Reminder: given $(X,\mathcal{T})$, $x \in X$:

-1-

- $\mathcal{T}(x)$ consists of opens U s.t. $x \in U$ (open neighborhoods of x)
- $\mathcal{N}(x)$ consists of all $V \subseteq X$ containing some $U \in \mathcal{T}(x)$ (neighbds of x)
- $\underset{\underset{X}{\cap}}{x_n} \longrightarrow x$ in $(X,\mathcal{T})$ if: $(\forall)\, V \in \cancel{\mathcal{N}(x)}$ $\exists\, n_V \in \mathbb{N}$ s.t. $x_n \in V$ $(\forall)\, n \geq n_V$.

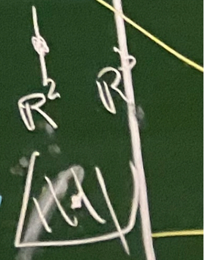
  given $\mathcal{B}_x$ basis $\quad B \in \mathcal{B}_x$

- basis of neighbds of x: any collection $\mathcal{B}_x \subseteq \mathcal{N}(x)$ s.t.

  $(\forall)\, N \in \mathcal{T}(x)$, $\exists\, B \in \mathcal{B}_x$ s.t. $B \subseteq N$.

- call $(X,\mathcal{T})$ 1st countable if $(\forall)\, x \in X$

  $\exists$ basis of neighbds of x which is countable: $\mathcal{B}_x = \{B_x^1, B_x^2, \dots\}$



Example: (X,d), $\underset{\mathcal{T}}{\mathcal{T}_d}$ $\Rightarrow$ can take

$\mathcal{B}_x = \{B_d(x,\varepsilon) : \varepsilon > 0\}$, $\quad \mathcal{B}'_x = \{B_d(x,\tfrac{1}{n}) : n \in \mathbb{N}\}$

$\exists$ basis of neighbds of x which is countable: $\mathcal{B}_x = \{B_x^1, B_x^2, \dots\}$

Example: (X,d), $\mathcal{T}_d$ $\Rightarrow$ can take

$\mathcal{B}_x = \{B_d(x,\varepsilon) : \varepsilon > 0\}$, $\quad \mathcal{B}'_x = \{B_d(x,\tfrac{1}{n}) : n \in \mathbb{N}\}$

$\Rightarrow$ any metric topo... is 1st counta...

-2-

Given $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$, $f : X \to Y$ a function, $x \in X$:

- f = continuous if: $f^{-1}(U) \in \mathcal{T}_X \; (\forall)\, U \in \mathcal{T}_Y$
- f = continuous at x: $f^{-1}(V) \in \mathcal{N}_{\mathcal{T}_X}(x) \; (\forall)\, V \in \cancel{\mathcal{N}_{\mathcal{T}_Y}(f(x))}$ $\quad B \in \mathcal{B}_{f(x)}$
- f = sequentially continuous at x: $\begin{pmatrix} x_n \to x \\ \text{in } (X, \mathcal{T}_X) \end{pmatrix} \Rightarrow \begin{pmatrix} f(x_n) \to f(x) \\ \text{in } (Y, \mathcal{T}_Y) \end{pmatrix}$

Thm (2.25, 2.29, 2.35):

f = continuous $\iff$ f = continuous at all $x \in X$

$\Downarrow$ $\quad \Updownarrow$ if X is 1st countable

f = sequentially continuous at all $x \in X$

$\leadsto$ Moral/slogan: 1st countabil... when one use sequences ... their full power

Reminder: given $(X, \mathcal{T})$, $x \in X$: [-1-]

- $\mathcal{T}(x)$ consists of opens U s.t. $x \in U$ (open neighborhoods of x)
- $\mathcal{N}(x)$ consists of all $V \subseteq X$ containing some $U \in \mathcal{T}(x)$ (neighbds of x)
- $x_n \to x$ in $(X, \mathcal{T})$ if: $(\forall)\, V \in \mathcal{N}(x)$ $\exists\, n_V \in \mathbb{N}$ s.t. $x_n \in V$ $(\forall)\, n \geq n_V$.
  $x_n \in X$; given $\mathcal{B}_x$ basis; $B \in \mathcal{B}_x$
- basis of neighbds of x: any collection $\mathcal{B}_x \subseteq \mathcal{N}(x)$ s.t. $(\forall)\, N \in \mathcal{T}(x)$, $\exists\, B \in \mathcal{B}_x$ s.t. $B \subseteq N$.
- call $(X, \mathcal{T})$ **1st countable** if $(\forall)\, x \in X$ $\exists$ basis of neighbds of x which is countable $\mathcal{B}_x = \{B_x^1, B_x^2, \dots\}$

generally, when local conditions are imposed (involving neighbds): it suffices to check them on a basis of neighbds

Example: (X, d), $\mathcal{T}_d$ $\Rightarrow$ can take $\mathcal{B}_x = \{B_d(x, \varepsilon) : \varepsilon > 0\}$, $\mathcal{B}'_x = \{B_d(x, \frac{1}{n}) : n \in \mathbb{N}\}$

$\Rightarrow$ any metric topology is 1st countable!

Given $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$, $f : X \to Y$ a function, $x \in X$: [-2-]

- f = **continuous** if: $f^{-1}(U) \in \mathcal{T}_X$ $(\forall)\, U \in \mathcal{T}_Y$
- f = **continuous at x**: $f^{-1}(V) \in \mathcal{N}_{\mathcal{T}_X}(x)$ $(\forall)\, V \in \mathcal{N}_{\mathcal{T}_Y}(f(x))$; $B \in \mathcal{B}_{f(x)}$
- f = **sequentially continuous at x**: $\begin{pmatrix} x_n \to x \\ \text{in } (X, \mathcal{T}_X) \end{pmatrix} \Rightarrow \begin{pmatrix} f(x_n) \to f(x) \\ \text{in } (Y, \mathcal{T}_Y) \end{pmatrix}$

Thm (2.25, 2.29, 2.35):

f = continuous $\iff$ f = continuous at all $x \in X$

f = continuous at all $x \in X$ $\Downarrow$ f = sequentially continuous at all $x \in X$

$\Updownarrow$ if X is 1st countable

$\rightsquigarrow$ Moral/slogan: 1st countability is when one use sequences in their full power

-3-

Def: $(X, \mathcal{T})$ is said to be Hausdorff if:

$(\forall)\, x, y \in X,\ x \neq y \quad \exists\, V \in \mathcal{N}(x),\ \exists\, W \in \mathcal{N}(y)$ s.t. $V \cap W = \emptyset$.

Theorem: $(X, \mathcal{T})$ = Hausdorff $\Longrightarrow$ (any sequence $(x_n)_{n\geq 1}$ in X has at most one limit in $(X, \mathcal{T})$)

$\Longleftarrow$ if X is 1^{st} countable.

Rk: (X, d) metric, $\mathcal{T} = \mathcal{T}_d \Rightarrow (X, \mathcal{T}_d)$ automatically Hausdorff

pf: Let $x, y \in X$, $\boxed{x \neq y} \to \boxed{r = d(x,y) > 0}$

look at $B(x, \frac{r}{2}) \cap B(y, \frac{r}{2}) = \emptyset$

if $z \in$

$\Rightarrow \begin{cases} d(x,z) < \frac{r}{2} \\ d(y,z) < \frac{r}{2} \end{cases}$ $\Big/ +$ $\Rightarrow d(x,y) \leq d(x,z) + d(z,y) < \frac{r}{2} + \frac{r}{2} = r$

$\underset{r}{\shortparallel}$ $\Rightarrow$ Impossible $\Rightarrow$ no such z exists $\Rightarrow$

-4-

Interesting: $(X, \mathcal{T})$ if metrizable (i.e. $\exists\, d$ s.t. $\mathcal{T} = \mathcal{T}_d$)

$\Rightarrow (X, \mathcal{T})$ must be both Hausdorff & 1^{st} countable

pf: "$\Rightarrow$" Assume $\exists\, (x_n)_{n \geq 1}$ $\begin{cases} \bullet\ x_n \to x \\ \bullet\ x_n \to y \end{cases}$ $x \neq y$ By Hausdorfness $\exists\, V \in \mathcal{N}(x)$, $W \in \mathcal{N}(y)$ s.t. $V \cap W = \emptyset$.

$\exists\, n_V \in \mathbb{N}$: $x_n \in V$ $(\forall)\, n \geq n_V$

$\exists\, n_W \in \mathbb{N}$ $x_n \in W$ $(\forall)\, n \geq n_W$

Contradiction

-5-

"$\Leftarrow$" Assume X = 1^{st} countable, any seq. has at most one limit.

To prove: $(X, \mathcal{T})$ = Hausdorff.

Assume $\boxed{x, y \in X,\ x \neq y}$. To prove $V \cap W = \phi$ for some $V \in \mathcal{N}(x)$, $W \in \mathcal{N}(y)$

Assume that is not the case $\Rightarrow$

$\Rightarrow \boxed{V \cap W \neq \phi \quad \forall\ V \in \mathcal{N}(x),\ W \in \mathcal{N}(y)}$

1^{st} count $\Rightarrow \exists$ $\mathcal{B}_x = \{B_x^1, B_x^2, \dots\}$ basis of nghbds of x

$\mathcal{B}_y = \{B_y^1, B_y^2, \dots\}$ —— " —— y.

for each n we have $B_x^n \cap B_y^n \neq \phi$

i.e. $\exists\ x_n \in B_x^n \cap B_y^n$.

Now: $x_n \in B_x^n\ (\forall) n \Rightarrow x_n \to x$

$x_n \in B_y^n\ \forall n \Rightarrow x_n \to y$

CONTRADICTION.

Def: Given $(X,\mathcal{T})$, a basis of $\mathcal{T}$ (or of $(X,\mathcal{T})$) is a family of opens:

$$\mathcal{B} \subseteq \mathcal{T}$$

such that: any $U \in \mathcal{T}$ can be written as a union of opens that belong to $\mathcal{B}$.

Say that $(X,\mathcal{T})$ is <u>2nd countable</u> if $\exists$ a basis $\mathcal{B}$ of $(X,\mathcal{T})$ s.t. $\mathcal{B}$ is countable.

Ex: $(X,\mathcal{T}_d)$ metric topology; one interesting basis is

$\mathcal{B}$ basis of $(X,\mathcal{T}_d)$ $\longrightarrow$ $\mathcal{B} = \{B(x,\varepsilon) \,/\, x\in X, \varepsilon>0\}$ $\qquad$ $\mathcal{B}' = \{B(x,\tfrac{p}{q}) : x\in X,\ p,q\in\mathbb{Q}_{>0}\}$

Ex: $(\mathbb{R}, \mathcal{T}_{Eucl})$ ✓ $\mathcal{B} = \{(a,b) \,/\, a,b\in\mathbb{R}, a<b\}$ a basis of $(\mathbb{R},\mathcal{T}_{Eucl})$

requires a proof $\longrightarrow$ $\mathcal{B}' = \{(p,q) \,/\, p,q\in\mathbb{Q}\}$ a countable basis of $(\mathbb{R},\mathcal{T}_{Eucl})$

Ex: $(\mathbb{R}, \mathcal{T}_\ell)$ ✓ $\mathcal{B} = \{[a,b) : a,b\in\mathbb{R}, a<b\}$ a basis of $(\mathbb{R},\mathcal{T}_{Eucl})$

requires a more difficult proof $\longrightarrow$ $\mathcal{B}' = \{[p,q) : p,q\in\mathbb{Q}\}$ ... no longer a basis !!

-7-

Ex: $(\mathbb{R}^n, \mathcal{T}_{Eucl}^{\mathbb{R}^n})$, $\boxed{(A, \mathcal{T}_{Eucl}^A)}$ for $A\subseteq\mathbb{R}^n$

countable a basis of $(\mathbb{R}^n, \mathcal{T}_{Eucl})$

Good to know $\forall A\subseteq\mathbb{R}^n$ with Euclidean top. are 2nd countable

Ex: $(\mathbb{R}, \mathcal{T}_{Eucl})$ ✓ $\mathcal{B} = \{(a,b) / a,b \in \mathbb{R}, a<b\}$ a basis of $(\mathbb{R}, \mathcal{T}_{Eucl})$

requires a proof $\longrightarrow$ $\mathcal{B}' = \{(p,q) / p,q \in \mathbb{Q}\}$ a countable basis of $(\mathbb{R}, \mathcal{T}_{Eucl})$

Ex: $(\mathbb{R}, \mathcal{T}_{\ell})$ ✓ $\mathcal{B} = \{[a,b) : a,b \in \mathbb{R}, a<b\}$ a basis of $(\mathbb{R}, \mathcal{T}_{Eucl})$

requires a more difficult proof $\longrightarrow$ $\mathcal{B}' = \{[p,q) : p,q \in \mathbb{Q}\}$ ... no longer a basis !!

Ex: $(\mathbb{R}^n, \mathcal{T}_{Eucl}^{\mathbb{R}^n})$, $\boxed{(A, \mathcal{T}_{Eucl}^{A})}$ for $A \subseteq \mathbb{R}^n$ — ? — countable

Good to know: $\forall A \subseteq \mathbb{R}^n$ with Euclidean top. are 2^{nd} countable

$\mathcal{B} = \{B(x, q) / x \in \mathbb{Q}^n, q \in \mathbb{Q}_{>0}\}$ ... a basis of $(\mathbb{R}^n, \mathcal{T}_{Eucl})$

Ex: Given $(X, \mathcal{T})$, $A \subseteq X$ $\Rightarrow$ $\mathcal{T}|_A := \{A \cap U / U \in \mathcal{T}\}$ $(A, \mathcal{T}|_A)$

If $\underset{\text{is countable}}{\mathcal{B}}$ = basis for $(X, \mathcal{T})$ $\Rightarrow$ $\mathcal{B}|_A := \underset{\text{countable as well}}{\{A \cap B / B \in \mathcal{B}\}}$ a basis of $(A, \mathcal{T}|_A)$

$\Rightarrow$ Conseq: a subspace of a 2^{nd} countable space must be 2^{nd} countable.

Rk: Given $\mathcal{B} \subseteq \mathcal{T}$ denote $\mathcal{B}(x) = \{B \in \mathcal{B} : x \in B\}$

Then:

$\mathcal{B}$ = basis for $(X, \mathcal{T})$ $\Longleftrightarrow$ $\mathcal{B}(x)$ is a basis of neighborhoods of x in $(X, \mathcal{T})$ $(\forall) x \in X$

In particular: 2^{nd} countable $\Rightarrow$ 1^{st} countable.

-6-

Def: Given $(X,\mathcal{T})$, a basis of $\mathcal{T}$ (or of $(X,\mathcal{T})$) is a family of opens:
$$\mathcal{B} \subseteq \mathcal{T}$$
such that: any $U \in \mathcal{T}$ can be written as a union of opens that belong to $\mathcal{B}$.

Say that $(X,\mathcal{T})$ is 2nd countable if $\exists$ a basis $\mathcal{B}$ of $(X,\mathcal{T})$ s.t. $\mathcal{B}$ is countable.

Ex: $(X, \mathcal{T}_d)$ metric topology: one interesting basis is

$\mathcal{B}$ basis of $(X,\mathcal{T}_d)$ $\longrightarrow$ $\mathcal{B} = \{ B(x,\varepsilon) / x\in X, \varepsilon>0\}$ $\qquad \mathcal{B}' = \{B(x, \frac{p}{q}) \mid x\in X,\ p,q\in\mathbb{Q}_{>0}\}$

Ex: $(\mathbb{R}, \mathcal{T}_{Eucl})$ ✓ $\mathcal{B} = \{(a,b) / a,b\in\mathbb{R}, a<b\}$ a basis of $(\mathbb{R}, \mathcal{T}_{Eucl})$

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Ex: $(\mathbb{R}, \mathcal{T}_\ell)$ ✓ $\mathcal{B} = \{[a,b) : a,b\in\mathbb{R}, a<b\}$ a basis of $(\mathbb{R}, \mathcal{T}_{Eucl})$

requires a more difficult proof $\longrightarrow$ $\mathcal{B}' = \{[p,q) : p,q\in\mathbb{Q}\}$ ... no longer a basis !!

-7-

Ex $(\mathbb{R}^n, \mathcal{T}^{\mathbb{R}^n}_{Eucl})$, $\boxed{(A, \mathcal{T}^A_{Eucl})}$ for $A \subseteq \mathbb{R}^n$ countable

Good to know $\forall A\subseteq\mathbb{R}^n$ with Euclidean top. are 2nd countable

$\mathcal{B} = \{B(x,q) / x\in\mathbb{Q}^n, q\in\mathbb{Q}_{>0}\}$ ... a basis of $(\mathbb{R}^n, \mathcal{T}_{Eucl})$

Ex: Given $\left.\begin{array}{l}(X,\mathcal{T})\\ A\subseteq X\end{array}\right\} \Rightarrow \mathcal{T}|_A = \{A\cap U / U\in\mathcal{T}\}$ $\quad (A, \mathcal{T}|_A)$

If $\underset{\text{if countable}}{\mathcal{B} = \text{basis for } (X,\mathcal{T})} \Rightarrow \mathcal{B}|_A := \underset{\text{countable as well}}{\{A\cap B / B\in\mathcal{B}\}}$ a basis of $(A, \mathcal{T}|_A)$

$\Rightarrow$ Learn: a subspace of a 2nd countable space must be 2nd countable.

Rk: Given $\mathcal{B} \subseteq \mathcal{T}$ denote $\mathcal{B}(x) = \{B\in\mathcal{B} : x\in B\}$

Then:
$$\mathcal{B} = \text{basis for } (X,\mathcal{T}) \iff \mathcal{B}(x) \text{ is a basis of neighborhoods of } x \text{ in } (X,\mathcal{T}) \ (\forall)\, x\in X$$

In particular: 2nd countable $\Rightarrow$ 1st countable.

Inside a topological space $(X,\mathcal{J})$: $A \subseteq X$ subset

-8-

$$\mathring{A} := \text{the largest open of } (X,\mathcal{J}) \text{ contained in } A$$

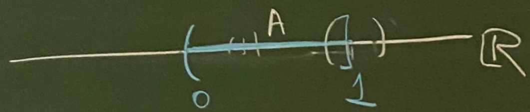
$$= \bigcup_{\substack{U \text{ open in } (X,\mathcal{J}) \\ \text{with } U \subseteq A}} U \quad \text{open}$$

$$\bar{A} := \text{the smallest closed of } (X,\mathcal{J}) \text{ containing } A$$

$$= \bigcap_{\substack{F \text{ closed in } (X,\mathcal{J}) \\ \text{with } A \subseteq F}} F$$

-9-

Ex: $A=(0,1]$ in $X=\mathbb{R}$ with $\mathcal{J}_{Eucl}$



$\mathring{A}=(0,1)$

$\bar{A}=[0,1]$

The largest closed inside A: $\nexists$

Lemma: For $x \in X$ one has

(i) $x \in \mathring{A} \iff \exists\, U \in \mathcal{T}(x)$ s.t. $U \subseteq A$

(ii) $x \in \bar{A} \iff (\forall)\, U \in \mathcal{T}(x)$ one has $A \cap U \neq \emptyset$

(iii) if $(X, \mathcal{T})$ is 1^{st} countable then

$x \in \bar{A} \iff \exists$ sequence $(a_n)_{n \geq 1}$ in A st $a_n \to x$ in $(X, \mathcal{T})$.
$(a_n \in A)$

Def: (X, J) is said to be Hausdorff if
-3-
Theorem: (X, J) = Hausdorff ⇒ any sequence (x_n) in X has at most one limit in (X, J)
1st countable
Rk (X, d) metric, J = J_d ⇒ (X, J_d) automatically Hausdorff
Lemma: For x ∈ X one has
-10-
(i) x ∈ Å ⇔ ∃ U ∈ J(x) s.t. U ⊆ A
(ii) x ∈ Ā ⇔ (∀) U ∈ J(x) one has A ∩ U ≠ ∅
(iii) if (X, J) is 1st countable then
Reminder: given (X, J), x ∈ X:
• J(x) consists of opens U s.t. x ∈ U (open neighborhoods of x)
• N(x) consists of all V ⊆ X containing some U ∈ J(x) (neighbds of x)
• basis of neighbds of x: any collection
• call (X, J) 1st countable if (∀) x ∈ X
∃ basis of neighbds of x which is countable
Example (X, d), J_d ⇒ can take
Def Given (X, J), a basis of J (or of (X, J)) is a family of opens
B ⊆ J
such that: any U ∈ J can be written as a union of opens that belong to B.
Say that (X, J) is 2nd countable
Ex (X, J_d) metric topology: one interesting basis is
a countable basis of (R, J_Eucl)
... no longer a basis !!
Given (X, J_X), (Y, J_Y), f: X → Y a function, x ∈ X
• f = continuous if
• f = continuous at x
• f = sequentially continuous at x
Thm (2.25, 2.29, 2.35)
f = continuous ⇔ f = continuous at all x ∈ X
if X is 1st countable
f = sequentially continuous at all x ∈ X
In particular 2nd countable ⇒ 1st countable
Ex A = (0, 1]
Å = (0, 1)
Ā = [0, 1]