$ such that

$$W_y \times V_y \subset U.$$

Now, $\{V_y : y \in Y\}$ will be an open cover of Y, hence we find $y_1, \ldots, y_n \in Y$ such that

$$Y = V_{y_1} \cup \ldots \cup V_{y_n}.$$

Choose

$$W = W_{y_1} \cap \ldots \cap W_{y_n}.$$

This is a finite intersection of open neighborhoods of x_0 hence it is itself an open neighborhood of x_0. To see that $W \times Y \subset U$, let $(x, y) \in W \times Y$. Since the V_{y_i}'s cover Y, we find i such that $y \in V_{y_i}$. But $x \in W$ implies $x \in W_{y_i}$, hence $(x, y) \in W_{y_i} \times V_{y_i}$. But $W_z \times V_z \subset U$ for all $z \in Y$, hence $(x, y) \in U$. □

THEOREM 7.3. *If X and Y are compact spaces, then $X \times Y$ is compact.*

PROOF. Let $\mathcal{U}$ be an open cover of $X \times Y$. For each $x \in X$,

$$\{x\} \times Y \subset X \times Y$$

is compact (why?), hence we find a $\mathcal{U}_x \subset \mathcal{U}$ finite such that

$$\{x\} \times Y \subset \bigcup_{U \in \mathcal{U}_x} U. \tag{1.1}$$

Using the previous lemma, we find W_x open containing x such that

$$W_x \times Y \subset \bigcup_{U \in \mathcal{U}_x} U. \tag{1.2}$$

Now, $\{W_x : x \in X\}$ is an open cover of X, hence we find a finite subcover

$$X = W_{x_1} \cup \ldots \cup W_{x_p}.$$

Then

$$\mathcal{V} = \mathcal{U}_{x_1} \cup \ldots \cup \mathcal{U}_{x_p}$$

is finite union of finite collections, hence finite. Moreover, $\mathcal{V}$ still covers $X \times Y$: given (x, y) arbitrary, using (1.2), we find i such that $x \in W_{x_i}$. Hence $(x, y) \in W_{x_i} \times Y$, and using (1.1) we find $U \in \mathcal{U}_{x_i}$ such that $(x, y) \in U$. Hence we found $U \in \mathcal{V}$ such that $(x, y) \in U$, □

1.3. Compactness and continuous functions: As we have already pointed out, if $f : X \longrightarrow Y$ is a continuous map, it may fail to send open subsets U of X to open subsets $f(U)$ of Y, and the same remark holds for closed sets as well. Compact spaces however are much better behaved from this point of view.

THEOREM 7.4. *If $f : X \longrightarrow Y$ is a continuous function and $A \subset X$ is compact, then $f(A) \subset Y$ is compact.*

PROOF. If $\mathcal{U}$ is an open cover of $f(A)$ in Y, then $f^{-1}(\mathcal{U}) := \{f^{-1}(U) : U \in \mathcal{U}\}$ is an open cover of A in X, hence it has a finite subcover $\{f^{-1}(U_i) : 1 \leq i \leq n\}$ with $U_i \in \mathcal{U}$. But then $\{U_i\}$ will be a finite subcover of $\mathcal{U}$. □

We have also seen that a bijective continuous map $f : X \longrightarrow Y$ between two topological spaces may fail to be a homeomorphism- f^{-1} may fail to be continuous. The role of compactness for this kind of "problem" is the following.

THEOREM 7.5. *If $f : X \longrightarrow Y$ is continuous and bijective, and if X is compact and Y is Hausdorff, then f is a homeomorphism.*

PROOF. We have to show that $g = f^{-1}$ is continuous. For this we show that if $A \subset X$ is closed then $g^{-1}(A) \subset Y$ is closed. Note that $g^{-1}(A) = f(A)$. Let A be closed. Since X is compact, A is compact (see Remark ?). By the previous theorem, $f(A)$ must be compact. Since Y is Hausdorff, ? implies that $f(A)$ is closed. $\square$

1.4. Compactness and sequences: When one deals with sequences, one often sees statements of type "we now consider a subsequence with this property". Compactness is related to the existence of convergent subsequences. In general, given a topological space $(X, \mathcal{T})$, one says that X is sequentially compact if any sequence $(x_n)_{n\geq 1}$ of elements of X has a convergent subsequence. Recall that a subsequence of $(x_n)_{n\geq 1}$ is a sequence $(y_k)_{k\geq 1}$ of type

$$y_k = x_{n_k}, \text{ with } n_1 < n_2 < n_3 < \dots .$$

However, we have already mentioned (and seen in various other cases) that topological properties involving sequences ususally require the axiom of first countability.

THEOREM 7.6. *Any first countable compact space is sequentially compact.*

PROOF. Let X be first countable and compact, and assume that $(x_n)_{n\geq 1}$ is an arbitrary sequence in X. For each integer $n \geq 1$ we put

$$U_n = X - \overline{\{x_n, x_{n+1}, \dots\}}.$$

Note that these define an increasing sequence of open subsets of X:

$$U_1 \subset U_2 \subset U_3 \subset \dots .$$

We now claim that $\cup_n U_n \neq X$. If this is not the case, $\{U_n : n \geq 1\}$ is an open cover of X hence we find a finite set F such that $\{U_i : i \in F\}$ covers X. Since our cover is increasing, we find that $U_p = X$ where $p = \max F$, and this is clearly impossible. In conclusion, $\cup_n U_n \neq X$. Hence there exists $x \in X$ such that, for all $n \geq 1$, $x \notin U_n$. Choose a countable basis of neighborhoods of x

$$V_1, V_2, V_3, \dots .$$

Since $x \notin U_1$, we have $V \cap \{x_1, x_2, \dots\} \neq \emptyset$ for all neighbrohoods V of x. Choosing $V = V_1$, we find n_1 such that

$$x_{n_1} \in V_1.$$

Next, we use the fact that $x \notin U_n$ for $n = n_1 + 1$. This means that $V \cap \{x_n, x_{n+1}, \dots\} \neq \emptyset$ for all neighbrohoods V of x. Choosing $V = V_2$, we find $n_2 > n_1$ such that

$$x_{n_2} \in V_2.$$

We continue this process inductively (e.g. the next step uses $x \notin U_n$ for $n = n_1 + n_2 + 1$) nad we find $n_1 < n_2 < \dots$ such that

$$x_{n_k} \in V_k$$

for all $k \geq 1$. Then $(x_{n_k})_{k\geq 1}$ is a subsequence of $(x_n)_{n\geq 1}$ converging to x. $\square$

1.5. Compactness and metric spaces: As we have already mentioned, a subspace $A \subset \mathbb{R}^n$ is compact if and only if it is closed and bounded. This is a result which re-writes topological property (compactness) in terms of another topological property (closedness) and a metric property (boundedness). We would like not only to prove this result, but also generalize it to general metric spaces (X, d). Let's look closer at the two properties we mentioned (boundeness and closedness).

Boundedness: We first have a closer look at the boundedness condition. Given the metric (X, d), the notion of boundedness (with respect to d) of a subset $A \subset X$ is clear: one requires that there exists $r > 0$ such that $d(a, b) < r$ for all $a, b \in r$ or, equivalently, that A can be included into a ball $B(x, R)$ (fro some $x \in X$, $R > 0$). However, this notion is not very useful because it depends too much on the metric (see the next exercise). One needs to replace this "obvious condition of boundedness in a metric spaces" with another condition which does not depend so strongly on the metric.

DEFINITION 7.7. *Given a metric space (X, d), we say that $A \subset X$ is totally bounded in X (or precompact in X) if for any $\epsilon > 0$, there exist a finite number of balls of radius ϵ covering A.*

If $A = X$, we say that (X, d) is totally bounded (or precompact).

EXERCISE 7.1. *For a metric space (X, d) we define $d_1 : X \times X \longrightarrow \mathbb{R}$ by*

$$\overline{d}(x, y) = min\{d(x, y), 1\}. \tag{1.3}$$

Show that

(1) *$\overline{d}$ is a metric inducing the same topology on X as d.*
(2) *any subset $A \subset X$ is bounded with respect to $\overline{d}$.*
(3) *A subset $A \subset X$ is totally bounded with respect to $\overline{d}$ if and only if it is totally bounded with respect to d.*
(4) *When $X = \mathbb{R}^n$ and d is the euclidean metric, then $A \subset \mathbb{R}^n$ is totally bounded with respect to d if and only if it is bounded (with respect to d).*

Closedness: We would like to understand what do we have to replace the condition "$A \subset \mathbb{R}^n$ is closed" by, in the case of metric spaces. This is a more subtle. First of all, it is worth remarking that the notion of totally boundedness is not really a relative notion: a subset A of a metric space (X, d) is totally bounded if and only if the metric space (A, d_A) itself (where d_A is the restriction of d to A) is totally bounded. Recall also that the same phenomena happened for the notion of compactness (see Remark 6.3). However, a statement like "A is closed in X" clearly does not have the same property. The key to this problem is the notion of completness of metric spaces. The point is that

- simiar to compactness and totally boundedness, completness is not a relative notion.
- $A \subset \mathbb{R}^n$ is closed if and only if it is complete.

(see the next exercise). Recall the notion of completness.

DEFINITION 7.8. *Given a metric space (X, d) and a sequence $(x_n)_{n\geq 1}$ in X, we say that $(x_n)_{n\geq 1}$ is a Cauchy sequence if*

$$\lim_{n,m\to\infty} d(x_n, x_m) = 0,$$

i.e., for each $\epsilon > 0$, there exists an integer n_ϵ such that

$$d(x_n, x_m) < \epsilon$$

for all $n, m \geq n_\epsilon$. One says that (X, d) is complete if any Cauchy sequence is is convergent. We say that $A \subset X$ is complete if A, together with the restriction of d to A, is complete.

EXERCISE 7.2. *Let (X, d) be a metric space. Show that*

(1) *Any $A \subset X$ which is complete must be closed.*
(2) *If X is complete, then $A \subset X$ is complete if and only if A is closed in X.*

We are now able to state the general compactness criteria for metric spaces.

THEOREM 7.9. *For a metric space (X, d), the following are equivalent:*
(i) *X is compact.*
(ii) *X is sequentially compact.*
(iii) *X is complete and totally bounded.*

THE REST (BELLOW) WILL BE DISCUSSED IN THE NEXT LECTURE

Before we turn to the proof, let us mention the following consequence

COROLLARY 7.10. *A subset A of a complete metric space (X, d) is compact if and only if it is closed (in X) and totally bounded (with respect to d).*

PROOF. (of Theorem 7.9 and Corollary 7.10) First the theorem.
(i)$\Longrightarrow$ (ii): Since metric spaces are first countable, this follows from Theorem 7.6.
(ii)$\Longrightarrow$ (iii): Assume now that X is sequentially compact. We first prove that X is complete. Let $(x_n)_{n\geq 1}$ be a Cauchy sequence. By hypothesis, we find a convergent subsequence $(x_{n_k})_{k\geq 1}$. Let x be its limit. We prove that the whole sequence (x_n) converges to x. Let $\epsilon > 0$. We look for an integer N_ϵ such that $d(x_n, x) < \epsilon$ for all $n > N_\epsilon$. Since (x_n) is Cauchy we find N'_ϵ such that

$$d(x_n, x_m) < \epsilon/2$$

for all $n, m \geq N'_\epsilon$. Since $(x_{n_k})_{k\geq 1}$ converges to x, we find k_ϵ such that

$$d(x_{n_k}, x) < \epsilon/2$$

for all $k \geq k_\epsilon$. Put

$$N_\epsilon = \max\{N'_\epsilon, n_{k_\epsilon}\}.$$

Then, for $n > N_\epsilon$, choosing k such that $n_k > n$ (such a k exists since $n_1 < n_2 < \ldots$ is a sequence that tends to ∞), we must have $k > k_\epsilon$ and $n_k > N'_\epsilon$, hence

$$d(x_n, x_{n_k}) < \epsilon/2, \ d(x_{n_k}, x) < \epsilon/2.$$

Using the triangle inequality, we obtain $d(x_n, x) < \epsilon$, and this holds for all $n \geq N_\epsilon$. This proves that (x_n) converges to x. We now prove that X is totally bounded. Assume it is not. Then we find $r > 0$ such that X cannot be covered by a finite number of balls of radius r. Construct a sequence $(x_n)_{n\geq 1}$ as follows. Start with any $x_1 \in X$. Since $X \neq B(x_1, r)$, we find $x_2 \in X - B(x_1, r)$. Since $X \neq B(x_1, r) \cup B(x_2, r)$, we find $x_3 \in X - B(x_1, r) \cup B(x_2, r)$. Continuoing like this we find a sequence with $x_n \notin B(x_m, r)$ for $n > m$. Hence $d(x_n, x_m) > r$ for all $n \neq m$. But by hypothesis, (x_n) has a subsequence $(x_{n_k})_{k\geq 1}$ which converges to some $x \in X$. But then we find N such that $d(x_{n_k}, x) < r/2$ for all $k \geq N$, hence

$$d(x_{n_k}, x_{n_l}) \leq d(x_{n_k}, x) + d(x, x_{n_l}) < r$$

for all $k, l \geq N$, and this contradicts the condition "$d(x_n, x_m) > r$ for all $n \neq m$".
(iii)$\Longrightarrow$ (i): Assume that (X, d) is complete and totally bounded. The last condition ensures that for each integer $n \geq 0$, there is a finite set $F_n \subset X$ such that

$$X = \bigcup_{x\in F_n} B(x, \frac{1}{2^n}).$$

Let $\mathcal{U} = \{U_i : i \in I\}$ be an open cover of X, and we want to prove that we can extract a finite subcover of $\mathcal{U}$. Assume this is not possible. We construct a sequence $(x_n)_{n\geq 1}$ inductively as follows. Since $\cup_{x\in F_1} B(x, \frac{1}{2}) = \cup_i U_i$, and the first union is a finite union (F_1 is finite), we find $x_1 \in F_1$ such that $B(x_1, \frac{1}{2})$ cannot be covered by of finite number of opens from $\mathcal{U}$. Now, since

$$B(x_1, \frac{1}{2}) = \bigcup_{x\in F_2} (B(x_1, \frac{1}{2}) \cap B(x, \frac{1}{4}))$$

we find $x_2 \in F_2$ such that

$$B(x_1, \frac{1}{2}) \cap B(x_2, \frac{1}{4}) \neq \emptyset$$

and $B(x_2, \frac{1}{4})$ cannot be covered by a finite numbers of opens from $\mathcal{U}$. Continuing this, at step n we find $x_n \in F_n$ such that

$$B(x_{n-1}, \frac{1}{2^{n-1}}) \cap B(x_n, \frac{1}{2^n}) \neq \emptyset$$

and $B(x_n, \frac{1}{2^n})$ cannot be covered by of finite number of opens from $\mathcal{U}$. Note that, choosing an element y in the (non-empty) intersection above, the triangle inequality implies that

$$d(x_{n-1}, x_n) < \frac{1}{2^{n-1}} + \frac{1}{2^n} = \frac{3}{2^n},$$

from which we deduce that $(x_n)_{n\geq 1}$ is a Cauchy sequence (why?). By hypothesis, it will converge to en element $x \in X$. Choose $U \in \mathcal{U}$ such that $x \in U$. Since U is open, we find $\epsilon > 0$ such that $B(x, \epsilon) \subset U$. Since $x_n \to x$, we find n_ϵ such that $d(x_n, x) < \epsilon/2$ for all $n > n_\epsilon$. Using the triangle inequality, we deduce that $B(x_n, \epsilon/2) \subset U$ for all $n \geq n_\epsilon$. Choosing n so that also $1/2^n < \epsilon/2$, we deduce that $B(x_n, 1/2^n) \subset U$, which contradicts the fact that $B(x_n, \frac{1}{2^n})$ cannot be covered by of finite number of opens from $\mathcal{U}$.

This ends the proof of the theorem. We are left with proving Corollary 7.10. But this follows immediately from the equivalence between (i) and (iii) in the theorem (applied to the metric space (A, d_A)) and the fact that a subset of a complete metric space is complete if and only if it is closed (see Exercise 7.2). ☐

1.6. Compactness and metrizability:

THEOREM 7.11. *If X is a compact Haudorff space, then the following are equivalent*

(i) *X is metrizable.*

(ii) *X is second countable.*

PROOF. (i)$\Longrightarrow$ (ii): Let d be a metric inducing the topology of X. Since (X, d) is compact, Theorem 7.9 implies that X is totally bounded, hence, for each n, we find a finite set F_n such that

$$X = \bigcup_{x\in F_n} B(x, \frac{1}{n}).$$

The set $A = \cup_n F_n$ is a countable union of finite sets, hence it is countable. We deduce that

$$\mathcal{B} = \{B(a, \frac{1}{n} : a \in A, n \geq 1 \text{ integer}\}$$

is a countable family of open sets of X. We claim it is a basis for the topology of X. Let U be an arbitrary open and $x \in U$. We have to prove that there exists $B \in \mathcal{B}$ such that $x \in B \subset U$. Since $x \in U$, we find an integer n such that

$$B(x, \frac{1}{n}) \subset U.$$

Using the defining property for F_{2n}, we see that there exists $a \in A$ such that

$$x \in B(a, \frac{1}{2n}).$$

Using the triangle inequality, we deduce that for each $y \in B(a, \frac{1}{2n})$,

$$d(x, y) \leq d(x, a) + d(a, y) < \frac{1}{2n} + \frac{1}{2n} = \frac{1}{n},$$

hence $y \in B(x, \frac{1}{n}) \subset U$. In conclusion,

$$B = B(a, \frac{1}{2n}) \in \mathcal{B} \text{ satisfies } x \in B \subset U.$$

(ii)$\Longrightarrow$ (i): Since X is compact, we know that X is normal (see Theorem 6.7), hence the result follows from Urysohn's metrization theorem (see Theorem 5.5). $\square$