–4–

Reminder:

• Ex: $\mathcal{T} = \mathcal{T}_d$ (whenever d – metric on X) where for $U \subset X$

–1–

Reminder:

- <u>topological space</u> $(X, \mathcal{T})$: $\emptyset, X$ opens &:
  - $\to$ U, V – open $\Rightarrow$ $U \cap V$ is open
  - $\to$ U_i – open $(\forall)\, i \in I$ $\Rightarrow$ $\bigcup_{i \in I} U_i$ is open
- <u>basis for $(X, \mathcal{T})$</u>: collection $\mathcal{B} \subseteq \mathcal{T}$ s.t. each $U \in \mathcal{T}$ is a union of opens from $\mathcal{B}$
- $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ <u>continuous</u> if

  $f^{-1}(U)$ – open in X $(\forall)$ U – open in Y.
- <u>Slogan</u>: IF X IS A SET & WE LOOK FOR AN INTERESTING TOPOLOGY on X ... THINK WHAT "<u>CANONICAL MAPS</u>" SHOULD BE CONTINUOUS!

PRODUCTS

What

Want

$\Rightarrow$

-4-

GENERAL PROCEDURE : Set X

-5-

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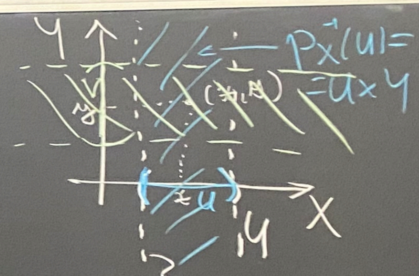
PRODUCTS: X, Y - sets $\Rightarrow$ we can construct

$$X \times Y = \{(x,y) : x \in X, y \in Y\}$$



What if $(X, J_X), (Y, J_Y)$ are topological spaces?

Want p_X, p_Y - continuous.

$\Rightarrow$ $(\forall)\ U \in J_X$: $p_X^{-1}(U) = U \times Y$ should be open in $X \times Y$

$(\forall)\ V \in J_Y$: $p_Y^{-1}(V) = X \times V$ should be open in $X \times Y$.

$\Rightarrow$ $U \times V$ - should be open in $X \times Y$ $\quad (\forall)\ U \in J_X, V \in J_Y$

$$U \times V = (U \times Y) \cap (X \times V)$$

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Define the product topology on $X\times Y$, denoted $\mathcal{T}_X \hat{\times} \mathcal{T}_Y$, as follows: it consists of those $D \subseteq X\times Y$ such that:

D is a union of products $U\times V$ of opens $U\in\mathcal{T}_X$, $V\in\mathcal{T}_Y$

$$\left(\Longleftrightarrow \quad (\forall)(x,y)\in D\,,\ \exists \begin{cases} U\in\mathcal{T}_X \\ V\in\mathcal{T}_Y \end{cases} \text{s.t. } (x,y)\in U\times V\,,\ U\times V\subseteq D. \right)$$

requires a little proof

RE : set X

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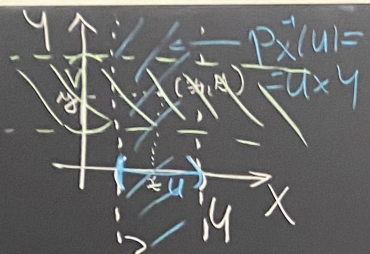

$$\begin{tikzcd} X \times Y \arrow[r, "p_X"] \arrow[d, "p_Y"'] & X \\ Y & \end{tikzcd} \qquad \begin{tikzcd} (x,y) \arrow[r, mapsto, "p_X"] \arrow[d, mapsto, "p_Y"'] & x \\ y & \end{tikzcd}$$

gical spaces?

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uld be open in $X \times Y$.

in $X \times Y$

$U \in \mathcal{T}_X$, $V \in \mathcal{T}_Y$

– 3' –

Proposition: $\mathcal{T}_X \times \mathcal{T}_Y$ is a topology on $X \times Y$, and it is the smallest one such that $p_X : X \times Y \to X$ continuous.
$p_Y : X \times Y \to Y$

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X) where for $U \subset X$:

GENERAL PROCEDURE: set X -5-

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PRODUCTS: X, Y - sets $\Rightarrow$ we can construct

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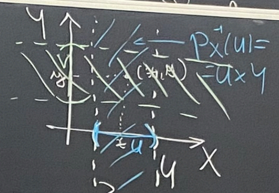

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-3-

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-4-

Reminder:

• Ex: $\mathcal{T}=\mathcal{T}_d$ (whenever d-metric on X) where for $U\subset X$:

-5-

GENERAL PROCEDURE: set X

-1-

Reminder:

• topological space $(X,\mathcal{T})$: $\emptyset, X$ opens &

→ U,V-open ⇒ $U\cap V$ is open

→ U_i-open $(\forall)\, i\in I$ ⇒ $\bigcup_{i\in I} U_i$ is open

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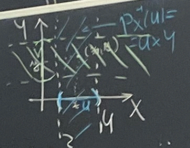
-2-

PRODUCTS: X, Y-sets ⇒ we can construct

$$X\times Y=\{(x,y): x\in X, y\in Y\}$$



$$\begin{array}{ccc} X\times Y & \xrightarrow{p_X} & X \\ {\scriptstyle p_Y}\downarrow & & \\ Y & & \end{array} \qquad \begin{array}{ccc} (x,y) & \xmapsto{p_X} & x \\ {\scriptstyle p_Y}\downarrow & & \\ y & & \end{array}$$

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-3'-

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-4-

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- Ex: $\mathcal{T}=\mathcal{T}_d$ (whenever d-metric on X) where, for $U\subseteq X$:

$$U\in\mathcal{T}_d \iff (\forall)\, x\in U\ \exists\, \varepsilon>0 \text{ s.t. } B_d(x,\varepsilon)\subseteq U$$
$$\iff U \text{ is a union of open balls}$$

- Ex: $\mathcal{T}=\mathcal{T}_{Eucl}$ on $X=\mathbb{R}$ where, for $U\subseteq\mathbb{R}$:

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Interesting: $\mathcal{T}_{Eucl}\subseteq\mathcal{T}_\ell$ $\quad\left((a,b)=\bigcup [a+\tfrac{1}{n},b)\right)$

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proof:

"$\subseteq$" Know

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-2-

PRODUCTS: X,Y-sets $\Rightarrow$

$X\times Y=\{(x$

What if $(X,\mathcal{T}_X),(Y$

-5-

GENERAL PROCEDURE: set X

Assume we have given a collection $\mathcal{B}$ of subsets of X.

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$$\mathcal{T}(\mathcal{B}) := \{ U \subseteq X : (\forall)\, x \in U \ \exists\, B \in \mathcal{B} \text{ s.t. } x \in B,\ B \subseteq U \}$$

$$= \{ U \subseteq X : U = \emptyset \text{ or } U \text{ is a union of members of } \mathcal{B} \}$$

proof: "$\supseteq$" $U = \bigcup_{i \in I} B_i$. Then for $x \in U \Rightarrow \exists\, i$ s.t. $x \in B_i, B_i \subseteq U$. $\boxtimes$

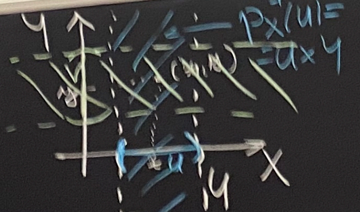
"$\subseteq$" Know: $(\forall)\, x \in U$, $\exists\, B_x \in \mathcal{B}$ s.t. $\{x\} \subseteq B_x \subseteq U$ apply $\bigcup_{x \in U}$ $\Rightarrow U \subseteq \bigcup_{x \in U} B_x \subseteq U$

$$\underbrace{\bigcup_{x \in U} \{x\}}_{U} \subseteq \bigcup_{x \in U} B_x$$

-2-

...ets $\Rightarrow$ we can construct

$\{ (x,y) : x \in X, y \in Y \}$



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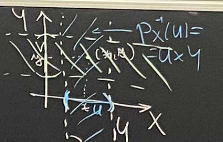

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-3-

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-4-

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  $\Leftrightarrow U$ is a union of open balls
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-5-

GENERAL PROCEDURE: set X

Assume we have given a collection $\mathcal{B}$ of subsets of X.

Construct

$$\mathcal{T}(\mathcal{B}) := \{U\subseteq X : (\forall)\, x\in U\ \exists\, B\in\mathcal{B} \text{ s.t. } x\in B, B\subseteq U\}$$

$$= \{U\subseteq X : U=\phi \text{ or } U \text{ is a union of members of } \mathcal{B}\}$$

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$$\underbrace{\bigcup_{x\in U}\{x\}}_{U}\subseteq\bigcup_{x\in U} B_x$$

where, for $U \subseteq X$:

$\varepsilon > 0$ s.t. $B_d(x,\varepsilon) \subseteq U$

open balls

$\mathbb{R}$:

s.t. $x \in (a,b) \subseteq U$

GENERAL PROCEDURE: Set X -5-

Assume we have given a collection $\mathcal{B}$ of subsets of X.

Construct

$$\mathcal{T}(\mathcal{B}) := \{ U \subseteq X : (\forall) x \in U \ \exists B \in \mathcal{B} \text{ s.t. } x \in B, B \subseteq U \}$$

$$\overset{\text{ex}}{=} \{ U \subseteq X : U = \emptyset \text{ or } U \text{ is a union of members of } \mathcal{B} \}$$

Prop 3.17: The following are equivalent: -6-

(2) $\mathcal{T}(\mathcal{B})$ is a topology on X

(1) $\mathcal{B}$ satisfies:

$$\boxed{\begin{array}{l} (B1) \text{ for each } x \in X \ \exists B \in \mathcal{B} \text{ s.t. } x \in B \\ (B2) \text{ for each } B_1, B_2 \in \mathcal{B}, x \in B_1 \cap B_2 \\ \qquad \exists B \in \mathcal{B} \text{ s.t. } x \in B, B \subseteq B_1 \cap B_2 \end{array}}$$

Def: A collection $\mathcal{B}$ of subsets of X satisfying these is called: A TOPOLOGY BASIS ON X

Rk: $(B1) \iff X \in \mathcal{T}(\mathcal{B})$

Rk: $(B2) \iff (\forall) B_1, B_2 \in \mathcal{B}, B_1 \cap B_2 \in \mathcal{T}(\mathcal{B})$

Moreover, in this case:

- $\mathcal{T}(\mathcal{B})$ = smallest topology on X that contains $\mathcal{B}$.
- $\mathcal{B}$ is a basis for $(X, \mathcal{T}(\mathcal{B}))$

-3'-

Proposition: $\mathcal{T}_X \times \mathcal{T}_Y$ is a topology on $X \times Y$, and it is the smallest one such that $p_X : X \times Y \to X$, $p_Y : X \times Y \to Y$ continuous.

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Reminder:

- Ex1: $\mathcal{T} = \mathcal{T}_d$ (whenever d-metric on X) where, for

$U \in \mathcal{T}_d \iff (\forall)\ x \in U\ \exists\ \varepsilon > 0$ s.t.

$\iff U$ is a union of open ball

- Ex2: $\mathcal{T} = \mathcal{T}_{Eucl}$ on $X = \mathbb{R}$ where, for $U \subseteq \mathbb{R}$:

$U \in \mathcal{T}_{Eucl} \iff (\forall)\ x \in U\ \exists\ a, b$ s.t. $x \in (a,$

-7-

Examples:

Ex1: $\mathcal{B}$ - collection of all balls w.r.t. d

(B2) $x \in B(x_1, \varepsilon_1) \cap B(x_2, \varepsilon_2)$

$\exists\ \varepsilon$ s.t. $B(x, \varepsilon) \subset B(x_1, \varepsilon_1) \cap B(x_2, \varepsilon_2)$

$\mathcal{T}(\mathcal{B}) = \mathcal{T}_d$

Ex2: On $\mathbb{R}$: collection $\mathcal{B}$ of all open intervals.

Ex3: On $\mathbb{R}$ — " ——— intervals of type $[a, b)$

Ex4: For any two top. spaces $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$ we have a topology basis $\mathcal{B}$ on $X \times Y$: the one consistings of products $U \times V$ $\quad U \in \mathcal{T}_X$, $V \in \mathcal{T}_Y$

The resulting top. the product topology.

Pro

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$U = \phi$ or U is a union of members of $\mathcal{B}\}$

$x \in U \Rightarrow \exists i$ s.t. $x \in B_i$, $B_i \subseteq U$. $\boxed{}$

Rk : $(B1) \iff X \in \mathcal{T}(\mathcal{B})$

Rk : $(B2) \iff (\forall)\ B_1, B_2 \in \mathcal{B},\ B_1 \cap B_2 \in \mathcal{T}(\mathcal{B})$

$\in \mathcal{B}$)

$\cap B_2$

$B_1 \cap B_2$

contain $\mathcal{B}$

$(1) \Rightarrow (2)$

-8-

(T1) $\phi, \underline{X \in \mathcal{T}(\mathcal{B})}$ (B1) ✓

(T3) unions of members of $\mathcal{T}(\mathcal{B})$ is in $\mathcal{T}(\mathcal{B})$ follows from 2nd description of $\mathcal{T}(\mathcal{B})$

(T2) $\underbrace{U, V \in \mathcal{T}(\mathcal{B})} \Rightarrow U \cap V \in \mathcal{T}(\mathcal{B})$?

$$U = \bigcup_{i \in I} B_i,\quad V = \bigcup_{i' \in I'} B'_{i'}$$

$B_i \in \mathcal{B}$ $\qquad$ $B'_{i'} \in \mathcal{B}$

$$U \cap V = \bigcup_{i, i'} \underbrace{(B_i \cap B'_{i'})}_{\mathcal{T}(\mathcal{B})} \in \mathcal{T}(\mathcal{B}) \quad \boxed{}$$

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Generating topologies: X = set

$\mathcal{S}$ = any collection of subsets of X

Want: topology on X s.t. each $S \in \mathcal{S}$ becomes open.

$\langle \mathcal{S} \rangle$ (the topology generated by $\mathcal{S}$)

$:=$ the smallest topology on X containing $\mathcal{S}$

Remark: this $\exists$ because, in general, if we have topologies $\mathcal{T}_i$ on X, $i \in I$ $\Rightarrow$ $\bigcap_{i \in I} \mathcal{T}_i = \{ U \subseteq X \mid U \in \mathcal{T}_i \ (\forall) i \in I \}$ is again a topology on X

$$= \bigcap_{\substack{\mathcal{T}\text{-topology on } X \\ \text{containing } \mathcal{S}}} \mathcal{T}.$$

Prop:

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INITIAL

$X =$

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Examples:

Ex1: $\mathcal{B}$ - collection of all balls w.r.t. d

Prop 3.17: The following are eq

-11-

Prop: Defining

$$\mathcal{B}_{\mathcal{S}} := \{ B \subseteq X \mid B \text{ is a finite intersection of members of } \mathcal{S} \}$$

if each $x \in X$ belongs to at least one $S \in \mathcal{S}$ $\Longrightarrow$

$\Longrightarrow$ $\mathcal{B}_{\mathcal{S}}$ is a topology basis and $\mathcal{T}(\mathcal{B}_{\mathcal{S}}) = \langle \mathcal{S} \rangle$.

INITIAL TOPOLOGIES: General scenario:

$X = \text{set}$, $\quad f_i : \underline{X} \longrightarrow X_i$ $\quad$ each $(X_i, \mathcal{T}_i)$ is a topological space

topology? on X

$f_{i_1}^{-1}(U_{i_1}) \cap \dots \cap f_{i_k}^{-1}(U_{i_k}) \in \mathcal{T}$.

Answer:

$$\mathcal{S} = \{ U \subseteq X \mid U = f_i^{-1}(U_i) \text{ for some } i, \text{ some } U_i \in \mathcal{T}_i \} \subseteq \mathcal{T}$$

& consider $\langle \mathcal{S} \rangle$ - topology on X, the smallest one s.t. all f_i become continuous.

$[0,\infty)$ open in $(\mathbb{R}, \mathcal{T}_e)$? closed in $(\mathbb{R}, \mathcal{T}_e)$?

$f([a,b)) \subseteq \mathbb{R}$ $\quad$ (B1) $\Longleftrightarrow$ $X \in \mathcal{T}(\mathcal{B})$

... equivalent: $\quad$ -6- $\quad$ $\mathbb{R}_K$ $\quad$ $B_1, B_2 \in \mathcal{B}$, $\; B_1 \cap B_2 \in \mathcal{T}(\mathcal{B})$ $\quad$ -8-