

Reminder: -1-

- topological spaces: about convergence & continuity
- 1st attempt: metric spaces (X, d)
- various examples... and:

☞ Different metrics on a X may give rise to same convergence & contin.

☞ The (cool!) process of gluing may take us out of metric spaces.

Key remark: notion of open -2-

Def: (X, d) metric space:

• for $x \in X, \varepsilon > 0$: $B_d(x, \varepsilon) := \{y \in X / d(x, y) < \varepsilon\}$

• a set $U \subset X$ is said to be open in (X, d) if

$$(\forall) x \in U, \exists \varepsilon > 0 \text{ s.t. } B_d(x, \varepsilon) \subseteq U$$

• the collection of all such opens is denoted \mathcal{T}_d , and called the topology induced by d .

Prk: $U \subset X$ is open $\iff U$ can be written as a union of balls.

\Rightarrow $(\forall) x \in U \exists \varepsilon_x \text{ s.t. } B_d(x, \varepsilon_x) \subseteq U$

$$\Rightarrow \{x\} \subseteq B_d(x, \varepsilon_x) \subseteq U \text{ Apply } \bigcup_{x \in U} \Rightarrow U = \bigcup_{x \in U} \{x\} \subseteq \bigcup_{x \in U} B_d(x, \varepsilon_x) \subseteq U$$

1.3.5: Two metrics d and d' on X induce the same convergence if and only if $\mathcal{T}_d = \mathcal{T}_{d'}$.

$$f: X \longrightarrow \underset{U}{Y} \quad f^{-1}(U) = \{x \in X / f(x) \in U\}$$

$$(X \setminus U_1) \cap (X \setminus U_2) = X \setminus (U_1 \cup U_2)$$

$$(X \setminus U_1) \cup (X \setminus U_2) = X \setminus (U_1 \cap U_2)$$

DEFINITIONS

(prescribes the OPENS of the space (X, \mathcal{T})) [-3-]

• a topology on a set X : a collection \mathcal{T} of subsets of X satisfying:

(T1) $\emptyset, X \in \mathcal{T}$ | \emptyset, X are open

(T2) intersection of two members of \mathcal{T} is in \mathcal{T} | intersection of two opens is open
 $U_1, U_2 \in \mathcal{T} \Rightarrow U_1 \cap U_2 \in \mathcal{T}$

(T3) union of any collection of members of \mathcal{T} is in \mathcal{T} : | arbitrary unions of opens are open
 $U_i \in \mathcal{T} \forall i \in I \Rightarrow \bigcup_{i \in I} U_i \in \mathcal{T}$
index set

• a topological space: (X, \mathcal{T}) set, \mathcal{T} -topology on X

• given (X, \mathcal{T}) :

→ a subset $U \subseteq X$ called open in (X, \mathcal{T}) if $U \in \mathcal{T}$.

→ a subset $A \subseteq X$ called closed in (X, \mathcal{T}) if $X \setminus A \in \mathcal{T}$.

Given two topologies $\mathcal{T}_1, \mathcal{T}_2$ on X , say \mathcal{T}_1 is smaller than \mathcal{T}_2 if $\mathcal{T}_1 \subseteq \mathcal{T}_2$.

Example

1. \mathcal{T}
- 2.
- 3.
- 4.

• given top. spaces $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$ [-4-]

continuous fct. $f: (X, \mathcal{T}_X) \rightarrow (Y, \mathcal{T}_Y)$ is

$\exists \neq \emptyset \cap \cdot V \rightarrow Y$ s.t.

Example 5: metric topologies

Prop Given (X, d) metric space \mathcal{T} is a topology

• given (X, \mathcal{T}) :

→ a subset $U \subseteq X$ called open in (X, \mathcal{T}) if $U \in \mathcal{T}$.

→ a subset $A \subseteq X$ called closed in (X, \mathcal{T}) if $X \setminus A \in \mathcal{T}$.

• given top. spaces $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$ -4-
 continuous fct. $f: (X, \mathcal{T}_X) \rightarrow (Y, \mathcal{T}_Y)$ is
 any function $f: X \rightarrow Y$ s.t.

$$f^{-1}(U) \in \mathcal{T}_X \quad (\forall) \quad \underbrace{U \in \mathcal{T}_Y}_{U \in \mathcal{B}}$$

Rk: (T2) for arbitrary intersections would
 be too much to ask: $\bigcap_{n \in \mathbb{N}} \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}$

pf of Prop (b) This says that: if \mathcal{T}
 topology on X s.t. $\mathcal{B}, (x, \varepsilon) \in \mathcal{T} \quad (\forall) \quad x$

Exam

Prop

pf: (T2)

Examples 1-4: On any set X :

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1. $\mathcal{T}_{\text{triv}}$: consists only of \emptyset, X .

2. \mathcal{T}_{dis} : — " — of all the subsets of X

3. \mathcal{T}_{cf} (cofinite): \emptyset and $U = X \setminus F$ with F finite

4. \mathcal{T}_{cc} (cocountable): \emptyset and $U = X \setminus C$
with $C \subseteq X$ at most countable

$$\mathcal{T}_{\text{triv}} \subseteq \mathcal{T}_{\text{cf}} \subseteq \mathcal{T}_{\text{cc}} \subseteq \mathcal{T}_{\text{dis}}$$

Why is \mathcal{T}_{cf} a topology?

$$(T2) \quad U_1, U_2 \in \mathcal{T}_{\text{cf}} \Rightarrow \begin{cases} U_1 = \emptyset \text{ or } U_1 = X \setminus F_1 \text{ with } F_1 \text{ finite} \\ U_2 = \emptyset \text{ or } U_2 = X \setminus F_2 \text{ — " — } F_2 \text{ — " —} \end{cases}$$

$$\Rightarrow U_1 \cap U_2 = \emptyset \text{ or } U_1 \cap U_2 = X \setminus \underbrace{(F_1 \cup F_2)}_{\text{finite}} \Rightarrow U_1 \cap U_2 \in \mathcal{T}_{\text{cf}}$$

(T3) similarly.

Examples 6-7-8: on \mathbb{R} , next to $\mathcal{T}_{\text{triv}}, \mathcal{T}_{\text{dis}}, \mathcal{T}_{\text{cf}}, \mathcal{T}_{\text{cc}}$

is a more natural one

$\tau_{\mathbb{R}} = \{U \subseteq \mathbb{R} : U = \emptyset \text{ or } U = \bigcup_{i \in \mathbb{N}} (a_i, b_i) \text{ for some } a_i, b_i \in \mathbb{R}\}$ induced by

Example 5: metric topologies

Prop ^{Given} (X, d) metric space:

- (a) \mathcal{T}_d is a topology
- (b) \mathcal{T}_d is the smallest topology on X that contains all the balls $B_d(x, \epsilon)$.

Prf: (a) \mathcal{T}_3 follows from Remark

(T2) $U_1, U_2 \in \mathcal{T}_d \implies U_1 \cap U_2 \in \mathcal{T}_d$

i.e. to prove

$$\forall x_1 \in U_1 \exists \epsilon_1 > 0 : B_d(x_1, \epsilon_1) \subseteq U_1$$

$$\forall x_2 \in U_2 \exists \epsilon_2 > 0 : B_d(x_2, \epsilon_2) \subseteq U_2$$

Apply \uparrow for $x_1 = x, x_2 = x \implies \exists \epsilon_1, \epsilon_2 > 0$ st $B(x, \epsilon) \subseteq U_1$

$\Rightarrow \{a\} \in \overline{J_0} \forall a \in \mathbb{R} \Rightarrow$ any $A \subseteq \mathbb{R}$ must be closed.

Interesting question: given (X, \overline{J}) when
is it metrizable, i.e.,
 \exists metric d s.t. $\overline{J} = \overline{J}_d$?

$$A = \bigcup_{a \in A} \{a\}$$

Example: Similarly on \mathbb{R}^n and on any $A \subseteq \mathbb{R}^n$ we
 $d_{\text{Eucl}}: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$

(T3) similarly.

finite

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-6- Examples 6-7-8: on \mathbb{R} , next to \mathcal{T}_{triv} , \mathcal{T}_{dis} , \mathcal{T}_{cf} , \mathcal{T}_{cc} there is a more natural one:

$\mathcal{T}_{Eucl}^{\mathbb{R}}$ = the topology induced by the Euclidean metric $d_{Eucl}^{\mathbb{R}}(x, y) = |x - y|$
= the smallest topology on \mathbb{R} containing all open intervals

Hence, for $U \subseteq \mathbb{R}$:

$U \in \mathcal{T}_{Eucl}^{\mathbb{R}} \iff U$ is a union of open intervals

$\iff (\forall) x \in U, \exists$ open interval (a, b) s.t. $x \in (a, b) \subseteq U$.

prove: $(\forall) x \in U_1 \cap U_2, \exists \varepsilon > 0$ s.t. $B(x, \varepsilon) \subseteq U_1 \cap U_2$,
fix x $U = [a, b) \in \mathcal{T}_e$

$\varepsilon_1 \in U_1, B(x, \varepsilon_1) \subseteq U_1. \varepsilon_2 \in U_2, B(x, \varepsilon_2) \subseteq U_2. \text{ Take } \varepsilon = \min\{\varepsilon_1, \varepsilon_2\} \implies B(x, \varepsilon) \subseteq U_1 \cap U_2 \square$

Questions:

(i) what if we replace open intervals by intervals of type $[a, b)$? \Rightarrow a topology \mathcal{T}_ℓ
 lower limit topology.

(ii) " " " " $[a, b]$? \Rightarrow a topology \mathcal{T}_u

(iii) " " " " $[a, b]$? \Rightarrow get a topology \mathcal{T}_o
 but not really new

Cute remarks:

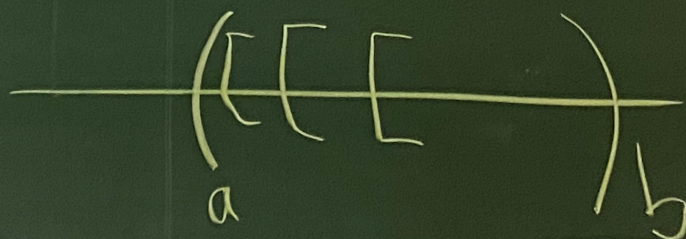
(i) $\mathcal{T}_{\text{Eucl}} \subseteq \mathcal{T}_\ell$

$(a, b) \in \mathcal{T}_\ell \quad \forall a, b$

$\mathcal{T}_{\text{Eucl}}$ = smallest containing (a, b)

$\Rightarrow \mathcal{T}_{\text{Eucl}} \subseteq \mathcal{T}_\ell$

$(a, b) = \bigcup_{n \in \mathbb{N}} [a + \frac{1}{n}, b) \in \mathcal{T}_\ell$



(iii) $[a-1, a] \cap [a, a+1] = \{a\} \Rightarrow$

$\Rightarrow \{a\} \in \mathcal{T}_o \quad \forall a \in \mathbb{R} \Rightarrow$ any $A \subseteq \mathbb{R}$ must be in $\mathcal{T}_o \Rightarrow \overline{\mathcal{T}_o} = \mathcal{T}_{\text{disc}}$

$$A = \bigcup_{a \in A} \{a\}$$

Example: Similarly on \mathbb{R}^n and on any $A \subseteq \mathbb{R}^n$, using

$$d_{\text{Eucl}}^{\mathbb{R}^n} : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}, \quad d_{\text{Eucl}}^A = d_{\text{Eucl}}^{\mathbb{R}^n} \Big|_{A \times A} : A \times A \rightarrow \mathbb{R}$$

$$\Rightarrow (\mathbb{R}^n, J_{\text{Eucl}}), \quad (A, J_{\text{Eucl}}^A)$$

Example 9: Given (X, \mathcal{J}) a space } we have an induced topology on A
 $A \subseteq X$

Exercise: check this is a topology.

$$\mathcal{J}|_A := \{A \cap U : U \in \mathcal{J}\}$$

Def: Given (X, \mathcal{J}) , $A \subseteq X$, a subset $B \subseteq A$ is called:

- B is open in A if $B \in \mathcal{J}|_A$ (if B is open in $(A, \mathcal{J}|_A)$)
- B is closed in A if $A \setminus B \in \mathcal{J}|_A$ (if B is closed in $(A, \mathcal{J}|_A)$)

Ex. $[0, 1) \subseteq \mathbb{R}$

open in $[0, \infty)$

closed in $(-1, 1)$

← 1st ex.
for next
work college.

Example 9: Given (X, \mathcal{J}) a space $\{ \dots \}$ we have an induced topology on $A \subseteq X$

Ensure check this is a topology $\mathcal{J}|_A = \{ A \cap U \mid U \in \mathcal{J} \}$

Def: Given $(X, \mathcal{J}), A \subseteq X$, a subset $B \subseteq A$ is called:

- B is open in A if $B \in \mathcal{J}|_A$ (if B is open in $(A, \mathcal{J}|_A)$)
- B is closed in A if $A \setminus B \in \mathcal{J}|_A$ (if $A \setminus B$ is open in $(A, \mathcal{J}|_A)$)

Ex: $[0, 2] \subseteq \mathbb{R}$

- open in $[0, \infty)$
- closed in $(-1, 1)$

for next week college

DEFINITIONS (MORSE-SMITH OPENNESS)

a topology on a set X is a collection \mathcal{J} of subsets of X satisfying:

- $\emptyset, X \in \mathcal{J}$ and X are open
- intersection of two members of \mathcal{J} is in \mathcal{J}
 $U_1, U_2 \in \mathcal{J} \Rightarrow U_1 \cap U_2 \in \mathcal{J}$ (intersection of two opens is open)
- union of any collection of members of \mathcal{J} is in \mathcal{J}
 $\{U_i\}_{i \in I} \subseteq \mathcal{J} \Rightarrow \bigcup_{i \in I} U_i \in \mathcal{J}$ (arbitrary unions of opens are open)

a topological space (X, \mathcal{J}) is a set X with a topology on X

given (X, \mathcal{J}) :

- a subset $U \subseteq X$ called open in (X, \mathcal{J}) if $U \in \mathcal{J}$
- a subset $A \subseteq X$ called closed in (X, \mathcal{J}) if $X \setminus A \in \mathcal{J}$

Examples on any set X :

- $\mathcal{J}_{\text{triv}}$: consists only of \emptyset, X
- \mathcal{J}_{dis} : all the subsets of X
- \mathcal{J}_{cf} (cofinite): \emptyset and $U = X \setminus F$ with F finite
- \mathcal{J}_{cc} (cocountable): \emptyset and $U = X \setminus C$ with $C \subseteq X$ of countable cardinality

Why is \mathcal{J}_{cc} a topology?

(T1) $U, V \in \mathcal{J}_{\text{cc}} \Rightarrow (U \cap V) = X \setminus (F_1 \cup F_2)$ with F_1, F_2 countable $\Rightarrow U \cap V \in \mathcal{J}_{\text{cc}}$

(T2) $\{U_i\}_{i \in I} \subseteq \mathcal{J}_{\text{cc}} \Rightarrow \bigcup_{i \in I} U_i = X \setminus \bigcap_{i \in I} F_i$ with F_i countable $\Rightarrow \bigcup_{i \in I} U_i \in \mathcal{J}_{\text{cc}}$

(T3) similar

Interesting question given (X, \mathcal{J}) when is it metrizable i.e. \exists metric d s.t. $\mathcal{J} = \mathcal{J}_d$?

$A = \bigcup_{a \in A} \{a\}$

Example: Subsets of \mathbb{R} and any $A \subseteq \mathbb{R}^n$ using d_{Eucl} $\mathbb{R} \rightarrow \mathbb{R}, d_{\text{Eucl}} = |x - y|$ $\mathbb{R}^n \rightarrow \mathbb{R}^n, d_{\text{Eucl}} = \sqrt{\sum_{i=1}^n (x_i - y_i)^2}$

$(\mathbb{R}, \mathcal{J}_{\text{Eucl}}), (A, \mathcal{J}_{\text{Eucl}})$

given top spaces $(X, \mathcal{J}_1), (Y, \mathcal{J}_2)$ a continuous fct $f: (X, \mathcal{J}_1) \rightarrow (Y, \mathcal{J}_2)$ is any function $f: X \rightarrow Y$ s.t.

$f^{-1}(U) \in \mathcal{J}_1 \quad \forall U \in \mathcal{J}_2$

Rk: (T2) for arbitrary intersections would be too much to ask: $\bigcap_{n \in \mathbb{N}} (-\frac{1}{n}, \frac{1}{n}) = \{0\}$

Prop (b) \mathcal{J}_1 is the smallest topology on X that contains all the balls $B_{\mathcal{J}_1}(x, r)$.

Prop (a) \mathcal{J}_1 follows from Remark (T2) $U_1, U_2 \in \mathcal{J}_1 \Rightarrow U_1 \cap U_2 \in \mathcal{J}_1$

$\mathcal{J}_1 \subseteq \mathcal{J}_2$ Proof: Remark & Axiom T3 for $\mathcal{J}_2 \Rightarrow \emptyset$

Apply fct $f: \mathbb{R} \rightarrow \mathbb{R}^2, f(x) = (x, x)$

Examples metric topologies

is a more natural one $\mathcal{J}_{\text{Eucl}}$

the topology induced by the Euclidean metric $d_{\text{Eucl}}(x, y) = |x - y|$

the smallest topology on \mathbb{R} containing all open intervals

Hence, for $U \subseteq \mathbb{R}$

$U \in \mathcal{J}_{\text{Eucl}} \iff U$ is a union of open intervals

$U \in \mathcal{J}_{\text{Eucl}} \iff \forall x \in U, \exists$ open interval $(a, b) \ni x \cap U = U$

$(a, b) \subseteq U$

$U = \bigcup_{i \in I} (a_i, b_i) \subseteq U$

$(a, b) \subseteq U$

Questions

(i) what if we replace open intervals by intervals of type $[a, b)$? \Rightarrow a topology \mathcal{J}_c from basic topology

(ii) \dots $(a, b]$? \Rightarrow a topology \mathcal{J}_c

(iii) \dots $[a, b]$? \Rightarrow a topology \mathcal{J}_c but not regular

Gate topology

(i) $\mathcal{J}_{\text{Eucl}} = \mathcal{J}_c$ $(a, b) = \bigcup_{n \in \mathbb{N}} [a + \frac{1}{n}, b)$

(ii) $\mathcal{J}_{\text{Eucl}} \neq \mathcal{J}_c$ $(a, b) \subseteq \mathcal{J}_{\text{Eucl}}$ but $(a, b) \notin \mathcal{J}_c$

(iii) $\mathcal{J}_{\text{Eucl}} \neq \mathcal{J}_c$ $(a, b) \subseteq \mathcal{J}_{\text{Eucl}}$ but $(a, b) \notin \mathcal{J}_c$

$\Rightarrow \mathcal{J}_{\text{Eucl}} \neq \mathcal{J}_c$ \Rightarrow any $A \subseteq \mathbb{R}$ must be in $\mathcal{J}_{\text{Eucl}}$

