s.t. $U \subseteq V,\ x \in V\}$

of x in $(X, \mathcal{T})$

ous at all $x \in X$

$V \in \mathcal{N}_{\mathcal{T}_1}($

prove

look

$U' \in \mathcal{T}(x)$

U'

e $U' = f$

rgument

$f^{-1}(U) \subseteq$

$x \in f^{-1}($

Reminder: Topology on X: collection $\mathcal{T}$ of subsets of X

(T1) $\emptyset, X \in \mathcal{T}$ — $\emptyset, X$ - opens

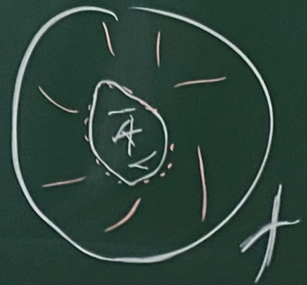
(T2) $U_1, U_2 \in \mathcal{T} \Rightarrow U_1 \cap U_2 \in \mathcal{T}$ — intersection of two opens is open

(T3) $U_i \in \mathcal{T}\ (\forall)\ i \in I \Rightarrow \bigcup_{i \in I} U_i \in \mathcal{T}$ — arbitrary union of opens is open

Terminology: $(X, \cancel{\mathcal{T}})$ = topological space

$U \subseteq X$ open in $(X, \cancel{\mathcal{T}})$ if $U \in \mathcal{T}$

$A \subseteq X$ closed in $(X, \cancel{\mathcal{T}})$ if $X \setminus A \in \mathcal{T}$



EXAMPLES:

- (X, d)-metric space $\rightsquigarrow$ topology $\boxed{\mathcal{T}_d}$ on X where, for $U \subseteq X$:

$$U \in \mathcal{T}_d \iff (\forall)\ x_0 \in U,\ \exists\ \varepsilon > 0 \text{ s.t. } \underset{\{y \in X / d(y, x_0) < \varepsilon\}}{\underset{\parallel}{B_d(x, \varepsilon)}} \subset U$$

$$\iff U \text{ is a union of } d\text{-balls.}$$

- on any set X: $\mathcal{T}_{triv} \subseteq \mathcal{T}_{cf} \subseteq \mathcal{T}_{cc} \subseteq \mathcal{T}_{disc}$
- on $\mathbb{R}$: the Euclidean topology induced by $d^{\mathbb{R}}_{Eucl}$. Hence, for $U \subseteq \mathbb{R}$:

$$U \in \mathcal{T}^{\mathbb{R}}_{Eucl} \iff (\forall)\ x \in U\ \exists\ a, b \in \mathbb{R} \text{ s.t. } x \in (a,b) \subseteq U \iff U \text{ is a union of } \underline{\text{open intervals}}$$

Variation: replace <u>open intervals</u> by those of type $\textcircled{[a,b)}$ $\Rightarrow$ new topology on $\mathbb{R}$, $\mathcal{T}_l$

- for any $\left.\begin{matrix} (X, \mathcal{T}) \\ \underbrace{A \subseteq X}_{\text{subset}} \end{matrix}\right\}$ $\rightsquigarrow$ induced topology on A: $\mathcal{T}_A := \{A \cap U : U \in \mathcal{T}\}$

$\boxed{-3-}$

Def 2.11 : $f:(X,\mathcal{T}_X)\longrightarrow(Y,\mathcal{T}_Y)$ continuous if:

$$f^{-1}(U)\in\mathcal{T}_X\ (\forall)\ U\in\mathcal{T}_Y\quad\left(\underset{\text{open}}{\overset{(\forall)}{U\subseteq Y}}\Rightarrow\underset{\text{open}}{f^{-1}(U)\subseteq X}\right)$$

Def 2.24 : $f:(\overset{\overset{x}{\cap}}{X},\mathcal{T}_X)\longrightarrow(\overset{\overset{f(x)}{\cap}}{Y},\mathcal{T}_Y)$ say f = continuous at x if:

$$\underset{\cup}{f^{-1}(V)}\in\mathcal{N}_{\mathcal{T}_X}(x)\quad(\forall)\ \underset{\shortparallel}{V}\in\mathcal{N}_{\mathcal{T}_Y}(f(x))$$

(in example: $\exists\ B_d(x,\delta)$ $\quad(\forall)\ B_d(f(x),\varepsilon)$)

Def 2.21. Given

$\mathcal{T}$

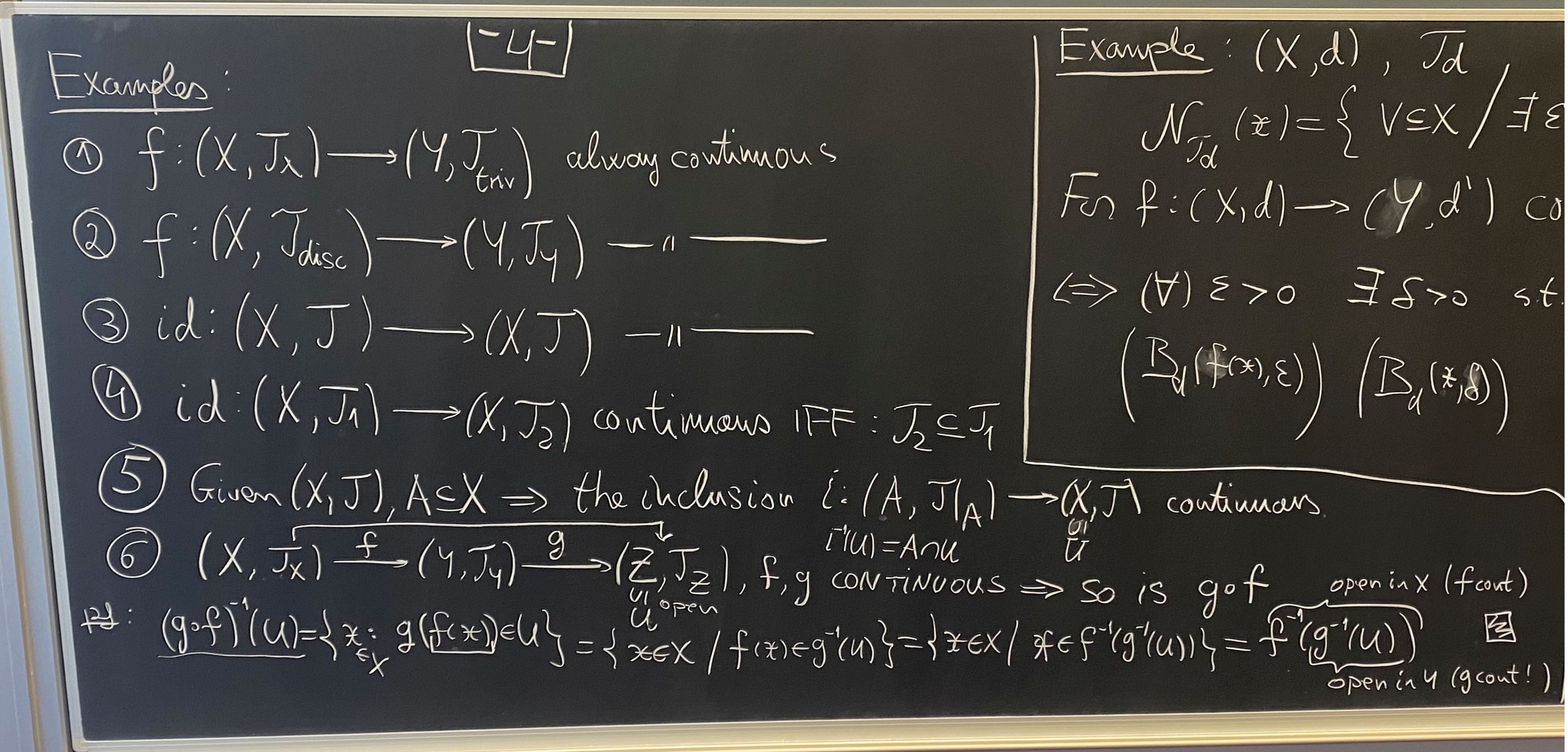
Theorem 2.25 : f

$\boxed{-4-}$

Examples :

Example : (X,d), $\mathcal{T}_d$

$\mathcal{N}_{\mathcal{T}_d}(x)=\{V\subseteq X\ /$



Examples:

① $f:(X,\mathcal{T}_X)\longrightarrow(Y,\mathcal{T}_{triv})$ alway continuous

② $f:(X,\mathcal{T}_{disc})\longrightarrow(Y,\mathcal{T}_Y)$ —"——

③ $id:(X,\mathcal{T})\longrightarrow(X,\mathcal{T})$ —"——

④ $id:(X,\mathcal{T}_1)\longrightarrow(X,\mathcal{T}_2)$ continuous IFF: $\mathcal{T}_2\subseteq\mathcal{T}_1$

⑤ Given $(X,\mathcal{T})$, $A\subseteq X\Rightarrow$ the inclusion $i:(A,\mathcal{T}|_A)\longrightarrow(X,\mathcal{T})$ continuous

$i^{-1}(U)=A\cap U$ $\quad\overset{\cup|}{U}$

⑥
$$\begin{tikzcd}(X,\mathcal{T}_X) \arrow[r, "f"] \arrow[rr, bend left] & (Y,\mathcal{T}_Y) \arrow[r, "g"] & (Z,\mathcal{T}_Z)\end{tikzcd}$$
$\underset{U\text{ open}}{\cup|}$, f, g CONTINUOUS $\Rightarrow$ so is $g\circ f$

Pf: $\underline{(g\circ f)^{-1}(U)}=\{\underset{\in X}{x}:g(\boxed{f(x)})\in U\}=\{x\in X/f(x)\in g^{-1}(U)\}=\{x\in X/x\in f^{-1}(g^{-1}(U))\}=\overbrace{f^{-1}(\underbrace{g^{-1}(U)}_{\text{open in }Y\ (g\text{ cont!})})}^{\text{open in }X\ (f\text{ cont})}$ ☒

Example: (X,d), $\mathcal{T}_d$

$\mathcal{N}_{\mathcal{T}_d}(x)=\{V\subseteq X/\exists\varepsilon$

For $f:(X,d)\longrightarrow(Y,d')$ co

$\Leftrightarrow(\forall)\varepsilon>0\ \exists\delta>0$ s.t.

$\left(B_{d'}(f(x),\varepsilon)\right)\ \left(B_d(x,\delta)\right)$

-3-

Def 2.11: $f:(X,\mathcal{T}_X)\longrightarrow(Y,\mathcal{T}_Y)$ <u>continuous</u> if:

$$f^{-1}(U)\in\mathcal{T}_X\ (\forall)\ U\in\mathcal{T}_Y\quad\left(\overset{(\forall)}{\underset{\text{open}}{U\subseteq Y}}\Rightarrow\underset{\text{open}}{f^{-1}(U)\subseteq X}\right)$$

Def 2.24: $f:(\overset{\overset{x}{\cap}}{X},\mathcal{T}_X)\rightarrow(\overset{\overset{f(x)}{\cap}}{Y},\mathcal{T}_Y)$ say $f=$ <u>continuous at x if</u>:

$$\underset{\underset{\exists\ B_d(x,\delta)}{\cup\!\shortmid}}{f^{-1}(V)}\in\mathcal{N}_{\mathcal{T}_X}(x)\qquad(\forall)\ \underset{\underset{(\forall)\ B_d(f(x),\varepsilon))}{\shortparallel}}{V}\in\mathcal{N}_{\mathcal{T}_Y}(f(x))$$

(in example:

Def 2.21 Given $(X,$

$\mathcal{T}(x)=$

Theorem 2.25: $f:(X,\mathcal{T}_X$

-4-

<u>Examples</u>:

① $f:(X,\mathcal{T}_X)\longrightarrow(Y,\mathcal{T}_{triv})$ alway continuous

② $f:(X,\mathcal{T}_{disc})\longrightarrow(Y,\mathcal{T}_Y)$ —"—

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$$i^{-1}(U)=A\cap U\qquad \underset{U}{\cup\!\shortmid}$$

⑥ $(X,\mathcal{T}_X)\xrightarrow{f}(Y,\mathcal{T}_Y)\xrightarrow{g}(\underset{\underset{U^{\text{open}}}{\cup\!\shortmid}}{Z},\mathcal{T}_Z)$, f,g CONTINUOUS $\Rightarrow$ so is $g\circ f$

Pf: $\underline{(g\circ f)^{-1}(U)}=\{\underset{\in X}{x}: g(\boxed{f(x)})\in U\}=\{x\in X\,/\,f(x)\in g^{-1}(U)\}=\{x\in X\,/\,x\in f^{-1}(g^{-1}(U))\}=\overbrace{f^{-1}(\underbrace{g^{-1}(U)}_{\text{open in }Y\text{ (g cont!)}})}^{\text{open in }X\text{ (f cont)}}$ ☒

-6-

<u>Example</u>: (X,d), $\mathcal{T}_d$

$$\mathcal{N}_{\mathcal{T}_d}(x)=\{V\subseteq X\,/\,\exists\,\varepsilon>0\text{ s.t. }B$$

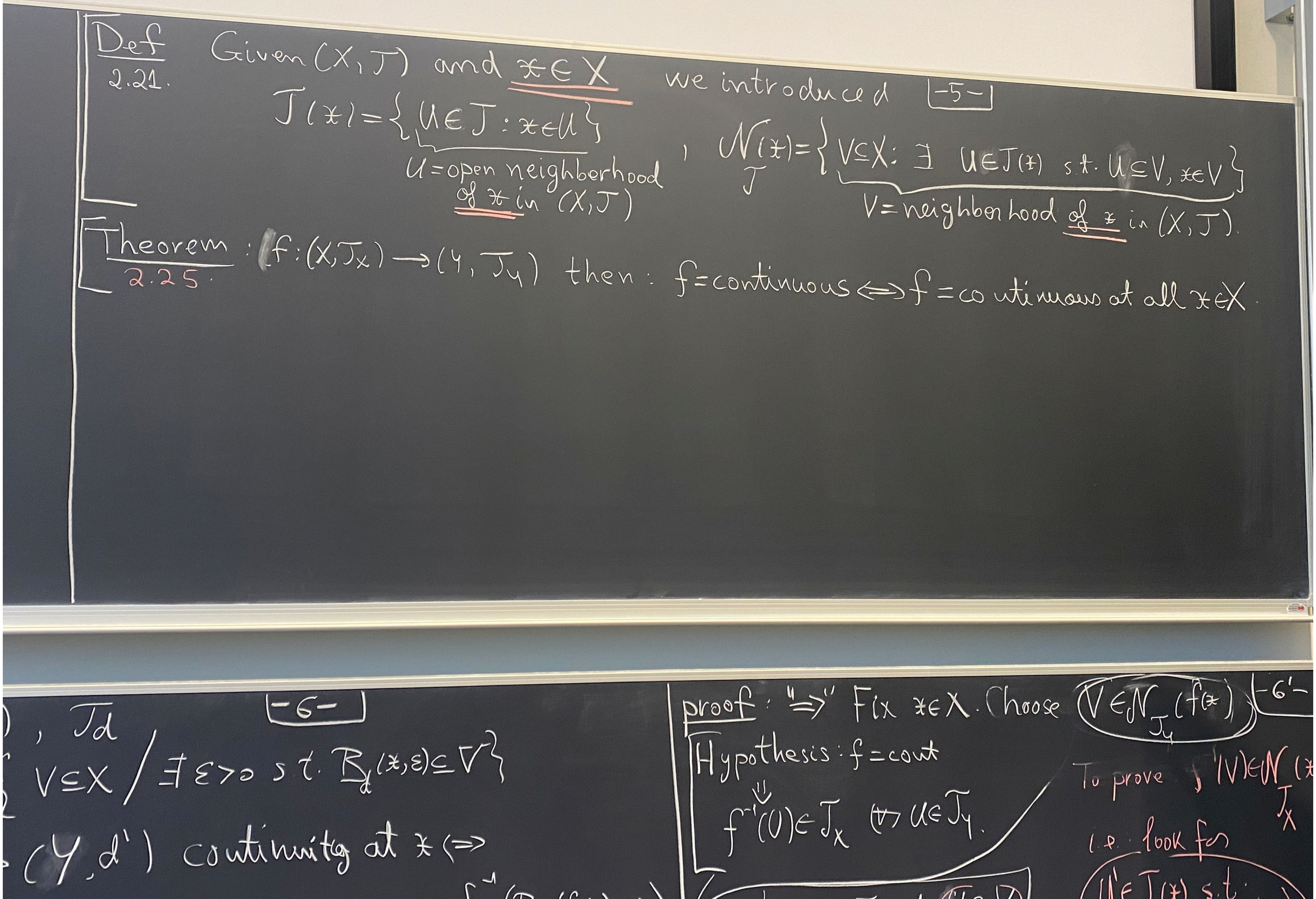
For $f:(X,d)\longrightarrow(Y,d')$ continuity

$\Longleftrightarrow(\forall)\ \varepsilon>0\quad\exists\,\delta>0$ s.t. $B_d(x,$

$$\big(B_d(f(x),\varepsilon)\big)\quad\big(B_d(x,\delta)\big)$$

i.e

$d'(f($



Def 2.21. Given $(X,\mathcal{T})$ and $x\in X$ we introduced

-5-

$$\mathcal{T}(x)=\{\underbrace{U\in\mathcal{T}: x\in U}_{U=\text{open neighberhood of } x \text{ in } (X,\mathcal{T})}\}, \quad \mathcal{N}_{\mathcal{T}}(x)=\{\underbrace{V\subseteq X:\ \exists\ U\in\mathcal{T}(x) \text{ s.t. } U\subseteq V,\ x\in V}_{V=\text{neighberhood of } x \text{ in } (X,\mathcal{T})}\}$$

Theorem 2.25: $f:(X,\mathcal{T}_X)\to(Y,\mathcal{T}_Y)$ then: $f=$ continuous $\Longleftrightarrow$ $f=$ continuous at all $x\in X$.

-6-

), $\mathcal{T}_d$

$\{V\subseteq X \,/\, \exists\, \varepsilon>0$ s.t. $B_d(x,\varepsilon)\subseteq V\}$

(Y,d') continuity at $x \Longleftrightarrow$

proof: "$\Longrightarrow$" Fix $x\in X$. Choose $V\in\mathcal{N}_{\mathcal{T}_Y}(f(x))$

-6'-

Hypothesis: $f=$ cont

$\Downarrow$

$f^{-1}(U)\in\mathcal{T}_X$ $\forall$ $U\in\mathcal{T}_Y$.

To prove $f^{-1}(V)\in\mathcal{N}_{\mathcal{T}_X}(x$

i.e. look for

$U'\in\mathcal{T}(x)$ s.t.

-6-

Example: (X,d), $\mathcal{T}_d$

$\mathcal{N}_{\mathcal{T}_d}(x) = \{ V \subseteq X \,/\, \exists\, \varepsilon > 0 \text{ s.t. } B_d(x,\varepsilon) \subseteq V \}$

For $f : (X,d) \longrightarrow (Y,d')$ continuity at $x \iff$

$\iff (\forall)\, \varepsilon > 0 \ \exists\, \delta > 0$ s.t. $B_d(x,\delta) \subseteq f^{-1}(B_{d'}(f(x),\varepsilon))$

$\left(B_d(f(x),\varepsilon)\right)$ $\left(B_d(x,\delta)\right)$

i.e.:

$d'(f(x),f(y)) < \varepsilon \ (\forall)\, y$ s.t. $d(x,y) < \delta$

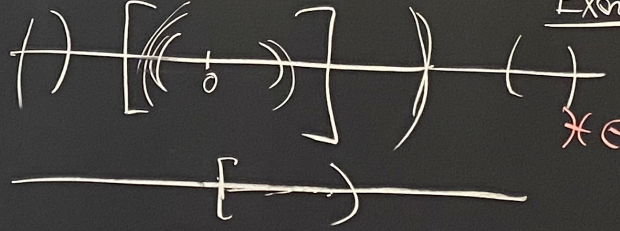
$\mathcal{T}_2 \subseteq \mathcal{T}_1$

$(A, \mathcal{T}|_A) \longrightarrow (\underset{\underset{U}{\cup\!\mid}}{X}, \mathcal{T})$ continuous.

$) = A \cap U$

NTINUOUS $\Rightarrow$ so is $g \circ f$

$)\} = \{ x \in X \,/\, x \in f^{-1}(g^{-1}(U)) \} = \overbrace{f^{-1}(\underbrace{g^{-1}(U)}_{\text{open in } Y \ (g \text{ cont!})})}^{\text{open in } X \ (f \text{ cont})}$ ☒



proof: "$\Rightarrow$" Fix $x \in$

Hypothesis: f = cont

$\Downarrow$

$f^{-1}(U) \in \mathcal{T}_X \ (\forall)\, U \in$

i.e. $\exists\, U \in \mathcal{T}_Y$ s.t.

$f^{-1}(U) \in \mathcal{T}(x)$

Exercise: make a pictu

$x \in f^{-1}(U) \iff f(x) \in U$.

$T, J_Y)$.

$R, J_{Eucl})$

in $(0,1), J_{Eucl})$

uences)

$x_n) \to f(x)$

$\in J_Y(f(x))$

$_V$ s.t.

$(\forall) n \geq n_V$

<u>proof</u>: "$\Rightarrow$" Fix $x \in X$ $\boxed{-7-}$

Let $\underbrace{V \in \mathcal{N}_{J_Y}(f(x))}_{\text{i.e.}}$. To prove: $f^{-1}(V) \in \mathcal{N}_{J_X}(x)$

Know: $\boxed{\exists\, U \in J_Y(f(x)) \text{ s.t. } U \subseteq V}$ i.e. $\boxed{\exists\, U' \in J_X(x) \text{ s.t. } U' \subseteq f^{-1}(V)}$

Take $U' := f^{-1}(U)$ open since f = cont.

"$\Leftarrow$" Let $U \in J_Y$. To prove: $f^{-1}(U) \in J_X$.

<u>Rk</u>: Since $U \in J_Y \Rightarrow U \in J_Y(y)$ $(\forall)\, y \in U$. Apply this to $y = f(x)$ with $x \in f^{-1}(U) \Rightarrow \boxed{U \in J_Y(f(x))\ (\forall)\, x \in f^{-1}(U)}$

$\Rightarrow f^{-1}(U) \in \mathcal{N}_{J_X}(x)\ (\forall)\, x \in f^{-1}(U)$.

$(\forall)\, x \in f^{-1}(U)$: $\exists\, V_x \in J_X(x)$ s.t. $\underline{\{x\}} \subseteq V_x \subseteq f^{-1}(U)$

on of OPENS $\Rightarrow$ OPEN.

$= f^{-1}(U)$

Take $f^{-1}(U) \subseteq \bigcup V_x \subseteq f^{-1}(U)$

◎ on any

◎ on $\mathbb{R}$: t

(1) $U \in J$

Variation: rep

● for any $(X$

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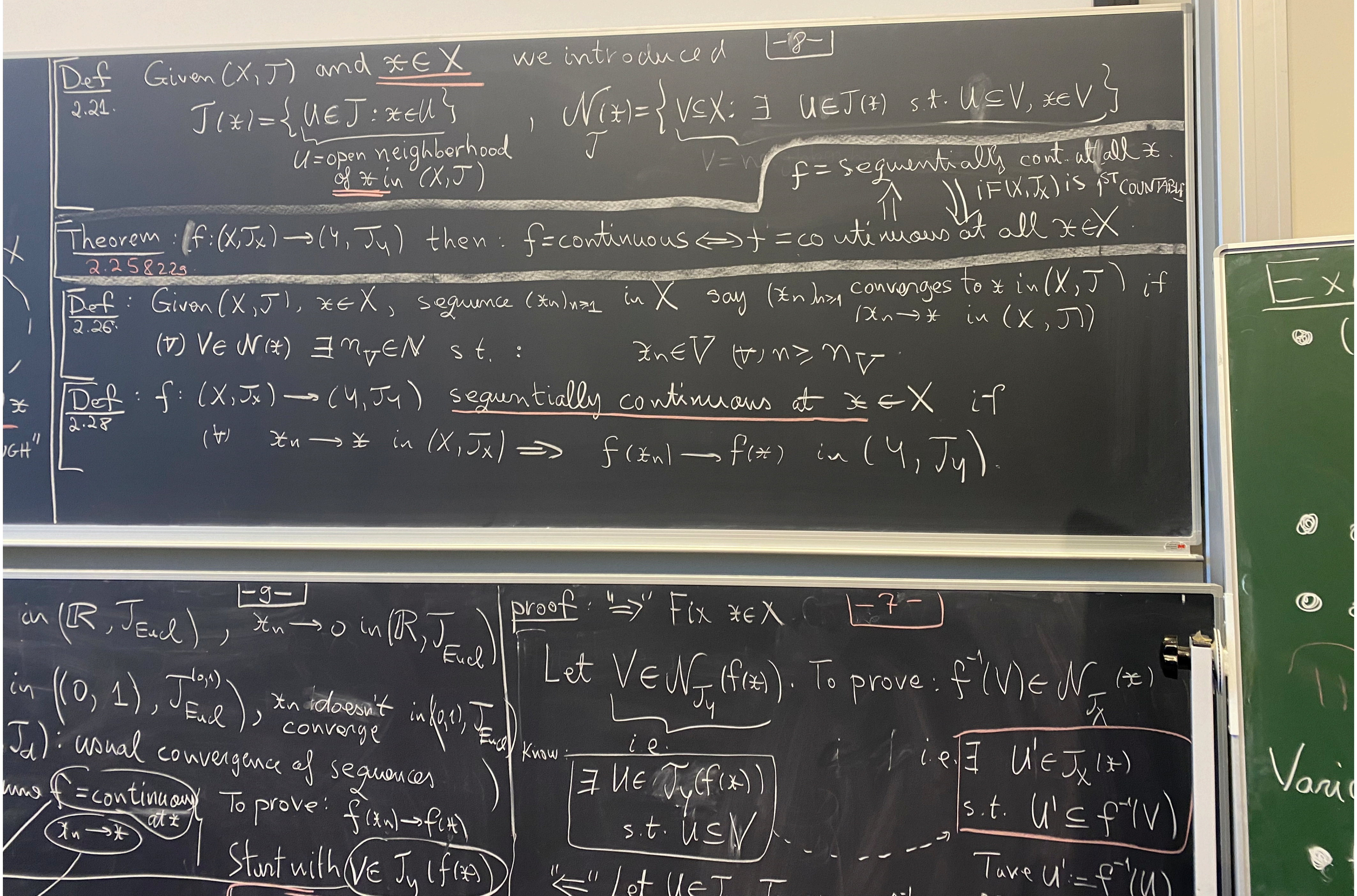

-8-

Def 2.21. Given $(X,\mathcal{T})$ and $x\in X$ we introduced

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$f=$ sequentially cont. at all x.

$\Uparrow$ $\Downarrow$ iF $(X,\mathcal{T}_X)$ is 1ST COUNTABLE

Theorem 2.25 8 2.29 : $f:(X,\mathcal{T}_X)\to(Y,\mathcal{T}_Y)$ then: $f=$ continuous $\Longleftrightarrow f=$ continuous at all $x\in X$.

Def 2.26 : Given $(X,\mathcal{T})$, $x\in X$, sequence $(x_n)_{n\geq 1}$ in X say $(x_n)_{n\geq 1}$ converges to x in $(X,\mathcal{T})$ if ($x_n\to x$ in $(X,\mathcal{T})$)

$(\forall)\ V\in\mathcal{N}(x)\ \exists\ n_V\in\mathbb{N}$ s.t. : $x_n\in V\ (\forall)\ n\geq n_V$.

Def 2.28 : $f:(X,\mathcal{T}_X)\to(Y,\mathcal{T}_Y)$ sequentially continuous at $x\in X$ if

$(\forall)\ x_n\to x$ in $(X,\mathcal{T}_X)\Longrightarrow f(x_n)\to f(x)$ in $(Y,\mathcal{T}_Y)$.

-9-

in $(\mathbb{R},\mathcal{T}_{Eucl})$, $x_n\to 0$ in $(\mathbb{R},\mathcal{T}_{Eucl})$

in $((0,1),\mathcal{T}_{Eucl}^{(0,1)})$, x_n doesn't converge in $((0,1),\mathcal{T}_{Eucl})$

$\mathcal{T}_d)$: usual convergence of sequences

ume $f=$ continuous at x To prove: $f(x_n)\to f(x)$

$x_n\to x$

Start with $V\in\mathcal{T}_Y(f(x))$

-7-

proof: "$\Rightarrow$" Fix $x\in X$.

Let $\underbrace{V\in\mathcal{N}_{\mathcal{T}_Y}(f(x))}$. To prove: $f^{-1}(V)\in\mathcal{N}_{\mathcal{T}_X}(x)$

Know: i.e. $\boxed{\exists\ U\in\mathcal{T}_Y(f(x)) \text{ s.t. } U\subseteq V}$ i.e. $\boxed{\exists\ U'\in\mathcal{T}_X(x) \text{ s.t. } U'\subseteq f^{-1}(V)}$

Take $U':=f^{-1}(U)$

"$\Leftarrow$" Let $U\in\mathcal{T}$

[illegible]

$\mathcal{B}_x \subseteq \underline{\underline{\mathcal{N}(x)}}$ which is "RICH ENOUGH"

$B \in \mathcal{B}_x$ s.t. $B \subseteq V$

2.28

$(\forall)\ x_n \to x$ in $(X, \mathcal{T}_X) \Rightarrow f(x_n) \to f($

---

neigh. of $x \in X$ $\boxed{-10-}$

} countable!

st

countable

as above & countable

$\mathcal{B}^2_x, \mathcal{B}^3_x, \dots$ }

$\boxed{-9-}$

Ex: $x_n = \frac{1}{n}$ in $(\mathbb{R}, \mathcal{T}_{Eucl})$, $x_n \to 0$ in $(\mathbb{R}, \mathcal{T}_{Eucl})$

$x_n = \frac{1}{n}$ in $((0,1), \mathcal{T}^{(0,1)}_{Eucl})$, x_n doesn't converge in $((0,1), \mathcal{T}_{Eucl})$

Ex (X, d): in $(X, \mathcal{T}_d)$: usual convergence of sequences

proof "⇑": Assume f = continuous at x, $x_n \to x$. To prove: $f(x_n) \to f(x)$

Start with $V \in \mathcal{T}_Y(f(x))$

$f^{-1}(V) \in \mathcal{N}(x)$

$\exists\ n_V$ s.t. $x_n \in f^{-1}(V)\ (\forall)\ n \geq n_V$

Look for n_V s.t. $f(x_n) \in V\ (\forall)\ n \geq n_V$

[illegible] g cont!)

Hence, for all $x \in f^{-1}(U)$: $\exists\ V_x \in \mathcal{T}(x)$ s.t. $\underline{\{x\}} \subseteq V_x \subseteq f^{-1}($

$\Rightarrow f^{-1}(U)$ = union of OPENS $\Rightarrow$ OPEN. x

proof: "⇒" Fix $x \in$

Let $V \in \mathcal{N}_{\mathcal{T}_Y}(f(x))$

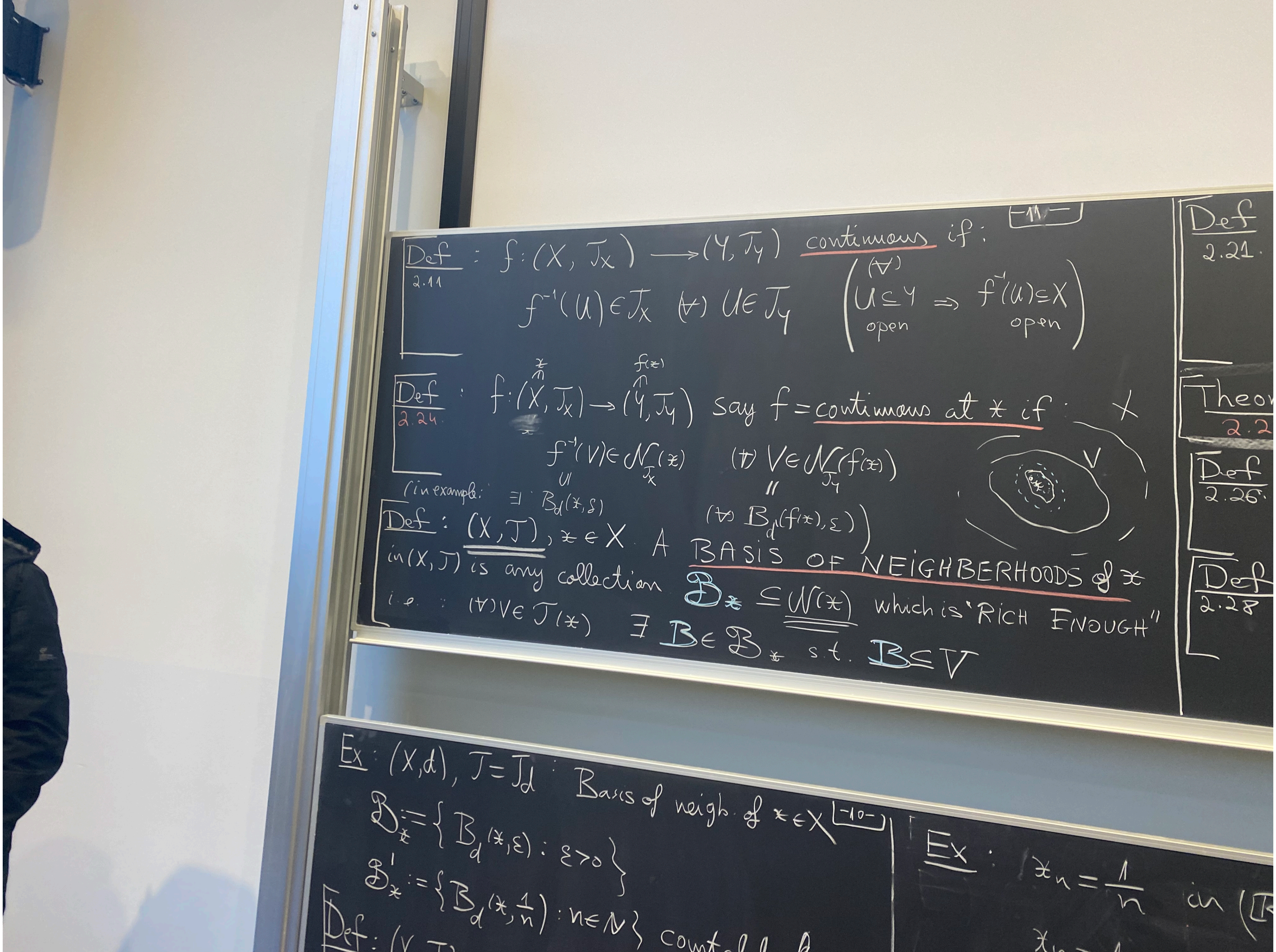
Know: i.e. $\exists\ U \in \mathcal{T}_Y(f(x))$ s.t. $U \subseteq V$

"⇐" Let $U \in \mathcal{T}_Y$. To pr

Rk: Since $U \in \mathcal{T}_Y \Rightarrow$

to $y = f(x)$ with $x \in f$

$f^{-1}(U) \in \mathcal{N}_{\mathcal{T}_X}(x)\ (\forall)\ x \in$



-11-

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Def 2.24: $f:(\overset{x}{\overset{\cap}{X}},\mathcal{T}_X)\to(\overset{f(x)}{\overset{\cap}{Y}},\mathcal{T}_Y)$ say f = continuous at x if:

$$\underset{\substack{\cup\!\shortmid\\ \exists\ B_d(x,\delta)}}{f^{-1}(V)}\in\mathcal{N}_{\mathcal{T}_X}(x)\qquad(\forall)\ \underset{\substack{\shortparallel\\ (\forall)\ B_d(f(x),\varepsilon)}}{V}\in\mathcal{N}_{\mathcal{T}_Y}(f(x))$$

(in example:

Def: $(X,\mathcal{T})$, $x\in X$. A BASIS OF NEIGHBERHOODS of x in $(X,\mathcal{T})$ is any collection $\mathcal{B}_x\subseteq\mathcal{N}(x)$ which is "RICH ENOUGH"

i.e.: $(\forall)V\in\mathcal{T}(x)\ \exists\ B\in\mathcal{B}_x$ s.t. $B\subseteq V$

Def 2.21.

Theor 2.2

Def 2.26.

Def 2.28

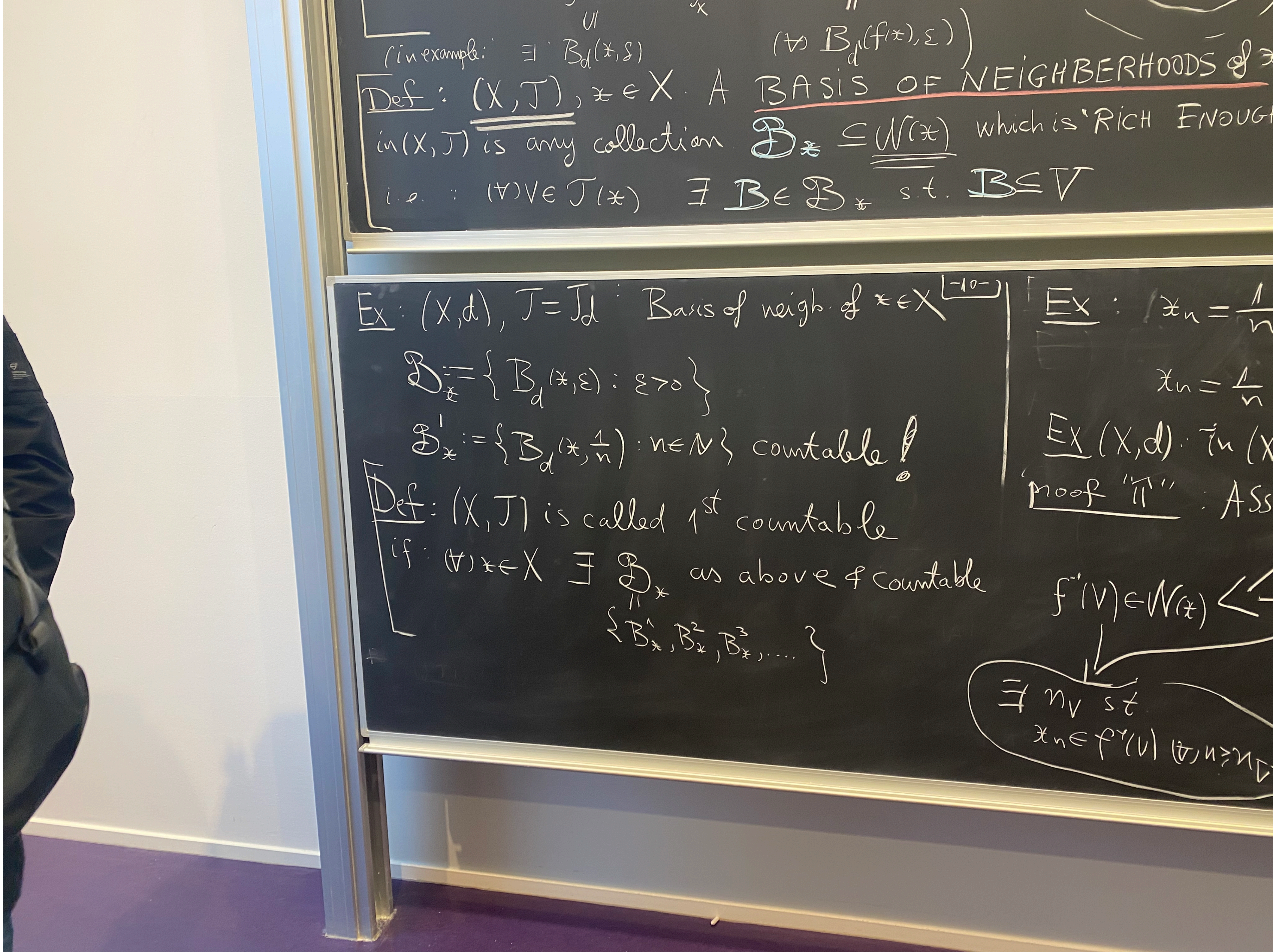
Ex: (X,d), $\mathcal{T}=\mathcal{T}_d$: Basis of neigh. of $x\in X$ -10-

$\mathcal{B}_x:=\{B_d(x,\varepsilon):\varepsilon>0\}$

$\mathcal{B}'_x:=\{B_d(x,\frac{1}{n}):n\in\mathbb{N}\}$ count

Def: $(X,$

Ex: $x_n=\frac{1}{n}$ in $($



(in example: $\exists$ $B_d(x,\delta)$ $\quad$ $(\forall)$ $B_d(f(x),\varepsilon)$)

Def: $(X,\mathcal{T})$, $x \in X$. A BASIS OF NEIGHBERHOODS of x in $(X,\mathcal{T})$ is any collection $\mathcal{B}_x \subseteq \mathcal{N}(x)$ which is "RICH ENOUG

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-10-

Ex: (X,d), $\mathcal{T}=\mathcal{T}_d$: Basis of neigh. of $x \in X$

$$\mathcal{B}_x := \{ B_d(x,\varepsilon) : \varepsilon > 0 \}$$

$$\mathcal{B}'_x := \{ B_d(x,\tfrac{1}{n}) : n \in \mathbb{N} \} \text{ countable !}$$

Def: $(X,\mathcal{T})$ is called 1^{st} countable if: $(\forall) x \in X$ $\exists$ $\mathcal{B}_x$ as above & countable

$$\mathcal{B}_x = \{ B^1_x, B^2_x, B^3_x, \dots \}$$

Ex: $x_n = \frac{1}{n}$

$x_n = \frac{1}{n}$

Ex (X,d). in $(X$

proof "$\Uparrow$": Ass

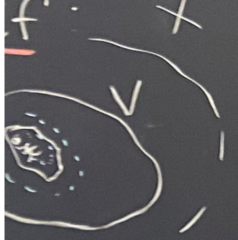
$f^{-1}(V) \in \mathcal{N}(x)$ $\Longleftarrow$

$\exists$ n_V s.t. $x_n \in f^{-1}(V)$ $(\forall) n \geq n_V$

$\in X$

en

f · X



HOODS of x

H ENOUGH"

Def 2.21. Given $(X, \mathcal{T})$ and $x \in X$ we introduced -8-

$$\mathcal{T}(x) = \{\underbrace{U \in \mathcal{T} : x \in U}_{U = \text{open neighborhood of } x \text{ in } (X,\mathcal{T})}\}, \quad \mathcal{N}_{\mathcal{T}}(x) = \{V \subseteq X : \exists\ U \in \mathcal{T}(x) \text{ s.t. } U \subseteq V, x \in V\}$$

f = sequentially cont. at all x.

$\Uparrow$ $\Downarrow$ iF $(X, \mathcal{T}_X)$ is 1st COUNTAB

Theorem 2.25 & 2.29: $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ then: f = continuous $\Leftrightarrow f$ = continuous at all $x \in X$

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$(\forall)\ V \in \mathcal{N}(x)\ \exists\, n_V \in \mathbb{N}$ s.t. : $x_n \in V\ (\forall)\, n \geq n_V$

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$(\forall)\quad x_n \to x$ in $(X, \mathcal{T}_X) \Rightarrow f(x_n) \to f(x)$ in $(Y, \mathcal{T}_Y)$.

EXAMPL

- $(X, d$

- on

- ov

Vari

-9-

$x_n = \frac{1}{n}$ in $(\mathbb{R}, \mathcal{T}_{Eucl})$, $x_n \to 0$ in $(\mathbb{R}, \mathcal{T}_{Eucl})$

$x_n = \frac{1}{n}$ in $((0,1), \mathcal{T}^{(0,1)}_{Eucl})$, x_n doesn't converge in $((0,1), \mathcal{T}_{Eucl})$)

d): in $(X, \mathcal{T}_d)$: usual convergence of sequences

tinuous To prove: $f(x_n) \to f(x)$

$\mathcal{T}\ f(x)$

-7-

proof: "$\Rightarrow$" Fix $x \in X$.

Let $\underbrace{V \in \mathcal{N}_{\mathcal{T}_Y}(f(x))}_{\text{i.e.}}$. To prove: $f^{-1}(V) \in \mathcal{N}_{\mathcal{T}_X}(x)$

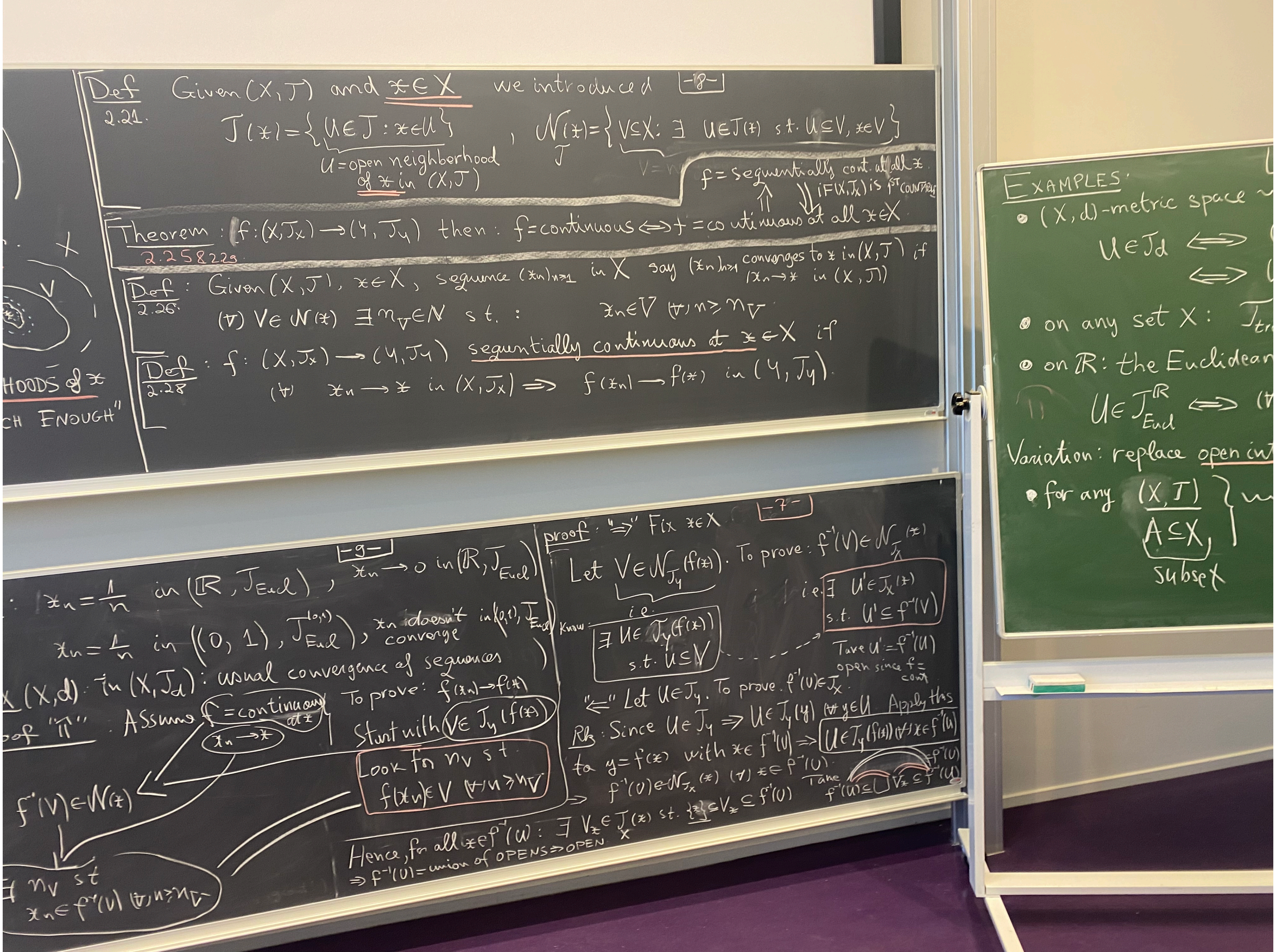
Know: $\boxed{\exists\ U \in \mathcal{T}_Y(f(x)) \text{ s.t. } U \subseteq V}$ i.e. $\boxed{\exists\ U' \in \mathcal{T}_X(x) \text{ s.t. } U' \subseteq f^{-1}(V)}$

Take $U' = f^{-1}(U)$ open since f = cont

"$\Leftarrow$" Let $U \in \mathcal{T}_Y$. To prove $f^{-1}(U) \in \mathcal{T}_X$.

$U \in \mathcal{T}_Y \Rightarrow U \in \mathcal{T}_Y(y)\ (\forall)\, y \in U$ Apply this

$\mathcal{T}(f(x))\ (\forall)\, x \in f^{-1}(U)$



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Def 2.21. Given (X, J) and $x \in X$ we introduced

$$J(x) = \{\underbrace{U \in J : x \in U}_{U = \text{open neighberhood of } x \text{ in } (X,J)}\}, \quad \mathcal{N}_J(x) = \{\underbrace{V \subseteq X}: \exists\ U \in J(x) \text{ s.t. } U \subseteq V, x \in V\}$$

V = N

f = sequentially cont. at all x.

$\Uparrow \quad \Downarrow$ IF (X, J_X) is 1ST COUNTABLE

Theorem 2.25 8 2.29. $f : (X, J_X) \rightarrow (Y, J_Y)$ then: f = continuous $\Longleftrightarrow f$ = continuous at all $x \in X$.

Def 2.26. Given (X, J), $x \in X$, sequence $(x_n)_{n \geq 1}$ in X say $(x_n)_{n\geq 1}$ converges to x in (X, J) ($x_n \rightarrow x$ in (X, J)) if

$(\forall)\ V \in \mathcal{N}(x)\ \exists\ n_V \in \mathbb{N}$ s.t.: $\quad x_n \in V\ (\forall)\ n \geq n_V$.

Def 2.28: $f : (X, J_X) \rightarrow (Y, J_Y)$ sequentially continuous at $x \in X$ if

$(\forall)\quad x_n \rightarrow x$ in $(X, J_X) \Rightarrow f(x_n) \rightarrow f(x)$ in (Y, J_Y).

X

V

HOODS of x

CH ENOUGH"

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: $x_n = \frac{1}{n}$ in $(\mathbb{R}, J_{Eucl})$, $x_n \rightarrow 0$ in $(\mathbb{R}, J_{Eucl})$

$x_n = \frac{1}{n}$ in $((0,1), J^{(0,1)}_{Eucl})$, x_n doesn't converge in $((0,1), J_{Eucl})$

X (X, d): in (X, J_d): usual convergence of sequences

oof "⇑": Assume f = continuous at x $\quad x_n \rightarrow x$

To prove: $f(x_n) \rightarrow f(x)$

Start with $V \in J_Y(f(x))$

Look for n_V s.t. $f(x_n) \in V\ (\forall)\ n \geq n_V$

$f^{-1}(V) \in \mathcal{N}(x)$

$\exists\ n_V$ st $x_n \in f^{-1}(V)\ (\forall)\ n \geq n_V$

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proof: "$\Rightarrow$" Fix $x \in X$

Let $\underbrace{V \in \mathcal{N}_{J_Y}(f(x))}$. To prove: $f^{-1}(V) \in \mathcal{N}_{J_X}(x)$

i.e. $\boxed{\exists\ U' \in J_X(x) \text{ s.t. } U' \subseteq f^{-1}(V)}$

Know: i.e. $\boxed{\exists\ U \in J_Y(f(x)) \text{ s.t. } U \subseteq V}$

Take $U' = f^{-1}(U)$ open since f = cont

"$\Leftarrow$" Let $U \in J_Y$. To prove: $f^{-1}(U) \in J_X$.

Rk: Since $U \in J_Y \Rightarrow U \in J_Y(y)\ (\forall)\ y \in U$. Apply this to $y = f(x)$ with $x \in f^{-1}(U) \Rightarrow \boxed{U \in J_Y(f(x))\ (\forall)\ x \in f^{-1}(U)}$

$\Rightarrow f^{-1}(U) \in \mathcal{N}_{J_X}(x)\ (\forall)\ x \in f^{-1}(U)$.

Take $f^{-1}(U) \subseteq \bigcup V_x \subseteq f^{-1}(U)$

Hence, for all $x \in f^{-1}(U)$: $\exists\ V_x \in J_X(x)$ st. $\{x\} \subseteq V_x \subseteq f^{-1}(U)$

$\Rightarrow f^{-1}(U)$ = union of OPENS $\Rightarrow$ OPEN.

EXAMPLES:

- (X, d)-metric space ~

$U \in J_d \Longleftrightarrow$ (

$\Longleftrightarrow$ (

- on any set X: J_{tr}

- on $\mathbb{R}$: the Euclidean

$U \in J^{\mathbb{R}}_{Eucl} \Longleftrightarrow$ ($\forall$

Variation: replace open int

- for any (X, J), $\underbrace{A \subseteq X}_{\text{subset}}$ } w