Topology and Geometry B (4 July, 2007)

Exercise 1. (*2.5 points exercise*) Which of the following pairs of spaces are homeomorphic. In each case explain your answer.

(i) $S^1 \times S^1$ and S^2. *(0.5p)*

(ii) $S^1 \times S^1$ and T^2. *(0.5p)*

(iii) S^2 and S^3. *(0.5p)*

(iii) $D^2 \times D^2$ and D^4. *(0.5p)*

(v) $S^2 \times S^2$ and S^4. *(0.5p)*

Exercise 2. (*1 point exercise*) Let $f : [0,1] \longrightarrow S^1$ be a continuous map with $f(0) = f(1) = 1$. If $deg(f) = n$, prove that the equation

$$f(x) = 1$$

has at least $n+1$ solutions.

Exercise 3. (*2.5 points exercise*) Recall that, given a topological space X and a subset $A \subset X$, a retraction of X into A is a continuous map $r : X \longrightarrow A$ with the property that $r(a) = a$ for all $a \in A$.

(i) Prove that if r is a retraction of X into A, then the map induced in the fundamental groups:

$$r_* : \pi(X, a) \longrightarrow \pi(A, a)$$

is surjective for all $a \in A$. What about the map induced by the inclusion of A into X? *(1p)*

(ii) Prove that the Moebius band cannot be retracted into its boundary. *(1.5p)*

Exercise 4. (*4 points exercise*) Consider the square $X = [0,1] \times [0,1]$ and let A be the subset consisting of its corners:

$$A = \{(0,0), (0,1), (1,0), (1,1)\}.$$

Consider the quotient space:

$$Y = X/A$$

(with the induced quotient topology).

(i) Make a picture of Y. *(0.5p)*

(ii) PROVE that, indeed, Y can be embedded in $\mathbb{R}^3$. *(1p)*

(iii) Describe a cell-decomposition of Y. *(1p)*

(iv) Compute the fundamental group of Y. *(1.5p)*