[-1-]

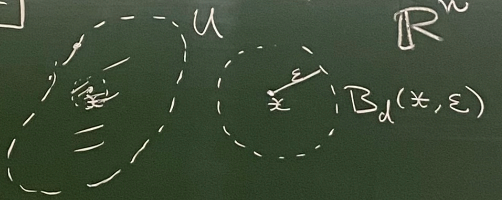
Reminder:

- topological spaces: about convergence & continuity
- 1st attempt: metric spaces (X,d)
- various examples ... and:

  - : Different metrics on a X may give rise to same convergence & contin.
  - : The (cool!) process of gluing may take us out of metric spaces.

[-2-]

Key remark: notion of open



Def: (X,d) metric space:

- for $x \in X$, $\varepsilon > 0$: $B_d(x,\varepsilon) := \{ y \in X \,/\, d(x,y) < \varepsilon \}$
- a set $U \subseteq X$ is said to be open in (X,d) if
  $$(\forall)\ x \in U,\ \exists\ \varepsilon > 0 \ \text{ s.t. } B_d(x,\varepsilon) \subseteq U$$
- the collection of all such opens is denoted $\mathcal{T}_d$, and called the topology induced by d.

Rk: $U \subseteq X$ is open $\iff$ U can be written as a union of balls.

Pf: "$\Rightarrow$" $(\forall)\ x \in U\ \exists\ \varepsilon_x$ s.t. $B_d(x,\varepsilon_x) \subseteq U$

$\Rightarrow \{x\} \subseteq B_d(x,\varepsilon_x) \subseteq U$. Apply $\bigcup_{x\in U} \Rightarrow U = \bigcup_{x\in U} \{x\} \subseteq \bigcup_{x\in U} B_d(x,\varepsilon_x) \subseteq U$

1.3.5: Two metrics d and d' on X induce the same convergence if and only if $\mathcal{T}_d = \mathcal{T}_{d'}$.

$$f: X \longrightarrow Y \supseteq U \qquad f^{-1}(U) = \{ x \in X \,/\, f(x) \in U \}$$

$$(X \setminus U_1) \cap (X \setminus U_2) = X \setminus (U_1 \cup U_2)$$

$$(X \setminus U_1) \cup (X \setminus U_2) = X \setminus (U_1 \cap U_2)$$

DEFINITIONS: -3-

- a topology on a set X (prescribes the OPENS of the space $(X,\mathcal{T})$): a collection $\mathcal{T}$ of subsets of X satisfying:

  (T1) $\emptyset, X \in \mathcal{T}$. | $\emptyset, X$ are open

  (T2) intersection of two members of $\mathcal{T}$ is in $\mathcal{T}$ | intersection of two opens is open

  $$U_1, U_2 \in \mathcal{T} \Rightarrow U_1 \cap U_2 \in \mathcal{T}.$$

  (T3) union of any collection of members of $\mathcal{T}$ is in $\mathcal{T}$: | arbitrary unions of opens are open

  $$U_i \in \mathcal{T} \ (\forall) i \in \underbrace{I}_{\text{index set}} \Rightarrow \bigcup_{i \in I} U_i \in \mathcal{T}.$$

- a topological space: $(X, \mathcal{T})$ set, $\mathcal{T}$ topology on X

- given $(X, \mathcal{T})$:
  - $\longrightarrow$ a subset $U \subseteq X$ called open in $(X,\mathcal{T})$ if $U \in \mathcal{T}$.
  - $\longrightarrow$ a subset $A \subseteq X$ called closed in $(X,\mathcal{T})$ if $X \setminus A \in \mathcal{T}$

Given two topologies $\mathcal{T}_1, \mathcal{T}_2$ on X, say $\mathcal{T}_1$ is smaller than $\mathcal{T}_2$ if $\mathcal{T}_1 \subseteq \mathcal{T}_2$.

Example

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- given top. spaces $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$ a continuous fct. $f: (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ is ... $f: X \to Y$ s.t.

Example 5: metric topolog

Prop: Given (X,d) metric spa ... topolo a topological space $(X, \mathcal{T})$

- given $(X, \mathcal{T})$:
  - $\longrightarrow$ a subset $U \subseteq X$ called open in $(X, \mathcal{T})$ if $U \in \mathcal{T}$
  - $\longrightarrow$ a subset $A \subseteq X$ called closed in $(X, \mathcal{T})$ if $X \setminus$

-4-
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- given top. spaces $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$

continuous fct. $f : (X, \mathcal{T}_X) \longrightarrow (Y, \mathcal{T}_Y)$ is any function $f : X \to Y$ s.t.

$$f^{-1}(U) \in \mathcal{T}_X \ (\forall)\ \underset{\substack{\parallel \\ Y \setminus B}}{U} \in \mathcal{T}_Y.$$

Rk: (T_2) for arbitrary intersections would be too much to ask: $\bigcap_{n \in \mathbb{N}} \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}$

pf of Prop: (b) This says that: if $\mathcal{T}$ ... topology on X s.t. $B_r(x, \varepsilon) \in \mathcal{T}$ $\forall x$

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Examples 1–4: On any set X: $\boxed{-5-}$

1. $\mathcal{T}_{triv}$: consists only of ϕ, X.
2. $\mathcal{T}_{dis}$: —"— of all the subsets of X
3. $\mathcal{T}_{cf}$ (cofinite): ϕ and $U = X \setminus F$ with F finite
4. $\mathcal{T}_{cc}$ (cocountable): ϕ and $U = X \setminus C$ with $C \subseteq X$ at most countable

$$\left(\mathcal{T}_{triv} \subseteq \mathcal{T}_{cf} \subseteq \mathcal{T}_{cc} \subseteq \mathcal{T}_{dis} \right)$$

Why is $\mathcal{T}_{cf}$ a topology?

(T2) $U_1, U_2 \in \mathcal{T}_{cf} \Rightarrow \begin{cases} U_1 = \phi \text{ or } U_1 = X \setminus F_1 \text{ with } F_1\text{-finite} \\ U_2 = \phi \text{ or } U_2 = X \setminus F_2 \text{ —"— } F_2 \text{ —"—} \end{cases}$

$\Rightarrow U_1 \cap U_2 = \phi$ or $U_1 \cap U_2 = X \setminus \underbrace{(F_1 \cup F_2)}_{\text{finite}} \Rightarrow U_1 \cap U_2 \in \mathcal{T}_{cf}$.

(T3) similarly.

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Examples 6–7–8: on $\mathbb{R}$, next to $\mathcal{T}_{triv}$, $\mathcal{T}_{dis}$, $\mathcal{T}_{cf}$, $\mathcal{T}_{cc}$

is a more natural one

$\mathcal{T}^{\mathbb{R}}$ ~~(the topology induced by~~

Example 5: metric topologies

Prop: Given (X,d) metric space:

(a) $\mathcal{T}_d$ is a topology

(b) $\mathcal{T}_d$ is the smallest topology on X that contains all the balls $B_d(x,\varepsilon)$.

Pf: (a) T3 follows from Remark

(T2) $\underbrace{U_1, U_2 \in \mathcal{T}_d}$ $\overset{?}{\Longrightarrow}$ $\boxed{U_1 \cap U_2 \in \mathcal{T}_d}$

i.e. to prove

$\boxed{(\forall)\, x_1 \in U_1\ \exists\, \varepsilon_1 > 0 : B_d(x_1,\varepsilon_1) \subseteq U_1}$

$\boxed{(\forall)\, x_2 \in U_2\ \exists\, \varepsilon_2 > 0 : B_d(x_2,\varepsilon_2) \subseteq U_2}$

Apply for $x_1 = x,\ x_2 = x \Rightarrow \exists\, \varepsilon_1, \varepsilon_2 > 0$ s.t. $B(x,\varepsilon_1) \subseteq U_1$,

$\Rightarrow \{a\} \in \mathcal{T}_0 \ (\forall) a \in \mathbb{R} \Rightarrow$ any $A \subseteq \mathbb{R}$ must be ...

$\boxed{6''}$

Interesting question: given $(X, \mathcal{T})$ when is it metrizable, i.e., $\exists$ metric d s.t. $\mathcal{T} = \mathcal{T}_d$?

$$A = \bigcup_{a \in A} \{a\}$$

Example: Similarly on $\mathbb{R}^n$ $\boxed{-9-}$ and on any $A \subseteq \mathbb{R}^n$ usi...

$d_{Eucl}^{\mathbb{R}^n} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$

(T3) similarly:

finite

-7-

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Examples 6-7-8: on $\mathbb{R}$, next to $\mathcal{T}_{triv}$, $\mathcal{T}_{dis}$, $\mathcal{T}_{cf}$, $\mathcal{T}_{cc}$ there is a more natural one:

$\mathcal{T}^{\mathbb{R}}_{Eucl}$:= the topology induced by the Euclidean metric $d^{\mathbb{R}}_{Eucl}(x,y)=|x-y|$

= the smallest topology on $\mathbb{R}$ containing all open intervals

Hence, for $U\subseteq\mathbb{R}$:

$U\in\mathcal{T}^{\mathbb{R}}_{Eucl} \iff$ U is a union of open intervals

$\iff (\forall)\, x\in U$, $\exists$ open interval (a,b) s.t. $x\in(a,b)\subseteq U$.

$U=[a,b)\in\mathcal{T}_e$

ology

e

$\mathcal{T}_d$:

prove: $(\forall)\, x\in U_1\cap U_2$, $\exists\, \varepsilon>0$ s.t. $B(x,\varepsilon)\subseteq U_1\cap U_2$;

fix x

$\varepsilon_1)\subseteq U_1$, $B(x,\varepsilon_2)\subseteq U_2$. Take $\varepsilon=\min\{\varepsilon_1,\varepsilon_2\}\Rightarrow B(x,\varepsilon)\subseteq U_1\cap U_2$ $\boxed{\varepsilon}$

Questions:

(i) what if we replace open intervals by intervals of type $[a,b)$? $\Rightarrow$ a topology $\mathcal{T}_\ell$
lower limit topology.

(ii) — " — $(a,b]$? $\Rightarrow$ a topology $\mathcal{T}_u$

(iii) — " — $[a,b]$? $\Rightarrow$ get a topology $\mathcal{T}_0$
but not really new

Cute remarks:

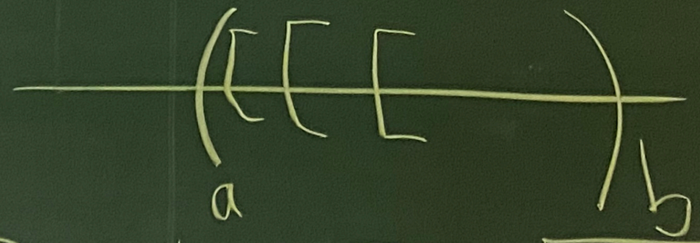
(i) $\mathcal{T}_{Eucl} \subseteq \mathcal{T}_\ell$

$(a,b) \in \mathcal{T}_\ell$ $\forall a, b$

$\mathcal{T}_{Eucl}$ = smallest containing (a,b)

$\Rightarrow \mathcal{T}_{Eucl} \subseteq \mathcal{T}_\ell$.

$$(a,b) = \bigcup_{n\in\mathbb{N}} \left[a+\tfrac{1}{n}, b\right) \in \mathcal{T}_\ell$$



(iii) $[a-1,a] \cap [a,a+1] = \{a\} \Rightarrow$
$\Rightarrow \{a\} \in \mathcal{T}_0$ $\forall a \in \mathbb{R}$ $\Rightarrow$ any $A \subseteq \mathbb{R}$ must be in $\mathcal{T}_0 \Rightarrow$ $\boxed{\mathcal{T}_0 = \mathcal{T}_{disc}}$

$A - \bigcup_{a \in A} \{a\}$

<u>Example</u>: Similarly on $\mathbb{R}^n$ $\boxed{-9-}$ and on any $A \subseteq \mathbb{R}^n$, using

$$d^{\mathbb{R}^n}_{Eucl} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, \quad d^A_{Eucl} = d^{\mathbb{R}^n}_{Eucl}\Big|_{A \times A} : A \times A \to \mathbb{R}$$

$$\Rightarrow (\mathbb{R}^n, \mathcal{T}_{Eucl}), \quad (A, \mathcal{T}^A_{Eucl})$$

Example 9: Given $(X,\mathcal{T})$ a space, $A \subseteq X$ $\Big\}$ $\rightsquigarrow$ we have an induced topology on A

$$\mathcal{T}|_A := \{A \cap U : U \in \mathcal{T}\}$$

Exercise: check this is a topology.

Def: Given $(X,\mathcal{T})$, $A \subseteq X$, a subset $B \subseteq A$ is called:

- B is open in A if $B \in \mathcal{T}|_A$ (if B is open in $(A, \mathcal{T}|_A)$)
- B is closed in A if $A \setminus B \in \mathcal{T}|_A$ (if B is closed in $(A, \mathcal{T}|_A)$)

Ex: $[0,1) \subseteq \mathbb{R}$ — open in $[0,\infty)$; closed in $(-1,1)$ ← 1^{st} ex. for next werkcollege.


DEFINITIONS
Examples 1-4: On any set X
Interesting question: given (X, J) when is it metrizable, i.e., ∃ metric d s.t. J = J_d ?
Example 5: metric topologies
Questions
Pangea