INTRO TO TOPOLOGY

-1-

Course	Main Objects	Maps	Isomorphisms	Key words
Group Theory	Groups $(G, \circ)$	Group morphisms $f:(G,\circ)\to(G',\circ')$	f is also bijective	multiplication, Subgroups, order of groups & of elements
Rings	Rings $(R,+,\cdot)$	Ring morphisms $f: R\to R'$	f is also bijective	
Linear ALGEBRA	Vector spaces (or just $\mathbb{R}^n$)	Linear maps f (matrices)	f is also bijective	matrices, dimension, norms, bases
ANALYSIS	opens $\subseteq \mathbb{R}^n$	Smooth maps	?	?
TOPOLOGY	Topological Spaces X	Continuous maps $f: X\to X'$	f is bijective & f^{-1} is continuous	convergence, open/closed subsets

1'

Temporary (& sloppy) def: set X & extra structure to be able to talk about convergence and continuity.

Called homeomorphisms

Plan: basic concepts
constructions
topological properties

$[0,1]$ $(0,1)$

-4-

Ex3: Any $X \subseteq \mathbb{R}^n$ is naturally a metric space together with the Euclidean metric. Some look different but they are homeomorphic

Ex1: $[0,1]$ homeom. to $[0,2]$

Ex2: $(0,1)$, $(0,\infty)$, $\mathbb{R}$ all homeom.

Ex3: $\mathbb{R}$ and $\mathbb{R}^2$ are not homeomorphic.

Example

INTRO TO TOPOLOGY

-1-

Course	Main Objects	Maps	Isomorphisms	Key words
Group Theory	Groups $(G, \circ)$	Group morphisms $f:(G,\circ)\to(G',\circ')$	f is also bijective	multiplication, subgroups, orders of groups & of elements
Rings	Rings $(R,+,\cdot)$	Ring morphisms $f: R\to R'$	f is also bijective	
Linear ALGEBRA	Vector spaces (or just $\mathbb{R}^n$)	Linear maps f (matrices)	f is also bijective	matrices, dimension, norms, bases
ANALYSIS	opens $\subseteq \mathbb{R}^n$	Smooth maps	?	?
TOPOLOGY	Topological Spaces X	Continuous maps $f: X\to X'$	f is bijective & f^{-1} is continuous	convergence, open/closed subsets

First metr

&

Conver

Conti

1'

emporary (& sloppy) def : set X

called

Example 4 :

ords

ation

ys &

ements

imension,

ses

d subsets

First (precise & pretty good) attempt:

metric spaces (X, d) set

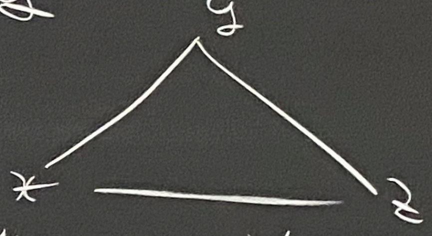
& $d : X \times X \to \mathbb{R}$ metric on X i.e.:

(M1) $d(x,y) \geq 0 \quad (\forall)\, x, y \in X$

(M2) $d(x,y) = 0 \iff x = y$

(M3) $d(x,y) = d(y,x)$

(M4) $d(x,y) + d(y,z) \geq d(x,z) \quad (\forall)\, x, y, z \in X$



Convergence: sequence $(x_k)_{k \geq 1}$, x in X

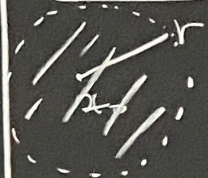
$\lim_{k \to \infty} x_k = x$ in (X,d) if $\lim_{k \to \infty} d(x_k, x) = 0$.

Continuity: $f : (X,d) \to (X', d')$ continuous if:

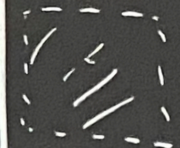
$\lim_{k \to \infty} x_k = x$ in $(X,d) \implies \lim_{k \to \infty} f(x_k) = f(x)$ in (X', d')

Ex1: A

Ex2:



Ex3:



Rk:

$(X, d$

$r > 0$,

ttempt

e

o

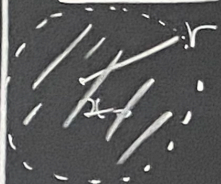
if:

(X', d')

Ex 1: Any set X carries the discrete metric

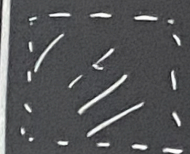
$$d(x,y) = \begin{cases} 0 & x = y \\ 1 & \end{cases}$$

Ex 2: $X = \mathbb{R}^n$ with Euclidean metric



$$d(x,y) = \sqrt{(x_1 - y_1)^2 + \dots + (x_n - y_n)^2} \qquad x = (x_1, \dots, x_n)$$

Ex 3: $X = \mathbb{R}^n$ with square metric ρ



$$\rho(x,y) = \max\{|x_1 - y_1|, \dots, |x_n - y_n|\}$$

Rk: they give rise to the same notion of convergence.

(X, d)

$r > 0$, $x_0 \in X$

$$B_d(x_0, r) = \{x \in X : d(x, x_0) < r\}$$

TOPOLOGY | Topological Spaces X | Continuous maps $f: X \to X'$ | f is bijective & f^{-1} is continuous | convergence open/closed subs

1'

Temporary (& sloppy) def: set X & extra structure to be able to talk about convergence and continuity.

called homeomorphisms

Exam

Plan: basic concepts
constructions.
topological properties

$[0,1]$ $(0,1)$

-4-

Ex 3: Any $X \subseteq \mathbb{R}^n$ is naturally a metric space together with the Euclidean metric. Some look different but they are homeomorphic

Ex 1: [0,1] homeom. to $[0,2]$

Ex 2: $(0,1)$, $(0,\infty)$, $\mathbb{R}$ all homeom.

Ex 3: $\mathbb{R}$ and $\mathbb{R}^2$ are not homeomorphic.

given by formulas?

$$\left\{\left(\left(R+u\sin\frac{\alpha}{2}\right)\cos\alpha,\ \left(R+u\sin\frac{\alpha}{2}\right)\sin\alpha,\ u\cos\frac{\alpha}{2}\right)\right.$$

EOMORPHISMS:

$X \to X'$

ective

f^{-1} CONTINUOUS

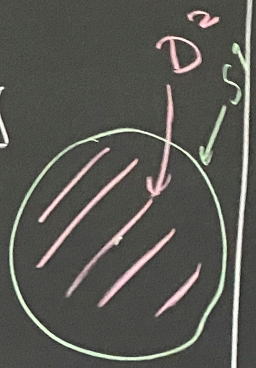
$\cos\alpha, \sin\alpha)$

-5-

Example 4: circles

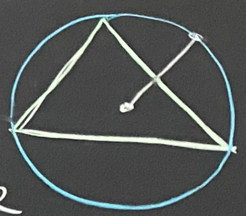
The unit circle $S^1=\{(x,y)\in\mathbb{R}^2 / x^2+y^2=1\}$

A circle: any topological space that is homeomorphic to S^1.



For instance:

- $S^1_r=\{(x,y)\in\mathbb{R}^2 / x^2+y^2=r^2\}$
- . . . also a triangle is a ... circle



- bended circles
- knotted circle.
- any walk without self intersections, returning to the start point.

• what

end

• S^1

S^1 is

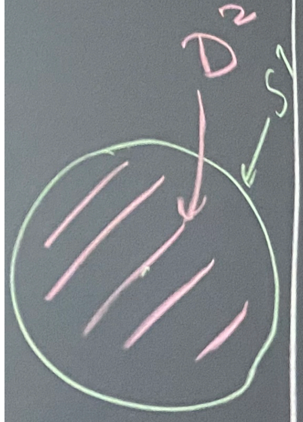
f

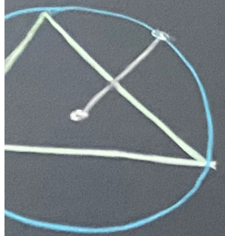
f(

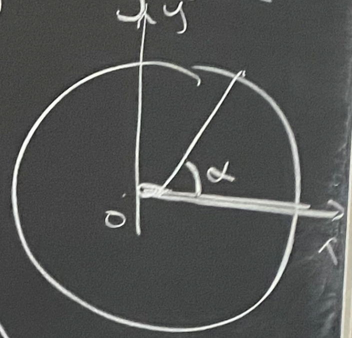
$\longrightarrow$ Klein bottle

-6-

- What you get from $[0,1]$ by gluing the end points $0, 1$







- $$S^1 = \{(\cos\alpha, \sin\alpha) \,/\, \alpha \in \mathbb{R}\}$$
$$= \{(\cos\alpha, \sin\alpha) \,/\, \alpha \in [0, 2\pi]\}$$
$$= \{(\cos\alpha, \sin\alpha) \,/\, \alpha \in [0, 2\pi)\}$$

S^1 is $\mathbb{R}$ modulo $2\pi\mathbb{Z}$

S^1 is obtained from $[0, 2\pi)$ by gluing 0 & 2π

$$f : [0, 2\pi) \longrightarrow S^1$$
$$f(\alpha) = (\cos\alpha, \sin\alpha)$$

continuous bijection but not homeomorphism able to talk about convergence and continuity.

bijective

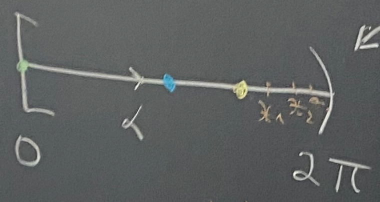
f & f^{-1} CONTINUOUS

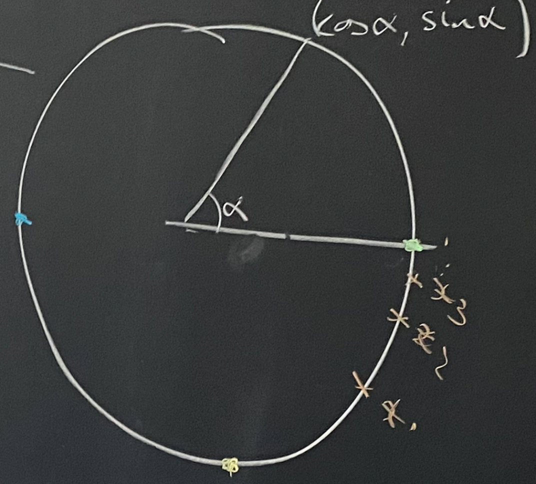
A circle

For instan

- $S^1_r =$
- . . .
- bended
- knotted
- any wal

$\boxed{-6'-}$





Not continuous.