-1-

Reminder: X = set

For: $\mathcal{S}$ = any collection of subsets of X

$\langle \mathcal{S} \rangle :=$ the smallest among all topologies $\mathcal{T}$ on X such that $\mathcal{S} \subseteq \mathcal{T}$.

Remark: $S_1 \cap \ldots \cap S_k \in \mathcal{T}$ $(\forall)\ S_1, \ldots, S_k \in \mathcal{S}$

any $U \subseteq X$ that is a union of such finite intersections or $U = X$ must be in $\mathcal{T}$

the collection of all such U does form a topology $\langle \mathcal{S} \rangle$

-1'-

Remark: $x_n \to x$ w.r.t. $\langle \mathcal{S} \rangle \iff (\forall)\ \underset{\text{s.t. } x \in S}{S \in \mathcal{S}}$, $\exists\, n_S \in \mathbb{N}$ s.t. $x_n \in S$ $(\forall)\ n \geq n_S$.

Particular case: When $\mathcal{S} = \mathcal{B}$ is a topology basis $\Rightarrow$ $\Rightarrow \langle \mathcal{S} \rangle = \mathcal{T}(\mathcal{B})$ with simpler description

Examples: $\mathcal{T}_d$, $\mathcal{T}^{\mathbb{R}}_{Eucl}$, $\mathcal{T}_e$.

$\boxed{-2-}$

Spaces of functions (sec 9/pp65): X, Y — topological space

$\mathcal{F}(X,Y) := \{\text{functions } f: X \to Y\}$

$\mathcal{C}(X,Y) := \{f \in \mathcal{F}(X,Y) \mid f = \text{continuous}\}$.

Focus on: $X = I \subseteq \mathbb{R}$ interval, $Y = \mathbb{R}^n$, $d = d_{Eucl}$ & the induced (Euclidean) top.

Known convergences: for $\boxed{(f_n)_{n\geq 1}, f \in \mathcal{F}(X,Y)}$:

- $f_n \xrightarrow{pt} f$ (pointwise): if $(\forall)\, x \in X$, $f_n(x) \to f(x)$ in Y, i.e.:

  $(\forall)\, x \in X, (\forall)\, \varepsilon > 0\ \exists\, n_\varepsilon^* \text{ s.t. } d(f_n(x), f(x)) < \varepsilon\ (\forall)\, n \geq n_\varepsilon^*$

- $f_n \rightrightarrows f$ (uniformly) if

  $(\forall)\, \varepsilon > 0\ \exists\, n_\varepsilon \text{ s.t. } d(f_n(x), f(x)) < \varepsilon\ (\forall)\, n \geq n_\varepsilon\ (\forall)\, x \in X$

- $f_n \xrightarrow{cp} f$ (uniformly on compacts) if:

  $(\forall)\, K \subseteq I$ compact interval $\quad f_n|_K \rightrightarrows f|_K$.

$\boxed{-2'-}$

Basic properties:

→ remark: $f_n \rightrightarrows f \Rightarrow f_n \xrightarrow{cp} f \Rightarrow f_n \xrightarrow{pt} f$.

→ proposition: If $f_n \in \mathcal{C}(X,Y)$ and $f_n \xrightarrow{cp} f \Rightarrow f \in \mathcal{C}(X,Y)$.

-3-

Pointwise convergence

For each $x \in X$, $U \subseteq Y$ open define: $\mathbb{R}^n$

$$S(x,U) = \{ f \in \mathcal{F}(X,Y) : f(x) \in U \} \subseteq \mathcal{F}(X,Y)$$

& consider

$$\mathcal{S} := \{ S(x,U) \,/\, x \in X, U \subseteq Y \text{ open} \}$$

$$\mathcal{T}_{pt} := \langle \mathcal{S} \rangle \text{ a topology on } \mathcal{F}(X,Y).$$

Proposition: $f_n \xrightarrow{pt} f \iff f_n \to f$ in $(\mathcal{F}(X,Y), \mathcal{T}_{pt})$

Pf: RHS $\iff$ $(\forall) x \in X$, $(\forall) U \in \mathcal{T}_Y$ s.t. $f(x) \in U$, $\exists\, n_U^*$ s.t. $f_n(x) \in U$ $(\forall) n \geq n_U^*$

$(\forall) U \in \mathcal{T}_Y(f(x))$ $\exists\, n_U^*$ s.t. — " —

i.e. $\boxed{f_n(x) \to f(x) \text{ in } Y}$ i.e. the LHS.

-4-

Uniform convergence

For $f, g \in \mathcal{F}(X,Y)$ define

The Euclidean distance for $Y = \mathbb{R}^n$

$$d_{sup}(f,g) := \sup \{ d(f(x), g(x)) : x \in X \}$$

$$\hat{d}_{sup}(f,g) := \min \{ 1, d_{sup}(f,g) \}$$

Exercise: Show this is indeed a metric on $\mathcal{F}(X,Y)$.

$B_{d_{sup}}(f, \varepsilon)$

$\mathcal{T}_{unif}$:= the induced metric topology on $\mathcal{F}(X,Y)$

Prop: $f_n \rightrightarrows f \iff f_n \to f$ in $(\mathcal{F}(X,Y), \mathcal{T}_{unif})$.

Pf: RHS $\iff \hat{d}_{sup}(f_n, f) \to 0$ in $\mathbb{R} \iff d_{sup}(f_n, f) \to 0$ in $\mathbb{R}$

$\iff (\forall) \varepsilon > 0\ \exists\, n_\varepsilon$ s.t. $\underbrace{d_{sup}(f_n, f) < \varepsilon}_{d(f_n(x), f(x)) \leq \varepsilon \ (\forall) n \geq n_\varepsilon,\ (\forall) x}$ $(\forall) n \geq n_\varepsilon$

-3-

Pointwise convergence

For each $x \in X$, $U \subseteq Y$ open define:

$$S(x,U) = \{ f \in \mathcal{F}(X,Y) : f(x) \in U \} \subseteq \underline{\mathcal{F}(X,Y)} \quad \mathbb{R}^{X}$$

& consider

$$\mathcal{S} := \{ S(x,U) \,/\, x \in X, U \subseteq Y \text{ open} \}$$

$$\mathcal{T}_{pt} := \langle \mathcal{S} \rangle \text{ a topology on } \mathcal{F}(X,Y).$$

Proposition: $f_n \xrightarrow{pt} f \iff f_n \to f$ in $(\mathcal{F}(X,Y), \mathcal{T}_{pt})$

Pf: RHS $\iff$ $(\forall) x \in X$, $(\forall) U \in \mathcal{T}_Y$ s.t. $f(x) \in U$, $\exists n_U^x$ s.t. $f_n(x) \in U$ $(\forall) n \geq n_U^x$

-4-

Uniform convergence

For $f, g \in \mathcal{F}(X,Y)$ define

$$d_{sup}(f,g) := \sup \{ d(f(x), g(x)) : x \in X \}$$

The distance for $Y = \mathbb{R}^n$ Euclidean

$$\hat{d}_{sup}(f,g) := \min \{1, d_{sup}(f,g)\} \longleftarrow$$

Exercise: Show this is indeed a metric on $\mathcal{F}(X,Y)$

$B_{sup}(f, \varepsilon)$

$\mathcal{T}_{unif} :=$ the induced metric topology on $\mathcal{F}(X,Y)$

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Pf: RHS $\iff \hat{d}_{sup}(f_n, f) \to 0$ in $\mathbb{R}$ $\iff d_{sup}(f_n, f) \to 0$ in $\mathbb{R}$

$\iff (\forall) \varepsilon > 0 \; \exists n_\varepsilon$ s.t. $\underbrace{d_{sup}(f_n, f) < \varepsilon}_{d(f_n(x), f(x)) \leq \varepsilon \; (\forall) n \geq n_\varepsilon, \; (\forall) x}$ $(\forall) n \geq n_\varepsilon$

-5-

Uniform convergence on compacts

For each $f \in \mathcal{F}(X,Y)$, $\underline{K \subseteq I}$ compact interval, $\varepsilon > 0$

$$B(f, \varepsilon) = \{ g \in \mathcal{F}(X,Y) : d(f(x), g(x)) < \varepsilon \; (\forall) x \in K \}$$

$$\mathcal{B} = \{ B_K(f, \varepsilon) \,/\, K, \varepsilon, f \text{ as above} \} \rightsquigarrow$$

Exercise: Check this is actually a topology

$$\mathcal{T}_{cp} := \langle \mathcal{B} \rangle$$

Prop: $f_n \xrightarrow{cp} f \iff f_n \to f$ in $(\mathcal{F}(X,Y), \mathcal{T}_{cp})$

−6−

Theorem: These three topologies on $\mathcal{F}(X,Y)$ compare as follows:

$$\mathcal{T}_{pt} \subseteq \mathcal{T}_{cp} \subseteq \mathcal{T}_{unif}$$

Lemma 3.18.

and $\mathcal{C}(X,Y)$ - closed in $(\mathcal{F}(X,Y), \mathcal{T}_{unif})$. (B)

Recall: For a metric space (Z, d_Z) and $A \subseteq Z$ we have:

A = closed $\Leftrightarrow$ $A = \bar{A}$ $\Leftrightarrow$ $(\forall)\, x \in X$ which is a limit of a sequence $(a_n)_{n\geq 1}$ in Z with $a_n \in A$ must be in A.

$\subseteq$

This shows: (A) $\Leftrightarrow$ (B).

−5−

Uniform convergence on compacts

For each $f \in \mathcal{F}(X,Y)$, $K \subseteq I$ compact interval, $\varepsilon > 0$

$$B(f,\varepsilon) = \{ g \in \mathcal{F}(X,Y) : d(f(x), g(x)) < \varepsilon \ (\forall)\, x \in K \}$$

$$\mathcal{B} = \{ B_K(f,\varepsilon) \,/\, K, \varepsilon, f \text{ as above} \}$$

$\rightsquigarrow$ Exercise: Check this is actually a topology basis

$$\mathcal{T}_{cp} := \langle \mathcal{B} \rangle$$

Prop: $f_n \xrightarrow{cp} f \Leftrightarrow f_n \to f$ in $(\mathcal{F}(X,Y), \mathcal{T}_{cp})$

$\mathbb{R}$
$\mathcal{F}(X,Y)$
$\}$

$(X,Y), \mathcal{T}_{pt})$
s.t.
n_U^*

the LHS.

-6-

Theorem: These three topologies on $\mathcal{F}(X,Y)$ compare as follows:

$$\mathcal{T}_{pt} \subseteq \mathcal{T}_{cp} \subseteq \mathcal{T}_{unif}.$$

Lemma 3.18

and $\mathcal{C}(X,Y)$ - closed in $(\mathcal{F}(X,Y), \mathcal{T}_{unif})$. (B)

Recall: For a metric space (Z, d_Z) and $A \subseteq Z$ we have

A = closed $\Leftrightarrow A = \bar{A} \Leftrightarrow (\forall) z \in X$ which is a limit of a sequence $(a_n)_{n\geq 1}$ with $a_n \in A$ must be in A.

This shows: (A) $\Leftrightarrow$ (B).

-6-

arbitrary.

$(\forall) x \in U$

-5-

Uniform convergence on compacts

For each $f \in \mathcal{F}(X,Y)$, $K \subseteq I$ compact interval, $\varepsilon > 0$

$$B(f,\varepsilon) = \{ g \in \mathcal{F}(X,Y) : d(f(x), g(x)) < \varepsilon \ (\forall) x \in K \}$$

$$\mathcal{B} = \{ B_K(f,\varepsilon) \ / \ K, \varepsilon, f \text{ as above} \} \rightsquigarrow$$

Exercise: Check this is actually a topology basis

RHS $\Longleftrightarrow$ $(\forall)\, x\in X$, $(\forall)\, U\in \mathcal{J}_Y$ st $f(x)\in U$, $\exists\, n_U$ s.t. $f_n(x)\in U$ $(\forall)\, n\geq n_U^*$

$(\forall)\, U\in \mathcal{J}_Y(f(x))$ $\exists\, n_U^*$ s.t. — " —

i.e. $\boxed{f_n(x)\to f(x) \text{ in } Y}$ i.e. the LHS.

Proof of A: Assume $\begin{cases} f_n \rightrightarrows f \\ f_n\in \mathcal{C}(X,Y)\end{cases}$ To prove: f = continuous at $x_0\in X$ arbitrary.

Fix $x_0\in X$

To prove: $(\forall)\,\varepsilon>0$ $\exists\, \underset{(x_0-\delta,\, x_0+\delta)}{U}\in \mathcal{J}_X(x_0)$ s.t. $d(f(x),f(x_0))<\varepsilon$ $(\forall)\, x\in U$.

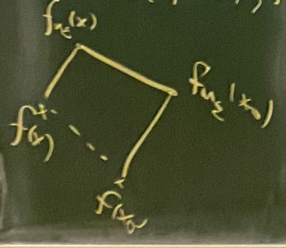
Now: fixing ε, since $f_n\rightrightarrows f \Rightarrow \exists\, n_\varepsilon$ s.t.

$\boxed{d(f_n(x),f(x))<\varepsilon/3}$ $(\forall)\, n\geq n_\varepsilon$, $(\forall)\, x\in X$.

Also: f_{n_ε} = continuous $\Rightarrow \exists\, U\in \mathcal{J}_X(x_0)$ st $\boxed{d(f_{n_\varepsilon}(x), f_{n_\varepsilon}(x_0))<\varepsilon/3}$ $(\forall)\, x\in U$

For: $x\in U$:

$$d(f(x),f(x_0))\leq \underbrace{d(f(x),f_{n_\varepsilon}(x))}_{<\varepsilon/3}+\underbrace{d(f_{n_\varepsilon}(x),f_{n_\varepsilon}(x_0))}_{<\varepsilon/3}+\underbrace{d(f_{n_\varepsilon}(x_0),f(x_0))}_{<\varepsilon/3}<\varepsilon.$$



he induced Euclidean) top.

n Y, i.e.:

$n\geq n_\varepsilon^*$

$n\geq n_\varepsilon$ $(\forall)\, x\in X$

A

(X,Y).

$\to f(x)=0$

$\in \mathcal{J}_{pt}(f)$

This s

Uniform

For

Prop

Quotient topologies

Def: Given $\begin{cases} \text{a topological space } (X, \mathcal{J}) \\ \text{a map } \pi : \underset{!!}{X} \longrightarrow \underset{???}{Y} \text{ which is surjective} \end{cases}$ ← a set

the quotient topology on Y induced by π is

$$\underline{\pi_*(\mathcal{J})} := \{ V \subseteq Y \;/\; \pi^{-1}(V) \in \mathcal{J} \}$$

$\cancel{Y \setminus \operatorname{Im} f}$

Thm: This is a topology on Y & it is the largest topology on Y making π continuous.

Pf: $\pi^{-1}(\emptyset) = \emptyset$, $\pi^{-1}(Y) = X$

$\pi^{-1}(V_1 \cap V_2) = \pi^{-1}(V_1) \cap \pi^{-1}(V_2)$

$\pi^{-1}\left(\bigcup_{i \in I} V_i\right) = \bigcup_{i \in I} \pi^{-1}(V_i)$

Rk: In practice, we are often interested in a given set Y and we want to make it into a topological space.

Approach: look for a space X that you do know and which is related to Y by a surjective function $\pi : X \to Y$.

the LHS.

arbitrary

$(\forall)\, x \in U$

$(\forall)\, x \in X$

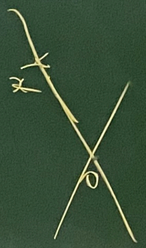
$(\forall)\, x \in U$

$< \varepsilon/3$

$< \varepsilon$

$\mathcal{T}_1 \subseteq \mathcal{T}_2$

$\boxed{-10-}$



Example 1: The projective space

$$\mathbb{P}^n = \{ \ell \,/\, \underbrace{\ell \subseteq \mathbb{R}^{n+1} \text{ is a line through the origin}}_{\text{i.e. } \ell = \mathbb{R}\cdot x = \{(\lambda x_0, \lambda x_1, \dots, \lambda x_n) : \lambda \in \mathbb{R}\}} \}$$

for some $x = (x_0, \dots, x_n) \in \mathbb{R}^{n+1} \setminus \{0\}$

Here we have

$$\pi : \mathbb{R}^{n+1} \setminus \{0\} \longrightarrow \mathbb{P}^n$$

$x \longmapsto$ the line through the origin and x.

$\Rightarrow$ an induced topology on $\mathbb{P}^n$: $\pi_*\left(\mathcal{T}_{Eucl}^{\mathbb{R}^{n+1} \setminus \{0\}}\right)$.

With this topology $\mathbb{P}^n$ is known as "the n-dimensional projective space"

d X $\Rightarrow \mathcal{T}_d$

d^A_{Eucl} $A \subseteq \mathbb{R}^{n+1}$ $\mathcal{T}^A_{Eucl} \ni U$ $U \subseteq A$ s.t. $(\forall)\, x \in U$ $\exists \varepsilon > 0$ s.t.

$$B_{d^A_{Eucl}}(x, \varepsilon) = \{ y \in A : d_{Eucl}(y, x) < \varepsilon \} \subseteq U$$