CHAPTER 10

Quotient topologies

The motivation behind quotient topologies is quite simple:

- many interesting topological spaces Y arise from simpler topological spaces X by identifying certain points of X. For instance, starting with the interval $X = [0, 1]$, after identifying the extremities 0 and 1, one arises at a space Y which is (or looks like) the unit circle S^1.
- often it happens that, starting from a topological space X, after identifying certain points of X, the result Y is not apriori a topological space- it is just a set which we would like to make into a topological space. Even more, it may happen that our interest is on the set Y which we would like to understand topologically, and then the role of (the topological space) X is just to help. For instance, consider the set of all lines through the origin in $\mathbb{R}^{n+1}$,

$$\mathbb{P}^n = \{l \subset \mathbb{R}^{n+1} : l - \text{one dimensional vector subspace}\}$$

(the projective space). At this point we have no (interesting) topology on $\mathbb{P}^n$. However, $\mathbb{P}^n$ can be viewed as obtained from $X = \mathbb{R}^{n+1} - \{0\}$ (which has an interesting topology!) by identifying certain points (see also below).

The motivating question is quite simple: starting from a topological space X after forming a quotient set Y of X (i.e. after identifying certain points of X), is there an "induced" (interesting) topology on Y? The answer is very simple, but before giving it we will first make more precise the meaning of "identifying points".

1. Quotient sets

We start with a set X. There are two (equivalent) ways of looking at quotients of X, each one having its own advantages (depending on the examples one looks at). We will exemplify this by looking at the set $\mathbb{P}^n$ consisting of all lines in $\mathbb{R}^{n+1}$ through the origin. First of all:

Definition 10.1. *Let X be a set. A quotient of X is a pair (Y, π) where Y is a set and*

$$\pi : X \longrightarrow Y$$

is a surjective map. We call π the quotient map and we call Y the quotient set; we also say that Y is a quotient of X with quotient map π.

If (Y, π) and (Y', π') are two quotients of X, we say that they are isomorphic if there exists a bijection $f : Y \longrightarrow Y'$ such that $f\pi = \pi'$.

For instance, $Y = \mathbb{P}^n$ is a quotient of $X = \mathbb{R}^{n+1} - \{0\}$, with quotient map

$$\pi : \mathbb{R}^{n+1} \longrightarrow \mathbb{P}^n, x \mapsto l_x,$$

where l_x is the line through the origin and x:

$$l_x = \mathbb{R}x = \{\lambda x : \lambda \in \mathbb{R}\} \subset \mathbb{R}^{n+1}.$$

DEFINITION 10.2. *Let X be a set. A relation on a set X is a subset*

$$R \subset X \times X.$$

For $x, y \in X$, we write xRy if $(x, y) \in R$. An equivalence relation on X is a relation $\sim$ on X with the properties:

- *$x \sim x$ for all $x \in X$.*
- *If $x \sim y$ then $y \sim x$.*
- *If $x \sim y$ and $y \sim z$, then $x \sim z$.*

Given the equivalence relation $\sim$, the equivalence class of an element $x \in X$, or the orbit of x (with respect to $\sim$), is

$$\sim (x) = \{y \in X : y \sim x\}.$$

The set of all such orbits is denoted by $X/\sim$ and is called the quotient set of X with respect to $\sim$:

$$X/\sim = \{\sim (x) : x \in X\}.$$

The quotient map with respect to $\sim$ is

$$\pi_\sim : X \longrightarrow X/\sim, \ x \mapsto \sim (x).$$

EXAMPLE 10.3. For instance, $\mathbb{P}^n$ can be realized as the quotient of $X = \mathbb{R}^{n+1} - \{0\}$ with respect to an equivalence relation. Namely we define the equivalence relation $\sim$ on $\mathbb{R}^{n+1} - \{0\}$ by declaring that two points x, y if they determine the same line through the origin; equivalently,

$$x \sim y \Longleftrightarrow x = \lambda y \text{ for some } \lambda \in \mathbb{R} - \{0\}.$$

Hence the $\sim$-equivalence class of $x \in \mathbb{R}^{n+1}$ is precisely the line through origin and x:

$$\sim (x) = l_x,$$

and then $X/\sim = \mathbb{P}^n$. The quotient map with respect to $\sim$ is precisely the map $x \mapsto l_x$ previously mentioned.

What happens in the previous example is much more general:

LEMMA 10.4. *Let X be a set.*

(i) *For any equivalence relation $\sim$ on X, $Y := X/\sim$ is a quotient of X with quotient map $\pi_\sim$.*

(ii) *For any quotient Y of X with quotient map $\pi : X \longrightarrow Y$, one has an equivalence relation $\sim_\pi$ on X defined by:*

$$x \sim_\pi x' \Longleftrightarrow \pi(x) = \pi(x'),$$

and (Y, π) is (isomorphic to) $(X/\sim_\pi, \pi_{\sim_\pi})$.

PROOF. Part (i) is clear. In (ii), it is easy to check that $\sim_\pi$ is an equivalence relation. Moreover,

$$f : X/\sim \longrightarrow Y, \sim (x) \mapsto \pi(x)$$

is well-defined (if $\sim (x) = \sim (x')$ then $\pi(x) = \pi(x')$), and it is bijective with inverse $f^{-1}(y) = \sim (x)$ where $x \in X$ is any point such that $\pi(x) = y$ (for any y, such a point x exists since π is surjective, and if x' is another one then $\sim (x) = \sim (x')$). $\square$

REMARK 10.5. In the previous lemma, it is easy to check that two quotients of X induce the same equivalence relation on X if and only if the induced equivalence relations on X coincide. In other words, there is a bijection between equivalence relations on X and isomorphism classes of quotients of X.

REMARK 10.6. Alternatively, equivalence relations $\sim$ on X can be viewed as (or are in 1-1 correspondence with) partitions $\mathcal{P}$ of X. Recall that a partition of X is a collection $\mathcal{P}$ of subsets of X with the property that any two distinct elements of $\mathcal{P}$ are disjoint, and any point in X lies in some set $P \in \mathcal{P}$. Equivalently, this means that for each $x \in X$ there exists exactly one set $P_x \in mathcal P$ such that $x \in P_x$. Note that a partition $\mathcal{P}$ of X is naturally a quotient set of X with quotient map

$$\pi_{\mathcal{P}} : X \longrightarrow \mathcal{P}, x \mapsto P_x.$$

In particular, any partition $\mathcal{P}$ of X induces an equivalence relation $\sim_{\mathcal{P}}$ on X:

$$x \sim_{\mathcal{P}} x \Longleftrightarrow \exists\, P \in \mathcal{P} : x, x' \in P.$$

Conversely, any equivalence relation $\sim$ on X defines a partition $\mathcal{P}_{\sim}$ consisting of all $\sim$-equivalence classes of elements of X. The correspondences $\mathcal{P} \mapsto \sim_{\mathcal{P}}$ and $\sim \mapsto \mathcal{P}_{\sim}$ define bijections (inverse to each other) between the set of partitions of X and the set of equivalence relations on X.

Next, we look at maps defined on quotient sets.

REMARK 10.7. Let X be a set and let Y be a quotient of X with quotient map π. Let Z be another set. Then there is a 1-1 correspondence between

(i) maps $f : Y \longrightarrow Z$.
(ii) maps $\tilde{f} : X \longrightarrow Z$ such that $\tilde{f}(x) = \tilde{f}(x')$ whenever $\pi(x) = \pi(x')$.

This correspondence is defined by $f \mapsto \tilde{f} = f \circ \pi$. Indeed, given $\tilde{f}$ satisfying (ii), we define $f : Y \longrightarrow Z$ as follows: for $y \in Y$, choose $x \in X$ such that $\pi(x) = y$ (possible since π is surjective) and put $f(y) = \tilde{f}(x)$ (and this is well defined because if x' is another point in X such that $\pi(x') = y$, then $\tilde{f}(x) = \tilde{f}(x')$). The two constructions are inverse to each other.

In the language of equivalence relations, if $(Y, \pi) = (X/\sim, \pi)$ with $\sim$-an equivalence relation on X, $f \mapsto f \circ \pi$ defines a 1-1 correspondence between

(i') maps $f : X/\sim \longrightarrow Z$.
(i") maps $\tilde{f} : X \longrightarrow Z$ which are constant on the $\sim$-orbits (i.e. $\tilde{f}(x) = \tilde{f}(x')$ whenever $x \sim x'$).

2. Quotient topologies

DEFINITION 10.8. *Let $(X, \mathcal{T}_X)$ be a topological space and let Y be a quotient set of X with quotient map $\pi : X \longrightarrow Y$. Define the induced quotient topology as the topology $\pi_*(\mathcal{T}_X)$ on Y defined by*

$$\pi_*(\mathcal{T}_X) = \{V \subset Y : \pi^{-1}(V) \in \mathcal{T}_X\}.$$

When there is no danger of confusion, we omit the topology from the notation.

If $\sim$ is an equivalence relation on X, the topological space $(X/\sim, \pi_*(\mathcal{T}_X))$ (where $\pi = \pi_{\sim} : X \longrightarrow X/\sim$ is the quotient map) is called the quotient of the topological space $(X, \mathcal{T}_X)$ with respect to the equivalence relation $\sim$.

EXERCISE 10.1. *Show that $\pi_*(\mathcal{T}_X)$ is a topology on Y.*

REMARK 10.9. The motivation for the quotient topology is very simple. Starting with the quotient Y of X, where X is a topological space not just a set, the question is: is there an induced "interesting" topology $\mathcal{T}_Y$ on Y? To be "interesting" we should at least require that the quotient map $\pi : X \longrightarrow Y$ becomes continuous. However, the trivial topology on Y (consisting of $\emptyset$ and Y only) has this property, and we would not call it interesting. Moreover, if a topology $\mathcal{T}$ on Y has this property, then so does any other topology $\mathcal{T}'$ on Y which is contained in $\mathcal{T}$.

So, the interesting topology $\mathcal{T}_Y$ on Y would be the largest topology on Y with the property that $\pi : X \longrightarrow Y$ is continuous. That π is continuous means that

$$\mathcal{T}_Y \subset \{V \subset Y : \pi^{-1}(V) \in \mathcal{T}_X\}.$$

What happens is that the right hand side of the previous inclusion is already a topology, hence, by the maximality of $\mathcal{T}_Y$, the inclusion above must become an equality.

The following result is useful for checking when a map defined on a quotient is continuous.

LEMMA 10.10. *Let X be a topological space and let Y be a quotient of X with quotient map $\pi : X \longrightarrow Y$; we endow Y with the quotient topology. Let Z be another topological space. Then a map $f : Y \longrightarrow Z$ is continuous if and only if $f \circ \pi : X \longrightarrow Z$ is continuous.*

PROOF. f being continuous is equivalent to $f^{-1}(U)$-open in Y for each U-open in Z. Since Y is endowed with the quotient topology, this is equivalent to: $\pi^{-1}(f^{-1}(U))$-open in Y for each U-open in Z. Since

$$\pi^{-1}(f^{-1}(U)) = (f \circ \pi)^{-1}(U),$$

the condition is equivalent to the continuity of $f \circ \pi$. □

COROLLARY 10.11. *Let X be a topological space and let Y be a quotient of X with quotient map $\pi : X \longrightarrow Y$; we endow Y with the quotient topology. Then, for any topological space Z, there is a 1-1 correspondence between*

(i) *continuous maps $f : Y \longrightarrow Z$.*
(ii) *continuous maps $\tilde{f} : X \longrightarrow Z$ such that $\tilde{f}(x) = \tilde{f}(x')$ whenever $\pi(x) = \pi(x')$.*

This correspondence is characterized by $\tilde{f} = f \circ \pi$.

PROOF. Follows from Remark 10.7 and the previous proposition. □

Here is the reformulation of the previous corollary using the language of equivalence relations.

COROLLARY 10.12. *Let $\sim$ be an equivalence relation on the topological space X and let $X/\sim$ be endowed with the quotient topology. Then, for any topological space Z, there is a 1-1 correspondence between*

(i') *continuous maps $f : X/\sim \longrightarrow Z$.*
(ii') *continuous maps $\tilde{f} : X \longrightarrow Z$ which are constant on the $\sim$-orbits.*

This correspondence is characterized by $\tilde{f}(x) = f(\sim(x))$ for all $x \in X$.

EXAMPLE 10.13. Look again at the projective space $\mathbb{P}^n$ and make the simple remark that a line in $\mathbb{R}^{n+1}$ through the origin is uniquely determined by its intersection with the unit sphere $S^n \subset \mathbb{R}^{n+1}$- which is a set consisting of two antipodal points. This indicates that $\mathbb{P}^n$ can be obtained from S^n by identifying its antipodal points. Let's prove this properly. Hence consider on S^n the equivalence relation $\sim$ defined by

$$x \sim y \Longleftrightarrow x = y \text{ or } x = -y$$

(This gives a precise meaning to "the space obtained from S^n by identifying its antipodal points"). We want to prove that $\mathbb{P}^n$ is homeomorphic to $S^n/\sim$ (where the last space is endowed with the quotient topology). As we have remarked, there is a set-theoretical bijection:

$$\phi : S^n/\sim \longrightarrow \mathbb{P}^n$$

which sends the equivalence class of $x \in S^n$ to the line l_x through x, with the inverse

$$\psi : \mathbb{P}^n \longrightarrow S^n/\sim$$

which sends the a line l to the equivalence class of one of the points in $S^n \cap l$. We have to check that these maps are continuous.

To see that ϕ is continuous, we have to check that the composition with the quotient map $S^n \longrightarrow S^n/\sim$ is continuous. But this composition is precisely the restriction of the quotient map $\mathbb{R}^{n+1} - \{0\} \longrightarrow \mathbb{P}^n$ to S^n, hence is continuous. In conclusion, ϕ is continuous.

To see that ψ is continuous, we have to check that its composition with the quotient map $\mathbb{R}^{n+1} - \{0\} \longrightarrow \mathbb{P}^n$ is continuous. But this composition- which is a map from $\mathbb{R}^{n+1} - \{0\}$ to $S^n/\sim$ can be written as the composition of two other maps which we know to be continuous:

- The map $\mathbb{R}^{n+1} - \{0\} \longrightarrow S^n$ sending x to $x/||x||$.
- The quotient map $S^n \longrightarrow S^n/\sim$.

In conclusion ψ is continuous, concluding the proof of the claim.

3. Particular classes of quotients

We now point out several particular classes of quotients. First of all, quotients by group actions. Let X be a topological space. We denote by Homeo(X) the set of all homeomorphisms from X to X. Together with composition of maps, this is a group. Let Γ be another group, whose operation we denote multiplicatively.

DEFINITION 10.14. *An action of the group Γ on the topological space X is a group homomorphism*

$$\phi : \Gamma \longrightarrow Homeo(X),\ \gamma \mapsto \phi_\gamma.$$

Hence, for each $\gamma \in \Gamma$, one has a homeomorphism ϕ_γ of X ("the action of γ on X"), so that

$$\phi_{\gamma\gamma'} = \phi_\gamma \circ \phi_{\gamma'} \ \ \forall\ \gamma, \gamma' \in X.$$

Sometimes $\phi_\gamma(x)$ is also denoted $\gamma(x)$, or simply $gammax.$, and one interprets the action as a map

$$\Gamma \times X \longrightarrow X,\ (\gamma, x) \longrightarrow \gamma x.$$

Given such an action one defines the orbit of a point $x \in X$ by

$$\Gamma x := \{\gamma x : \gamma \in \Gamma\}.$$

DEFINITION 10.15. *Let X be a topological space, and let Γ be a group acting on X. For $x \in X$ one defines the Γ-the orbit of x as:*

$$\Gamma x := \{\gamma x : \gamma \in \Gamma\}.$$

The set of all orbits is denoted by X/Γ:

$$X/\Gamma = \{\Gamma x : x \in X\}.$$

We view X/Γ as a quotient of X with quotient map

$$\pi : X \longrightarrow X/\Gamma,\ x \mapsto \Gamma x.$$

The quotient of X by the action of Γ is the space X/Γ endowed with the induced quotient topology.

Note that, in terms of equivalence relations, the action induces an equivalence relation $\sim_\Gamma$ on X defined by:

$$x \sim_\Gamma y \Longleftrightarrow\ \exists\ \gamma \in \Gamma \text{ s.t. } y = \gamma x.$$

With this, we have

$$X/\Gamma = X/\sim_\Gamma .$$

EXAMPLE 10.16. Looking at Example 10.3, we recognize that the equivalence relation $\sim$ on $X = \mathbb{R}^{n+1} - \{0\}$ associated to the projective space $\mathbb{P}^n$ comes from a group action: the group $\Gamma = \mathbb{R}^*$ (non-zero reals together with usual multiplication) acts on X by:

$$\phi_\lambda(x) = \lambda x \quad \text{for } \lambda \in \mathbb{R}^*, x \in \mathbb{R}^{n+1} - \{0\}.$$

So, with the new notations, we have

$$\mathbb{P}^n = (\mathbb{R}^{n+1} - \{0\})/\mathbb{R}^*.$$

Note also that the space $S^n/\sim$ which appears in Example 10.13 fits as well in this setting: the group $\Gamma = \mathbb{Z}_2 = \{1, -1\}$ acts on $X = S^n$: ϕ_1 is the identity map, while ϕ_{-1} is the map sending $x \in S^n$ to its antipodal point $-x$. Hence, what we have already shown in Example 10.13 can be rewritten as:

$$\mathbb{P}^n \cong S^n/\mathbb{Z}_2.$$

Another class of quotient spaces are quotients obtained by collapsing a subspace to a point.

DEFINITION 10.17. *Let X be a topological space and let $A \subset X$. We define X/A as the topological space obtained from X by collapsing A to a point (i.e. by identifying to each other all the points of A). Equivalently,*

$$X/A = X/\sim_A,$$

where $\sim_A$ is the equivalence relation on X defined by

$$x \sim_A y \Longleftrightarrow x = y \text{ or } x, y \in A.$$

EXAMPLE 10.18. Let X be a topological space. Consider the cylinder of X:

$$\mathrm{Cyl}(X) := X \times [0, 1],$$

endowed with the product topology (and the unit interval is endowed with the Euclidean topology). It contains two interesting copies of X: $X \times \{1\}$ and $X \times \{0\}$. Collapsing the first copy to a point will produce something which looks like a cone with basis X; for this reason, one uses the notation

$$\mathrm{Cone}(X) := X \times [0, 1]/X \times \{1\},$$

and one calls this space the cone of X. The cone contains the copy $X \times \{0\}$ of X (the basis of the cone). Collapsing it to a point will produce a space which topologically looks like "a sphere with great circle X". One uses the notation

$$\mathrm{S}(X) := \mathrm{Cone}(X)/X \times \{0\}$$

(called the suspension of X).

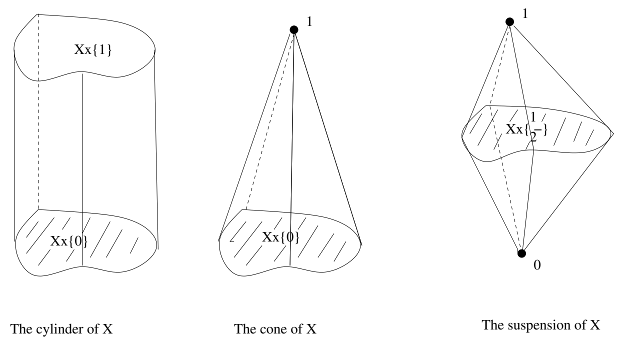


FIGURE 1.

4. Properties of quotient topologies

Before we look at some more interesting examples, we need to establish several preliminary results about quotient topologies. One can wonder which topological properties are preserved after passing to quotients. First of all, since continuous maps map compact sets to compact sets (and similarly for connected and path connected), we deduce:

PROPOSITION 10.19. *Assume that Y is a quotient of the topological space X (with the quotient topology). Then:*

(i) *If X is compact, then so is Y.*
(ii) *If X is connected, then so is Y.*
(iii) *If X is path connected, then so is Y.*

EXAMPLE 10.20. In Example 10.13 we have seen that $\mathbb{P}^n$ can be realized as a quotient of S^n. An immediate consequence is that $\mathbb{P}^n$ is compact.

EXAMPLE 10.21. Let us point out here that, in general, the quotient of a Hausdorff space need not be Hausdorff. For instance, look at $Y = X/A$ with $X = [0,1]$, $A = (0,1)$. In other words, $Y = X/\sim$, where $\sim$ is the equivalence relation on $[0,1]$ given by

$$t \sim t' \Longleftrightarrow t = t', \text{ or } t, t' \in (0,1).$$

Then $Y = \{a, b, c\}$ consists of three points: the class (or orbit) a of 0, the class b of the points inside $(0,1)$, and the class c of 1. The quotient topology on Y consist of the following opens:

$$\emptyset, \{b\}, \{a,b\}, \{b,c\}, \{a,b,c\},$$

hence Y is not Hausdorff (a and c cannot be separated).

As already indicated by the previous example, the Hausdorffness property is much more problematic: there is no criteria which insures that the quotient space is Hausdorff, and which is general enough to be relevant in all examples. Instead, there are various criterias for various classes of examples (see also below).

For instance, it is true that the projective space $\mathbb{P}^n$ is Hausdorff (we will shortly see two different proofs). However, proving this directly is not very pleasant (and certainly not satisfactory). Instead, we have Hausdorffness criteria which apply to quotients by finite group actions:

LEMMA 10.22. *If Γ is a finite group acting on the Hausdorff space X, then the quotient X/Γ is Hausdorff.*

PROOF. Let $\Gamma x, \Gamma y \in X/\Gamma$ be two distinct points ($x, y \in X$). That they are distinct means that, for each $\gamma \in \Gamma$, $x \neq \gamma y$. Hence, for each $\gamma \in \Gamma$, we find disjoint opens $U_\gamma, V_\gamma \subset X$ containing x, and γy, respectively. Note that

$$W_\gamma = \phi_\gamma^{-1}(V_\gamma)$$

is an open containing y, and what we know is that

$$U_\gamma \cap \phi_\gamma(W_\gamma) = \emptyset.$$

Since Γ is finite,

$$U = \cap_\gamma U_\gamma, W = \cap_\gamma W_\gamma,$$

will be open neighborhoods of x and y, respectively, with the property that

$$U \cap \phi_a(V) = \emptyset, \ \ \forall \ a \in \Gamma.$$

Using the quotient map $\pi : X \longrightarrow X/\Gamma$, we consider $\pi(U), \pi(V)$, and we claim that they are disjoint opens in X/Γ separating Γx and Γy. That they are disjoint follows from the previous property of U and V. To see that $\pi(U)$ is open, we have to check that $\pi^{-1}(\pi(U))$ is open, but

$$\pi^{-1}(\pi(U)) = \cup_{\gamma \in \Gamma} \phi_{\gamma}(U)$$

(check this!) is a union of opens, hence opens. Similarly, $\pi(V)$ is open. Clearly, $\Gamma x = \pi(x) \in \pi(U)$ and $\Gamma y = \pi(y) \in \pi(V)$. □

EXAMPLE 10.23. Since $\mathbb{P}^n$ has been shown to be homeomorphic to $S^n/\mathbb{Z}_2$, we deduce that $\mathbb{P}^n$ is Hausdorff.

The next aspect we discuss is the following: assume that we look at the quotient Y of a topological space X, where Y has an interesting topology topology on its own (independent of the quotient map), and we would like to prove that the two topologies on Y (the given one and the quotient one) coincide. For instance, start with the set $X = [0, 1]$ and consider the circle $Y = S^1$ viewed as a quotient of X with quotient map

$$\pi : [0, 1] \longrightarrow S^1,\ t \mapsto (cos(2\pi t), sin(2\pi t))$$

(i.e. we realize the circle as obtained from $X = [0, 1]$ by identifying the points 0 and 1). Hence we can consider on S^1 the quotient topology. But S^1 has already an interesting topology: the one induced from the Euclidean topology on $\mathbb{R}^2$. How can we show that the two topologies on S^1 are the same? The next proposition gives a criterion for proving such statements. But first some terminology:

DEFINITION 10.24. *Let X and Y be two topological spaces and let $\pi : X \longrightarrow Y$ be a continuous map. We say that π is a topological quotient map if π is surjective and the topology of Y coincides with the quotient topology on Y.*

PROPOSITION 10.25. *If X is a compact space and Y is a Hausdorff space, then any continuous surjection $f : X \longrightarrow Y$ is a topological quotient map.*

PROOF. That f is a topological quotient map means that:if $U \subset Y$ has the property that $f^{-1}(U)$-open in X, then U is open in Y. However, since closed sets are complement of opens, and since $U = f(f^{-1}(U))$ (because f is surjective), we see that the condition on f being a topological quotient map is equivalent to the condition: if $A \subset Y$ has the property that $\pi^{-1}(A)$-closed in X, then A is closed in Y. It is this property that we can easily check: assume that $A \subset X$ be closed; since A is closed in X and X is compact, A will be compact. Since f is continuous, $f(A) \subset Y$ will be compact. Since Y is Hausdorff, $f(A)$ must be closed in Y. □

COROLLARY 10.26. *Assume that $Y = X/\sim$ is a quotient of a compact space X with respect to an equivalence relation $\sim$, Z is a Hausdorff space, and*

$$\tilde{f} : X \longrightarrow Z$$

is a continuous map with the property that, for $x, x' \in X$, one has

$$\tilde{f}(x) = \tilde{f}(x') \Longleftrightarrow x \sim x'.$$

Then $\tilde{f}$ induces a homeomorphism

$$f : X/\sim \longrightarrow \tilde{f}(X) \subset Z.$$

EXAMPLE 10.27. Look at the motivating example $Y = S^1$ as a quotient of $X = [0, 1]$ mentioned above. Then the quotient map $\pi : [0, 1] \longrightarrow S^1$ is continuous as a map from $[0, 1]$-endowed with the usual topology (induced from $\mathbb{R}$) to S^1-endowed with the usual topology (induced from $\mathbb{R}^2$). Hence, using the previous proposition, we see that the quotient topology on S^1 coincides with the usual topology on S^1.

EXAMPLE 10.28. The previous proposition can be used to prove that certain quotient spaces are Hausdorff: as an example, let's give another proof to the Hausdorffness of $\mathbb{P}^n$. As before, we use the identification of $\mathbb{P}^n$ with $S^n/\mathbb{Z}_2$ (the reason for using this is that it ensures that $\mathbb{P}^n$ is compact- which we need in order to apply the proposition). We define

$$\tilde{f} : S^n \longrightarrow \mathbb{R}^N, \quad (N = \frac{(n+1)(n+2)}{2}),$$

$$(x_0, \ldots, x_n) \mapsto (x_0x_0, x_0x_1, \ldots, x_0x_n, x_1x_1, x_1x_2, \ldots, x_1x_n, \ldots, x_nx_n).$$

This is a continuous map and it is not difficult to see that

$$\tilde{f}(x) = \tilde{f}(y) \Longleftrightarrow x = y \text{ or } x = -y.$$

Using the previous proposition (or its corollary), we deduce that $\tilde{f}$ induces a homeomorphism between $S^n/\mathbb{Z}_2$ and $\tilde{f}(S^n)$ (with the topology induced from $\mathbb{R}^N$). This shows not only that $\mathbb{P}^n$ is Hausdorff, but also that $\mathbb{P}^n$ can be embedded in Euclidean spaces of large enough dimension (note however that the the dimension $N = (n+1)(n+2)/2$ is not the smallest one in general).

5. More examples

5.1. Example: the cylinder. We consider the square $X = [0,1] \times [0,1]$. We consider the two opposite vertical sides $\{0\} \times [0,1]$ and $\{1\} \times [0,1]$ and we identify them by identifying $(0,s)$ with $(1,s)$ for each $s \in [0,1]$ (hence we identify the two opposite sides with the same orientation). More precisely, we define on X the equivalence relation $\sim$ by saying that $(t_0,s_0) \sim (t_1,s_1)$ if and only if either $(t_0,s_0) = (t_1,s_1)$ or $\{t_0,t_1\} = \{0,1\}$ and $s_0 = s_1$. We form the quotient space $C = X/\sim$. Intuitively, the gluing process is described in the picture (Figure 2) and produces a subspace of $\mathbb{R}^3$ which looks like the standard cylinder $S^1 \times [0,1]$. This indicates that C can be embedded in $\mathbb{R}^3$ and can be identified (as a topological space!) with the standard cylinder.

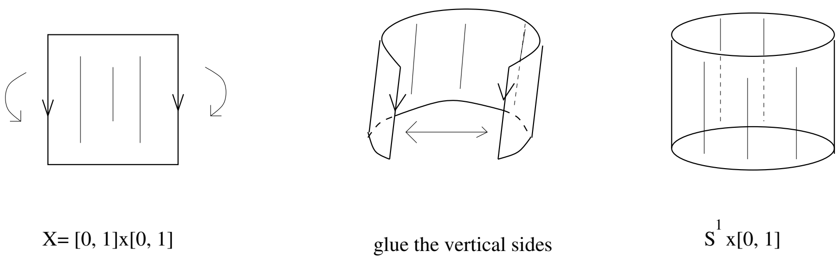


FIGURE 2.

To prove this, we define

$$\tilde{f} : [0,1] \times [0,1] \longrightarrow S^1 \times [0,1] \subset \mathbb{R}^3,$$

$$\tilde{f}(t,s) = (cos(2\pi t), sin(2\pi t), s).$$

This is a continuous map, which is surjective and which has the property that

$$\tilde{f}(t_0,s_0) = \tilde{f}(t_1,s_1) \Longleftrightarrow (t_0,s_0) \sim (t_1,s_1).$$

Also, C is compact, as the quotient of a compact space, while $\mathbb{R}^3$ is Hausdorff. Hence we can apply Corollary 10.26 and we deduce that $\tilde{f}$ induces a homeomorphism between C and $S^1 \times [0,1]$.

5.2. Example: the Moebius band. We consider the square $X = [0,1] \times [0,1]$. We consider the two opposite vertical sides $\{0\} \times [0,1]$ and $\{1\} \times [0,1]$ and we identify them, but this time with opposite orientations. I.e., we identify $(0,s)$ with $(1, 1-s)$ for each $s \in [0,1]$. Equivalently, we define on X the equivalence relation $\sim$ by saying that $(t_0, s_0) \sim (t_1, s_1)$ if and only if either $(t_0, s_0) = (t_1, s_1)$ or $\{t_0, t_1\} = \{0, 1\}$ and $s_1 = 1 - s_0$. The resulting quotient space $M = X/\sim$ is called the Moebius band. Intuitively, the gluing process is described in the picture (Figure 3) and produces a subspace of $\mathbb{R}^3$- this indicates that M can be embedded in $\mathbb{R}^3$ and can be identified (as a topological space!) with the space drawn in the picture.

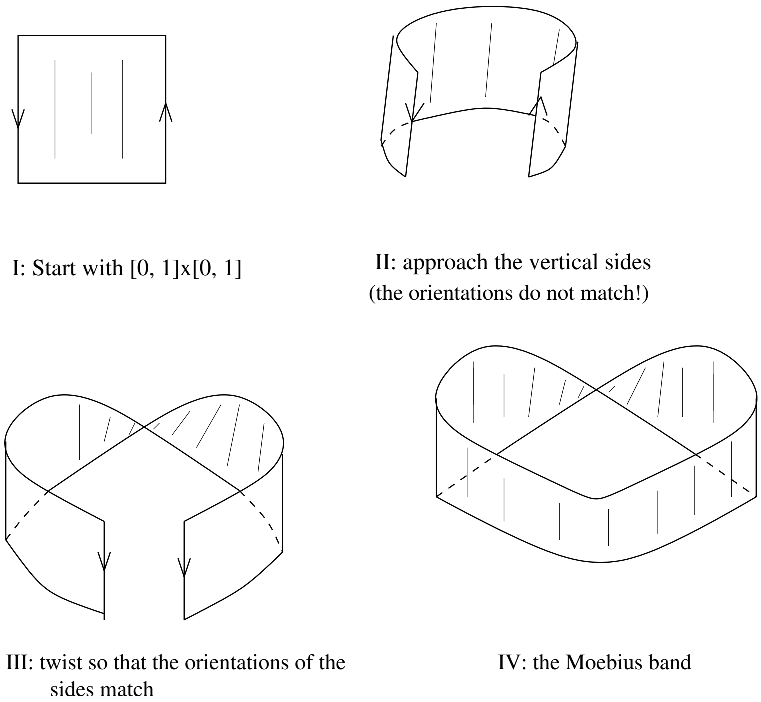


FIGURE 3.

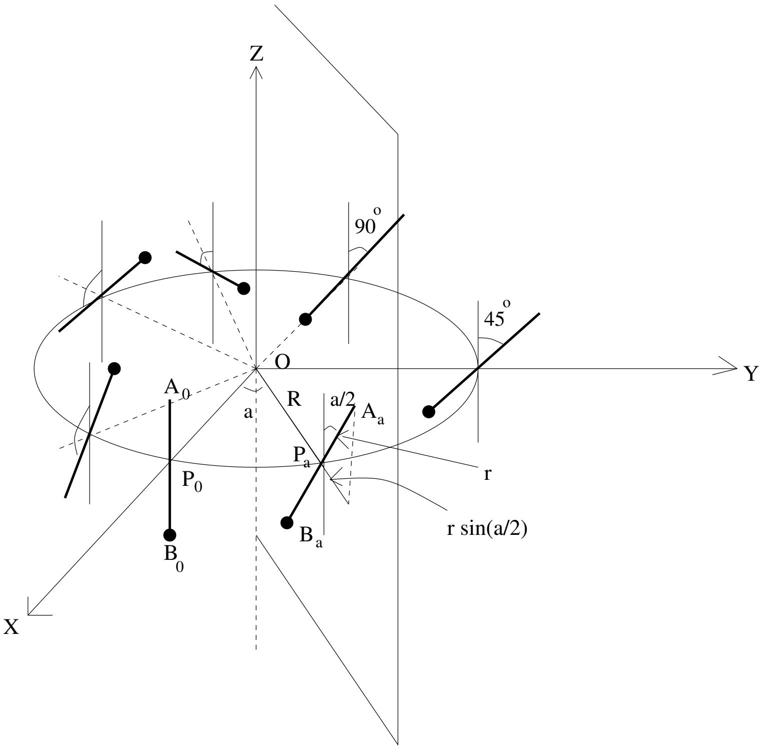
To prove this, let's first describe an explicit model for the space drawn in the picture. This space can be imagined as obtained by starting with a segment in $\mathbb{R}^3$ and rotating it around its middle point, while its middle point is being rotated on a circle. See Figure 4. The rotations take place at the same time (uniformly), and while the segment rotates by $180°$, the middle point rotates a full circle. To write down explicit formulas, assume that

- the circle is situated in the XOY plane, is centered at the origin, and has radius R.
- the length of the segment is $2r$ and the starting position A_0B_0 of the segment is perpendicular on XOY with middle point $P_0 = (R, 0, 0)$.
- at any moment, the segment stays in the plane through the origin and its middle point, which is perpendicular on the XOY plane.

We denote by $M_{R,r}$ the resulting subspace of $\mathbb{R}^3$ $(2R > r)$. To parameterize $M_{R,r}$, we parameterize the movement by the angle a which determines the middle point on the circle:

$$P_a = (Rcos(a), Rsin(a), 0).$$

At this point, the precise position of the segment, denoted A_aB_a, is determined by the angle that it makes with the perpendicular on the plane XOY through P_a; call it b. This angle depends



Explicit realization of the Moebius band in $\mathbb{R}^3$: $M_{R,r}$

FIGURE 4.

on a. Due to the assumptions (namely that while a goes from 0 to 2π, b only goes from 0 to π, and that the rotations are uniform), we have $b = a/2$ (see 4). We deduce

$$A_a = \{(R + rsin(a/2))cos(a), (R + rsin(a/2))sin(a), rcos(a/2)),$$

and a similar formula for B_a (obtained by replacing r by $-r$). Hence we obtain

$$M_{R,r} = \{(R + tsin(a/2))cos(a), (R + tsin(a/2))sin(a), tcos(a/2)) : a \in [0, 2\pi], t \in [-r, r]\}.$$

We view $M_{R,r}$ as a topological space with the topology induced from $\mathbb{R}^3$. Note that, although this depends on R and r, different choices of R and r produce spaces which are homeomorphic to $M_{R,r}$. So, we could as well assume that $R = r = 1$. With these in mind, we define

$$\tilde{f} : [0, 1] \times [0, 1] \longrightarrow \mathbb{R}^3,$$

$$\tilde{f}(t, s) = ((1 + (2t - 1)sin(s\pi))cos(2s\pi), (1 + (2t - 1)sin(s\pi))sin(2s\pi), (2t - 1)cos(s\pi)),$$

which is a continuous map; it is not difficult to check that

$$\tilde{f}(t_0, s_0) = \tilde{f}(t_1, s_1) \Longleftrightarrow (t_0, s_0) \sim (t_1, s_1).$$

We deduce that $\tilde{f}$ induces a continuous map

$$f : M \longrightarrow \mathbb{R}^3$$

which is injective, and whose image is $M_{1,1}$. Since M is compact (as quotient of a compact space) and $\mathbb{R}^3$ is Hausdorff, we deduce that f defines a homeomorphism between M and $M_{1,1}$.

5.3. Example: the torus. As before, we start with the square $X = [0,1] \times [0,1]$, but this time we make different identifications of its sides: we identify the opposite sides keeping the orientation- in other words, we identify $(t, 0)$ with $(t, 1)$ for each $t \in [0, 1]$, and we identify $(0, s)$ with $(1, s)$ for each $s \in [0, 1]$. We denote by T^2 the resulting topological space. It is called the torus. As before, this means that $T^2 = X/\sim$, where $\sim$ is an equivalence relation on X defined so that the point that we want to be identified are declared to be equivalent. Intuitively, the gluing process is described in the picture (Figure 5) and produces a subspace of $\mathbb{R}^3$ which looks like a doughnut- this indicates that T^2 can be embedded in $\mathbb{R}^3$ and can be identified (as a topological space!) with the space drawn in the picture.

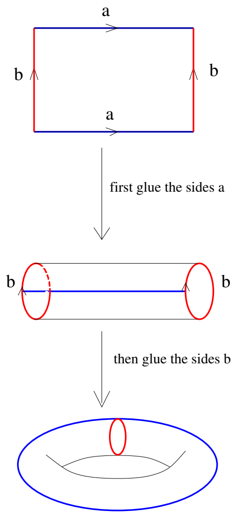


FIGURE 5.

To prove this, let's first describe an explicit model for the doughnut. This space can be imagined as obtained by starting with a circle in $\mathbb{R}^3$ and by rotating it around a line which is in the plane of the circle. See Figure 6. To describe the resulting space explicitly, we assume that the rotation line is the OZ axis, the initial position of the circle is in the XOZ plane, with center of coordinates $(R, 0, 0)$. Let r be the radius of the circle $(R > r)$. We denote by $T^2_{R,r}$ the resulting subspace of $\mathbb{R}^3$. A point on $T^2_{R,r}$ is uniquely determined by the angles a and b indicated on the picture (Figure 6), and we find the parametric description:

$$T^2_{R,r} = \{(R + rcos(a))cos(b), (R + rcos(a))sin(b), rsin(a)) : a, b \in [0, 2\pi]\} \subset \mathbb{R}^3.$$

(alternatively, writing down the equations of the circles- or eliminating α and β in the parametric description-, we find the implicit description: $(\sqrt{x^2 + y^2} - R)^2 + z^2 = r^2)$. Of course, we consider $T^2_{R,r}$ as a topological space with the topology induced from $\mathbb{R}^3$. We now have a function

$$\tilde{f} : [0,1] \times [0,1] \longrightarrow \mathbb{R}^3, \tilde{f}(t,s) = (R + rcos(2t\pi))cos(2s\pi), (R + rcos(2t\pi))sin(2s\pi), rsin(2t\pi)),$$

which is continuous, whose image is $T^2_{R,r}$, and which has the property that

$$\tilde{f}(t_0, s_0) = \tilde{f}(t_1, s_1) \Longleftrightarrow (t_0, s_0) \sim (t_1, s_1).$$

As in the case of the Moebius band (... but do fill in the details!), we deduce that $\tilde{f}$ induces a homeomorphism f between T^2 and $T^2_{R,r} \subset \mathbb{R}^3$.

One more comment about the torus: we have defined it abstractly (as a quotient), and then we identified it with its version in $\mathbb{R}^3$ which is easy to visualize- the doughnut. However, as a

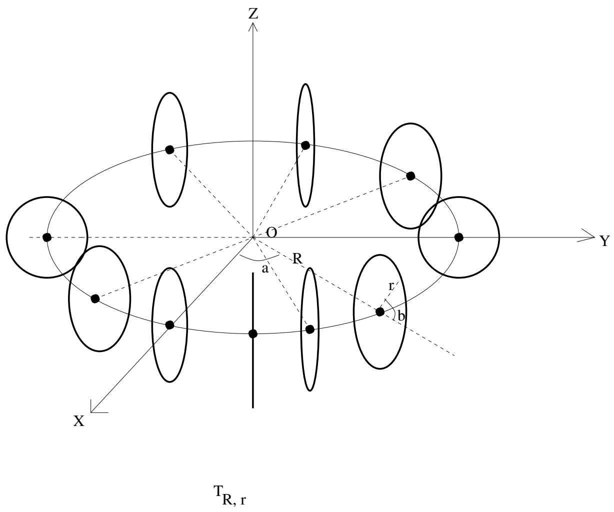


FIGURE 6.

topological space, one has an even simpler description: it is just the product of two circles. To see that T^2 is homeomorphic to $S^1 \times S^1$, one starts with

$$\tilde{g} : [0,1] \times [0,1] \longrightarrow \mathbb{R}^3, \tilde{g}(t,s) = (e^{2\pi t}, e^{2\pi s}),$$

which has properties similar to those of $\tilde{f}$, allowing us to conclude that $\tilde{g}$ induces a homeomorphism g between T^2 and $S^1 \times S^1$.

5.4. Example: the Klein bottle. The next in this list of example is another quotient of $X = [0,1] \times [0,1]$, obtained by identifying the opposite sides but reversing the orientation for one of them, as indicated in Figure 7. The resulting topological space is called the Klein bottle, denoted here by K. Trying to repeat what we have done in the previous examples, we have trouble when "twisting the cylinder". That is not really a problem; instead, it indicates that K cannot be pictured in $\mathbb{R}^3$ or, more precisely, that K cannot be embedded in $\mathbb{R}^3$ (but we will not prove that here). Instead, K can be embedded in $\mathbb{R}^4$, and the argument is very similar to the ones given in the previous exercises: consider the map

$$\tilde{f} : [0,1] \times [0,1] \longrightarrow \mathbb{R}^4,$$

$$\tilde{f}(t,s) = ((2 + 2cos(2\pi s)cos(4\pi t), (2 + 2cos(2\pi s)sin(4\pi t), sin(2\pi s)cos(2\pi t), sin(2\pi s)sin(2\pi t)).$$

Proceeding as in the previous examples, we deduce obtain a homeomorphism from K to the subspace $\tilde{f}(X) \subset \mathbb{R}^4$.

5.5. Example: the ball and the sphere. The general constructions of quotients, such as the quotient by collapsing a subspace to a point, the cone construction and the suspension construction, are nicely illustrated by the various relations between the closed unit balls $D^n \subset \mathbb{R}^n$ and the unit spheres $S^n \subset \mathbb{R}^{n+1}$. We mention here the following:

(i) D^n is homeomorphic to Cone(S^{n-1})- the cone of S^{n-1}.
(ii) S^n is homeomorphic to S(S^{n-1})- the suspension of S^n.
(iii) S^n is homeomorphic to D^n/S^{n-1}- the space obtained from D^n by collapsing its boundary to a point.

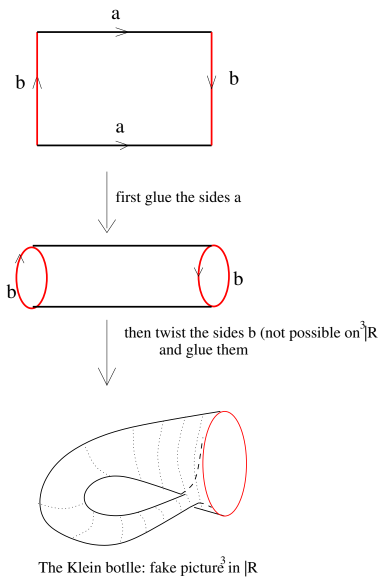


FIGURE 7.

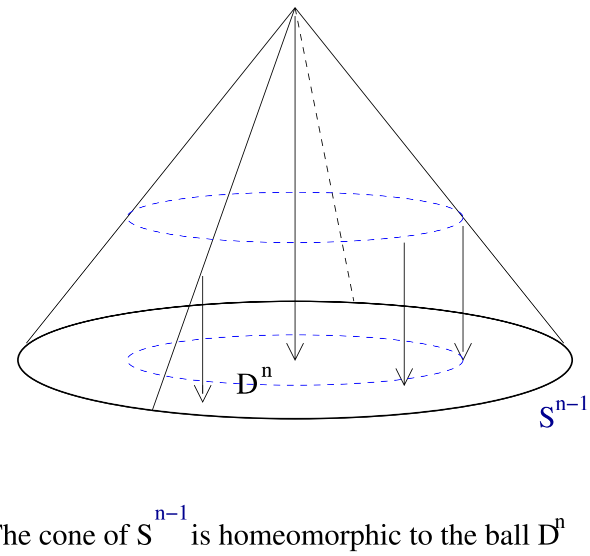


FIGURE 8.

The first homeomorphism is indicated in Figure 8 (project the cone down to the disk). It is not difficult to make this precise: we have a map

$$\tilde{f}: S^{n-1} \times [0,1] \longrightarrow D^n, \tilde{f}(x,t) = (1-t)x.$$

This is clearly continuous and surjective, and it has the property that

$$\tilde{f}(x,t) = \tilde{f}(x',t') \Longleftrightarrow (x,t) = (x',t') \text{ or} t = 1,$$

which is precisely the equivalence relation corresponding to the the quotient defining the cone. Since the cone of S^{n-1} is compact (as a quotient of a compact) and D^n is Hausdorff, we can apply Corollary 10.26 to deduce that $\tilde{f}$ induces a homeomorphism

$$f : \mathrm{Cone}(S^{n-1}) = S^{n-1} \times [0,1]/S^{n-1} \times \{1\} \longrightarrow D^n.$$

Note that this f sends $S^{n-1} \times \{1\}$ to the boundary of D^n, hence (ii) will follow from (iii). In turn, (iii) is clear on the picture (see Figure 9); the map from D^n to S^n indicated on the picture can be written explicitly as

$$\tilde{g} : D^n \longrightarrow S^n, x \mapsto (\frac{x_1}{||x||} sin(\pi||x||), \ldots, \frac{x_n}{||x||} sin(\pi||x||), cos(\pi||x||))$$

(well defined for $x \neq 0$) and which sends 0 to the north pole $(0, \ldots, 0, 1)$.

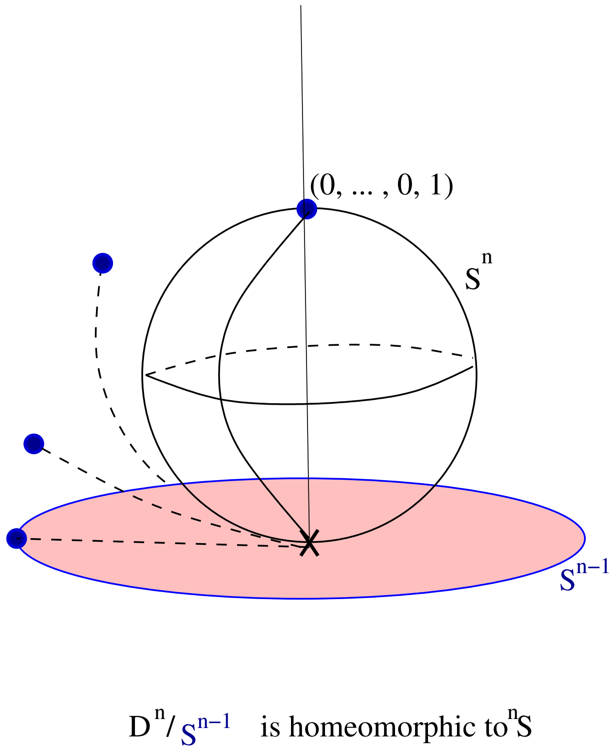


FIGURE 9.

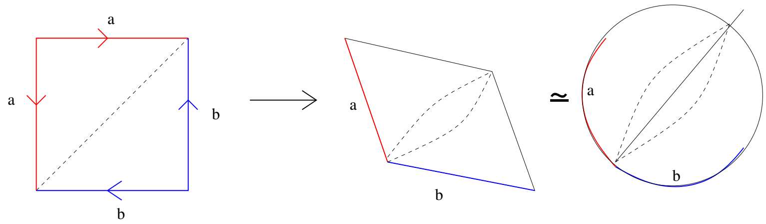
One can check directly that

$$\tilde{g}(x) = \tilde{g}(x') \Longleftrightarrow x = x' \text{ or } x, x' \in S^{n-1},$$

which is the equivalence relation corresponding to the quotient D^n/S^{n-1}. Applying again Corollary 10.26 (which is fine because D^n is compact and S^{n+1} is Hausdorff), we end up with the desired homeomorphism from D^n/S^{n-1} to S^n. In the spirit of the 2-dimensional examples presented above, note that D^2 is homeomorphic to the square $[0,1]\times[0,1]$, while S^2 can be obtained from the square by identifying two the sides a and the two sides b as shown in the picture (Figure 10). Indeed,

$$f : [0,1] \times [0,1] \longrightarrow D^2, f(x,y) = \frac{\max\{x,y\}}{\sqrt{x^2+y^2}}(x,y)$$

is a bijective continuous map, hence a homeomorphism (compact domain, and Hausdorff codomain!). The description of S^2 is indicated on the picture: by the gluing of the sides and and of the sides b, one obtains the suspension of S^1, which is homeomorphic to S^2.



The sphere S^2 obtained from the square by identifying its sides

FIGURE 10.

5.6. Example: the projective space $\mathbb{P}^n$. Let us summarize what we already know for $\mathbb{P}^n$:

1. We have shown (in Example 10.13) that it can be obtained from S^n by identifying the antipodal points, or, in other words, that it can be written as a quotient of the sphere modulo the action of $\mathbb{Z}_2$:
$$\mathbb{P}^n = S^n/\mathbb{Z}_2.$$
2. We have remarked that $\mathbb{P}^n$ is compact and (path) connected.
3. Using the general result about quotients by finite group actions (Lemma 10.22), we deduced that $\mathbb{P}^n$ is Hausdorff.
4. We have seen (see Example 10.28) that, for each n, there is a large enough N such that $\mathbb{P}^n$ can be embedded in $\mathbb{R}^N$. This also provides a (simpler) proof that $\mathbb{P}^n$ is Hausdorff. The "large N" that we found was $N = \frac{(n+1)(n+2)}{2}$, but it is in general to the smallest possible choice (see also below).

There is one more thing that we would like to mention here:

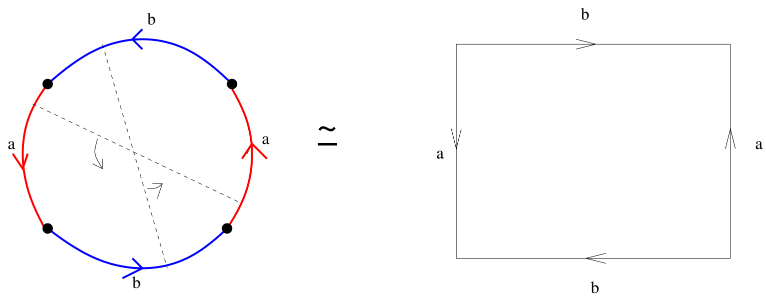
1. $\mathbb{P}^n$ is homeomorphic also with the space obtained from $\mathbb{D}^n$ by identifying the antipodal points on the boundary S^{n-1}, i.e.
$$\mathbb{P}^n = D^n/\sim$$
where $\sim$ is defined by
$$x \sim y \Longleftrightarrow (x = y) \text{ or } (x, y \in S^{n-1}, x = -y).$$

To prove this, we consider the surjection
$$\tilde{f} : D^n \longrightarrow \mathbb{P}^n,$$
which sends $x = (x_1, \ldots, x_n)$ to $\pi(\sqrt{1 - ||x||^2}, x)$, where π is the quotient map associated to π (either the one defined on $\mathbb{R}^{n+1} - \{0\}$, or its restriction to S^n). $\tilde{f}$ is continuous because it is the composition of π with a continuous map. Moreover, it is easy to see that
$$\tilde{f}(x) = \tilde{f}(y) \Longleftrightarrow x \sim y,$$
where $\sim$ is the equivalence relation on $\mathbb{D}^n$ defined above. Using again Corollary 10.26, we deduce the desired homeomorphism.

Although the proof itself is important, it is equally important to have in mind the intuitive idea behind this result and behind the proof. Here it is (See Figure 12). Think of $\mathbb{P}^n$ as obtained from S^n by identifying the antipodal points. Look at S^{n-1} inside S^n as the big circle
$$S^{n-1} \cong \{(x_0, x_1, \ldots, x_n) \in S^n : x_n = 0\} \subset S^n,$$

and we consider the two components that are left after removing this big circle: the north hemisphere S^n_+ and the south hemisphere S^n_-. Any point in S^n_- is identified with one in S^n_+, so one could just not use S^n_- when describing $\mathbb{P}^n$. This indicates that $\mathbb{P}^n$ can be obtained from the closed hemisphere $\overline{S^n_+} = S^n_+ \cup S^{n-1}$ by identifying only the antipodal points sitting on S^{n-1}. On the other hand, the projection on the horizontal hyperplane $(x_n = 0)$ produces a homeomorphism between $\overline{S^n_+}$ and D^n (whose inverse is precisely $x \mapsto (\sqrt{1-||x||^2}, x)$ used in the proof), hence the description above.

A few more words about the case $n = 2$, i.e. about the projective plane $\mathbb{P}^2$. First of all, in the spirit of the 2-dimensional examples presented above, note that $\mathbb{P}^2$ can be obtained (or is homeomorphic to the space obtained) by starting from the square $X = [0,1] \times [0,1]$ and identifying the opposite sides as shown in Figure 11; this follows from the description of $\mathbb{P}^2$ as obtained from D^2 by identifying the opposite points of its boundary, and the homeomorphism between D^2 and the square (e.g. the one described in the previous example).



The projective plane $|P^2$ obtained from the square by identifying opposite sides

FIGURE 11.

As in the case of the Klein bottle, it is difficult to picture $\mathbb{P}^2$ because it cannot be embedded in $\mathbb{R}^3$. By the general discussion above we know that it can be embedded in $\mathbb{R}^6$, but we can do better: it can be embedded in $\mathbb{R}^4$. To see this one uses

$$\tilde{f} : S^2 \longrightarrow \mathbb{R}^4 \ , \tilde{f}(x,y,z) = (x^2 - y^2, xy, xz, yz).$$

and proceeds exactly as in Example 10.28.

Still related to the previous two-dimensional examples, we mention here that the Moebius band can be embedded in the projective plane:

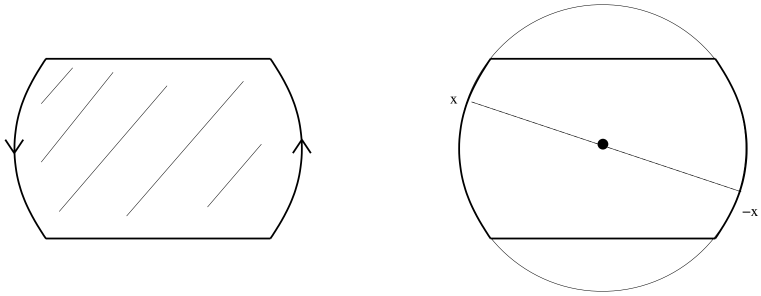
$$M \hookrightarrow \mathbb{P}^2.$$

To see this, realize again $\mathbb{P}^2$ as $D^2/\sim$- obtained from D^2 by identifying the antipodal points on $\partial D^2 = S^1$. Consider inside D^2 the "band"

$$B = \{(x,y) \in D^2 : -\frac{1}{2} \le y \le \frac{1}{2}\}. \subset D^2.$$

Considering also the restriction of the equivalence relation we are using to B, it corresponds to identifying the "opposite curved sides" of B as in the picture (Figure 12), and gives us the Moebius band. On the other hand, we have the obvious set-theoretical inclusion $B/\sim \subset D^2/\sim$, and we claim that this inclusion is a topological one, i.e., the quotient topology on $B/\sim$ (that is how the topology of the Moebius band is defined!) coincides with the restriction of the topology coming from $D^2/\sim$. But that follows again from Proposition 10.25: the restriction of the quotient map $\pi : D^2 \longrightarrow D^2/\sim$ to B gives a continuous surjection from $\pi|_B$ from B

to $\pi(B) = B/\sim$ (where the last space is endowed with the restriction topology). Since the proposition implies that $\pi|_B$ is a quotient map, the two topologies on $B/\sim$ coincide.



The Moebius band inside the projective plane $\mathbb{P}^2$

FIGURE 12.

5.7. Example: the cone of an open interval (or: how one has to be careful with the intuition). In part (iii) of Proposition 10.25, as well as in Corollary 10.26, the hypothesis that X is compact is essential. Here is an example which also shows that care is required when working with pictures. Take first $Y = \mathrm{Cone}([0,1])$, i.e. start with the square $X = [0,1] \times [0,1]$, and we collapse $[0,1] \times \{1\}$ to a point. In other words, $Y = X/\sim$ where $\sim$ is the equivalence relation on X given by

$$(t_0, s_0) \sim (t_1, s_1) \Longleftrightarrow (t_0, s_0) = (t_1, s_1) \text{ or } t_0 = t_1 = 1.$$

Intuitively, the identification process is described in the picture and produces a subspace of $\mathbb{R}^2$, which is just a closed triangle as shown in the picture, and which we denote by T. This indicates that Y can be embedded in $\mathbb{R}^2$ and can be identified (as a topological space!) with the triangle. The intuition is correct, but only because X is compact! (see below what can go wrong). Let us prove this properly: we consider the map

$$\tilde{f} : [0,1] \times [0,1] \longrightarrow T, \tilde{f}(t,s) = (t + s - ts, s)$$

(which is the intuition behind this formula?), which is a continuous surjective map. It is not difficult to check that

$$\tilde{f}(t_0, s_0) = \tilde{f}(t_1, s_1) \Longleftrightarrow (t_0, s_0) \sim (t_1, s_1).$$

Hence, using the previous corollary, we deduce that $\tilde{f}$ induces a homeomorphism between Y and the triangle T.

To see how essential the compactness of X is, let's now consider $Y' = \mathrm{Cone}((0,1))$. This means that we remove from X the vertical sides and consider

$$X' = (0,1) \times [0,1],$$

and Y' is obtained from X' by collapsing $(0,1) \times \{1\}$ to a point. Then all we said about X can be said about the X' (and T will be replaced by T'- which is T from which we remove the non-horizontal sides, without removing the common vertex), except one thing: we cannot apply Proposition 10.25 (or its corollary). That is no accident: what actually happens is that the map

$$\tilde{f} : [0,1] \times [0,1] \longrightarrow T',$$

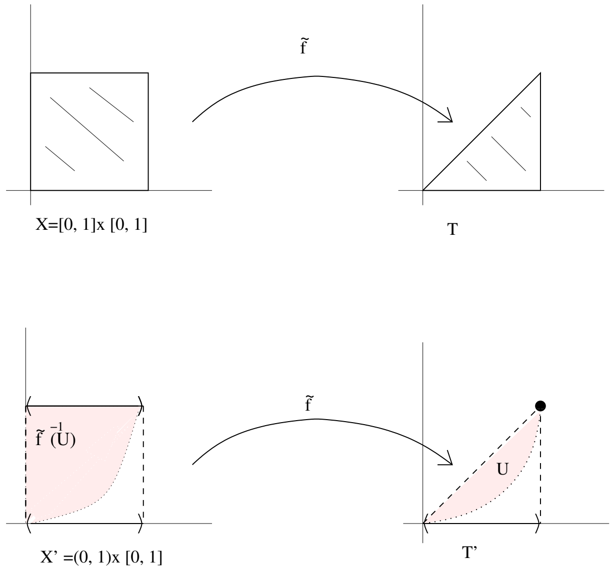


FIGURE 13.

although it is continuous and surjective, the induced quotient topology on T' does not coincide with the topology on T' as a subspace of $\mathbb{R}^2$. To see this, consider

$$U = (\overset{\circ}{B}((1,0),1) \cap X') \cup \{(1,1)\}.$$

Then $\tilde{f}^{-1}$ is open in X', hence U is open in T' with the quotient topology. However, U is clear not an open in the topology induced on T' by the Euclidean topology. In particular, this discussion shows that $\tilde{f}$ induces a continuous bijection

$$f : \mathrm{Cone}((0,1)) \longrightarrow T'$$

which is not a homeomorphism. Actually, one can show that Cone((0, 1)) is not homeomorphic to any subspace of (i.e. cannot be embedded in) any of the Euclidean spaces $\mathbb{R}^n$ (it is not even metrizable).

6. Exercises

EXERCISE 10.2. Consider the additive group $\mathbb{Z}$ acting on $\mathbb{R}$ by translations:

$$\phi_n(r) = n + r, \quad \text{for } n \in \mathbb{Z}, r \in \mathbb{R}.$$

Show that the resulting quotient is homeomorphic to S^1.

EXERCISE 10.3. Consider $X = [0, 2]$ and let Y be the space obtained from X by identifying 0 with 1, and 1 with 2. Describe Y explicitly.

EXERCISE 10.4. Let X be a topological space and assume that we find a compact Hausdorff topological space Y which contains X as an open dense subset. Let $A = Y - X$. Prove that X is locally compact Hausdorff and that the one-point compactification of X is (homeomorphic to) Y/A. Next, using this, describe the one-point compactification of $X = [0,1] \times [0,1)$.