Exercises Meetkunde en Topologie
June 19, 2006

Exercise 1 Draw a picture and describe a cell decomposition and the fundamental group of the space X in each of the following cases:

1. When X is the one point compactification of the space consisting of the xOy plane together with the upper half plane of the unit sphere (Figure 1).

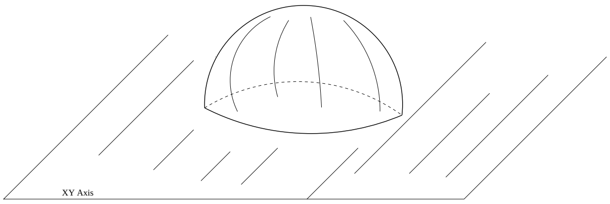

Figure 1:

2. When X is obtained from a square together with a segment whose endpoints are inside the square and does not intersect the square in any other point (Figure 2).

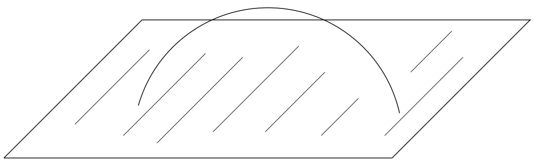

Figure 2:

3. When X is the one point compactification of the space consisting of the xOy plane together with the upper half of the Z axis (Figure 3).

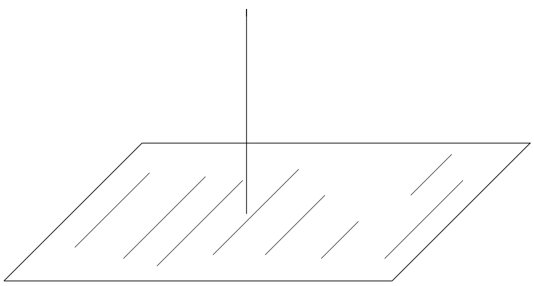

Figure 3:

Exercise 2 Consider the space consisting of two circles joined by a line (Figure 4).

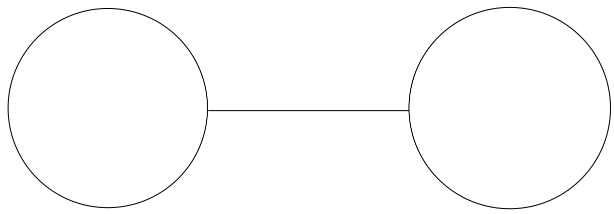

Figure 4:

1. Find a cell decomposition of X.
2. Prove that X is homotopic equivalent to a bouquet $S^1 \vee S^1$ of two circles.
3. Prove that X is not homeomorphic to S^1.
4. Compute the fundamental group of X with basepoint x shown in the picture; use the picture to describe the generators.

Exercise 3 Do the same as in the previous exercise for the space drawn in Figure 5.

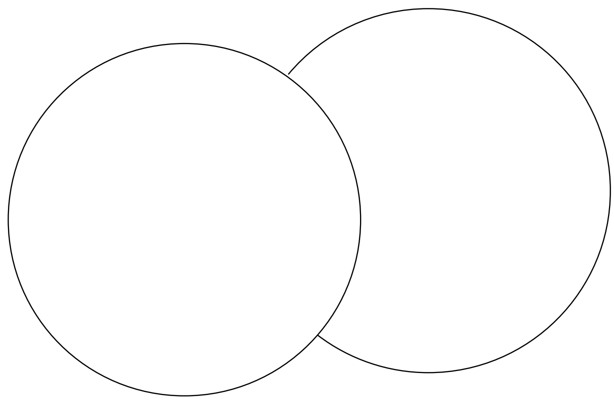

Figure 5:

Exercise 4 Consider the cylinder, $C = S^1 \times [0,1]$ (Figure 6).

1. Show that C is homotopic equivalent to S^1.
2. Describe a cell decomposition of C.
3. For any $p = (x,t) \in C$ with $t \notin \{0,1\}$, show that $C - \{p\}$ is homotopic equivalent to $S^1 \vee S^1$.

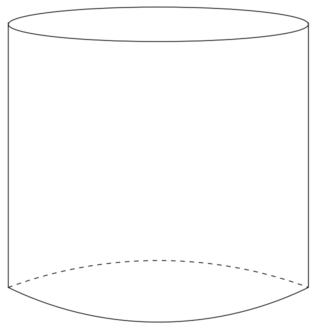

Figure 6:

4. For any $p = (x, t) \in C$ with $t \in \{0, 1\}$, show that $C - \{p\}$ is homotopic equivalent to S^1.

Exercise 5 Prove that

$$X = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \neq 0\}$$

is not homeomorphic to $\mathbb{R}^3$. Also, compute the fundamental group of X.

Exercise 6 Show that the space X consisting of the sides and the diagonal of a square is homotopic equivalent to a bouquet $S^1 \vee S^1 \vee S^1 \vee S^1$ of four circles (see Figure 7).

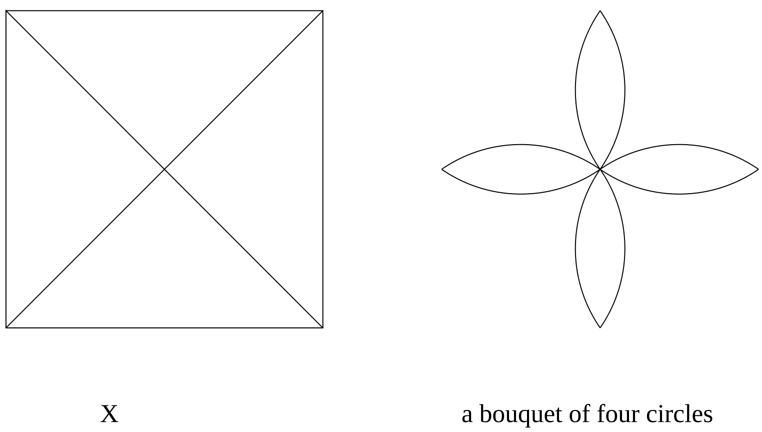

Figure 7:

Exercise 7 Recall that the torus $T = X/\sim$ is obtained from $X = [0, 1] \times [0, 1]$ by identifying the opposite sides. Consider T' obtained from T by removing a small disk (Figure 8).

1. Find $X_0 \subset X$ such that $T' = X_0/\sim$.

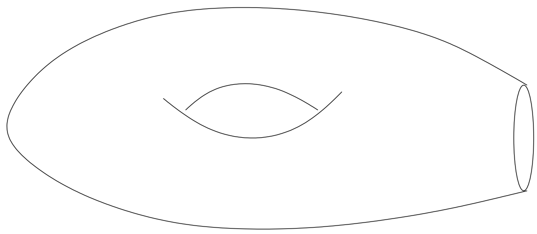

Figure 8:

2. Show that T' is homotopic equivalent to $S^1 \vee S^1$.

Exercise 8 Consider an octogon X in the plane and the equivalence relation $\sim$ on X which identifies the sides of ∂X as shown in the Figure 9.

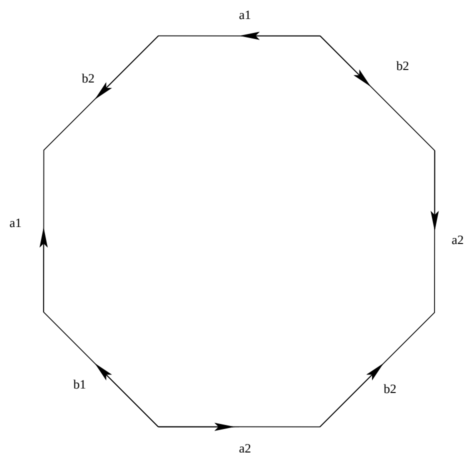

Figure 9:

1. Show that $\partial X/\sim$ is homeomorohic to a bouquet $S^1 \vee S^1 \vee S^1 \vee S^1$ of four circles (see Figure 7).

2. Show that $X/\sim$ us homeomorphic to the double torus T_2 (Figure 10).

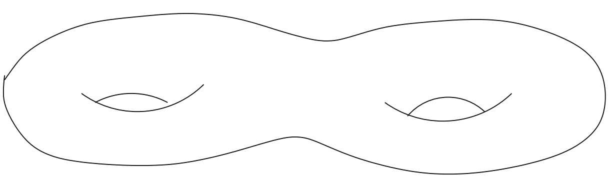

Figure 10:

3. Describe a cell decomposition of the double torus T_2.

4. Compute the fundamental group of the double torus T_2.

Exercise 9 Compute the Euler number of each of the spaces appearing in the previous exercises.