—1—

Reminder: a map $\pi:(X,\mathcal{T}_X)\longrightarrow(Y,\mathcal{T}_Y)$ continuous if, for $V\subseteq Y$:

$$V\in\mathcal{T}_Y \Longrightarrow \pi^{-1}(V)\in\mathcal{T}_X.$$

Def: π is called a topological quotient map if "$\Longleftrightarrow$" holds and π = surjective.

Slogan: close relationship between $(X,\mathcal{T}_X)$ and $(Y,\mathcal{T}_Y)$ (via π)

Proposition: If $\pi: X\to Y$ is a topological quotient map

$\Rightarrow$ for any other space $(Z,\mathcal{T}_Z)$ and a map $f:(Y,\mathcal{T}_Y)\longrightarrow(Z,\mathcal{T}_Z)$ one has:

$$(f = \text{continuous}) \Longleftrightarrow (f\circ\pi \text{ is continuous})$$

$$\begin{tikzcd}
(X,\mathcal{T}_X) \arrow[d, "\pi"'] \arrow[dr, dotted, "f\circ\pi"] & \\
(Y,\mathcal{T}_Y) \arrow[r, "f"'] & (Z,\mathcal{T}_Z)
\end{tikzcd}$$

Given

$V \subseteq Y.$

holds.

surjective

)

map

π

$\to (Z, \mathcal{T}_Z)$

—2—

Given $\left\{\begin{array}{l} Y \quad \text{a set} \\ (X, \mathcal{T}) \text{ a topological space} \\ \pi : X \to Y \text{ a surjective map} \end{array}\right\} \rightsquigarrow$ a topology on Y

$$\pi_*(\mathcal{T}) := \{ V \subseteq Y : \pi^{-1}(V) \in \mathcal{T} \}$$

In practice: you have Y and you look for an $(X, \mathcal{T})$ that you could use

- called <u>the quotint topology induced by π</u>.
- it makes π continuous (smallest such)
- it makes π a topological quotient map.

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Given $\left\{\begin{array}{l} Y \text{ a set} \\ (X,\mathcal{T}) \text{ a topological space} \\ \pi: X\longrightarrow Y \text{ a surjective map}\end{array}\right\} \rightsquigarrow$ -2- a topology on Y

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In practice: you have Y and you look for an $(X,\mathcal{T})$ that you could use

- 3 -

Example: $Y = \mathbb{P}^n = \{ \ell \,/\, \ell \subseteq \mathbb{R}^{n+1}$ is a line throug the origin $\}$ the projective space

This can be related to ... $X = \mathbb{R}^{n+1} \setminus \{0\}$ via a surjective map

$$\begin{array}{rcl} \pi : \mathbb{R}^{n+1} \setminus \{0\} & \longrightarrow & \mathbb{P}^n \\ \Psi & & \Psi \\ x & \longmapsto & \ell_x = \text{the line through the origin and } x. \\ \| & & \| \\ (x_0, x_1, \dots, x_n) & & \{ (\lambda x_0, \lambda x_1, \dots, \lambda x_n) : \lambda \in \mathbb{R} \} \end{array}$$

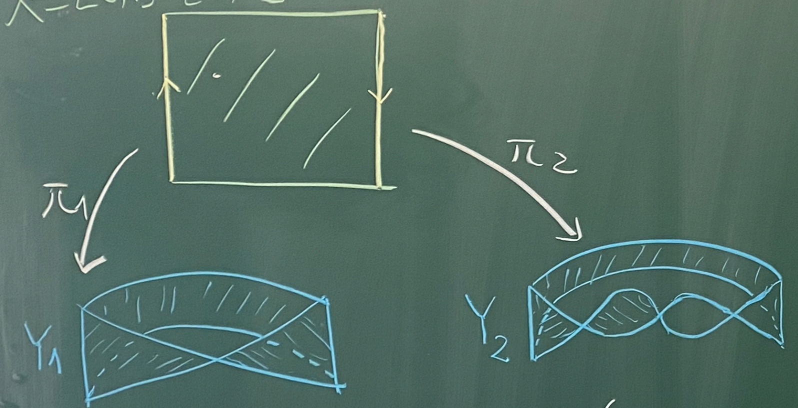
On $\mathbb{R}^{n+1} \setminus \{0\}$: the Euclidean topology $\Rightarrow$ the quotient topology on $\mathbb{P}^n$

(Hence $V \subseteq \mathbb{P}^n$ is open iff

$$\bigcup_{\ell \in \mathbb{P}^n} \ell = \{ x \in \mathbb{R}^{n+1} \setminus \{0\} : \exists \text{ line } \ell \in V \text{ such that } x \in \ell$$
)

Example: A result of a "Moebius gluing"

$X = [0,1]\times[0,1]$



Concrete model $M_{R,r} = \left\{ \left(\left(R+u\sin\frac{a}{2}\right)\cos a,\ \left(R+u\sin\frac{a}{2}\right)\sin a,\ u\cos\frac{a}{2} \right) : a\in[0,2\pi],\ u\in[-r,r] \right\} \subseteq \mathbb{R}^3$

$\pi_{R,r} : [0,1]\times[0,1] \longrightarrow M_{R,r}$, $(t,s) \longmapsto$ that expression with $a = 2\pi t$, $u = (2s-1)r$

$\Rightarrow$ for any other space $(Z, \mathcal{J}_Z)$ and a map $f : (Y, \mathcal{J}_Y) \to (Z, \mathcal{J}_Z)$

$(X, \mathcal{J}_X) \dashrightarrow^{f \circ \pi}$

$\pi \downarrow$

-5-

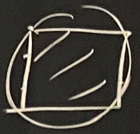
Def: X = a set. An *equivalence relation* on X is a subset $R \subseteq X \times X$ s.t.

- $(x,y) \in R, (y,z) \in R \Rightarrow (x,z) \in R$
- $(x,y) \in R \Rightarrow (y,x) \in R$
- $(x,x) \in R \quad (\forall)\, x \in X$

For $x, y \in X$, when $(x,y) \in R$

- we also say that x is *R-equivalent* to y
- we write $x \sim_R y$.

Rk: intuitively, R encodes a "set of gluing rules" on X.



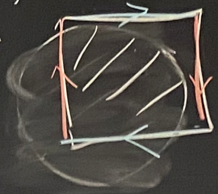
Example: Any $\pi : X \to Y$ surjective map (between sets)

$\Rightarrow$ an equivalence relation R_π on X:

$$x \sim_{R_\pi} y \iff \pi(x) = \pi(y).$$

$X' =$

R'



The outcome of the gluing looks like Y.

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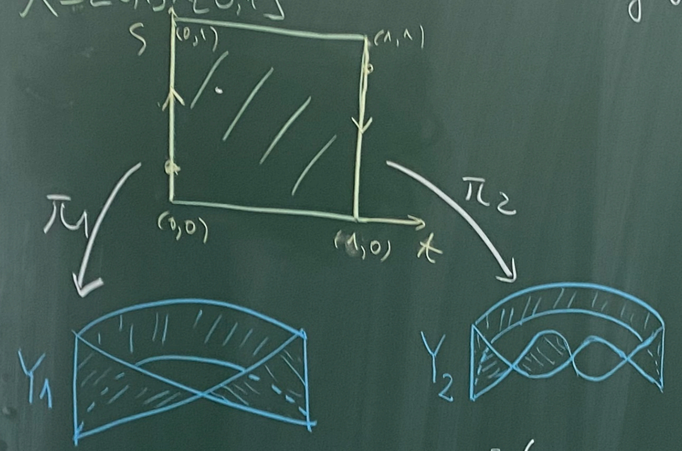

$\underbrace{\text{"Moebius gluing"}}$: the equivalence relation R_{Moebius} on $X = [0,1] \times [0,1]$ consisting of $(x,y) \in X \times X$ with

- $x = y$ **or**
- $\boxed{x = (0,t),\ y = (1,1-t) \text{ with } t \in [0,1]}$ **or**
- $x = (1,1-t),\ y = (0,t)$ — " —

Concrete model $M_{R,r} = \left\{ \underbrace{\left(\left(R + u \sin \frac{a}{2}\right) \cos a,\ \left(R + u \sin \frac{a}{2}\right) \sin a,\ u \cos \frac{a}{2} \right)} : a \in [0, 2\pi],\ u \in [-r, r] \right\} \subseteq \mathbb{R}^3$

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Look at 0.24, 0.25.

Example :

-6-

Def : Given R on X, a topological quotient of X modulo R is a pair (Y,π) consisting of:
- a ~~set Y~~ a space $(Y, \mathcal{T}_Y)$
- a ~~surjective map~~ topological quotient map $\pi : X \to Y$

such that, for $x, y \in X$:

$$(x,y) \in R \iff \pi(x) = \pi(y).$$

Rk: Intuitively, (Y,π) is a model for the outcome of the gluing.
also a topological version, when $(X, \mathcal{T}_X)$ is a topological space

Ex: Models of Moebius gluing.

Ex: On $X = \mathbb{R}^{n+1} \setminus \{0\}$ define R as follows: $R = \{ (x,x') \in X \times X : 0, x \text{ and } x' \text{ are colinear} \}$

or, equivalently:

$$\boxed{x \sim_R x' \iff x' = \lambda \cdot x \text{ for some } \lambda \in \mathbb{R}^*.}$$

Then $(\mathbb{P}^n, \pi)$ is a quotient of X modulo R.

$\boxed{-3-}$

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On $\mathbb{R}^{n+1} \setminus \{0\}$: the Euclidean topology $\Rightarrow$ the quotient topology on $\mathbb{P}^n$

$\pi(x) = \pi(x')$ YES! $\Leftrightarrow \ell_x = \ell_{x'} \overset{?}{\Leftrightarrow} x' = \lambda x$ for some $\lambda \in \mathbb{R}^*$?

(Hence $V \subseteq \mathbb{P}^n$ is open iff $\bigcup_{\ell \in \mathbb{P}^n} \ell = \{x \in \mathbb{R}^{n+1} \setminus \{0\} : \exists$ line $\ell \in V$ such that $x \in \ell\}$)

— 7 —

For general R on a set X:

- for each $x \in X$ define the R-equivalence class of x / or: the R-orbit through x:

$$R(x) = \{ y \in X : y \sim_R x \} \subseteq X$$

- set: $\boxed{X/R := \{ R(x) \,/\, x \in X \} \qquad \pi_R : X \to X/R \,,\ \pi_R(x) = R(x)}$

endowed with the q. top. induced by π_R

If (X, T) is a topological space:

THE (ABSTRACT) TOPOLOGICAL QUOTIENT OF ~~the set~~ the space X modulo R.

Proposition: If $\pi : X \to Y$ is a topological quotient map

$\Rightarrow$ for any other space (Z, T_Z) and a map $f : (Y, T_Y) \to (Z, T_Z)$ one has:

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— 5 —

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For $x, y \in X$, when $(x,y) \in R$

- we also say that x is R-equivalent …
- we write …

Rk: intuitively, R encodes …

Example: Any $\pi : X$ …

— 6 —

Def: Given …

such that …

-1-

Reminder: a map $\pi : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ continuous if, for $V \subseteq Y$:

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-2-

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-5-

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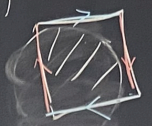

R'

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-6-

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such that, for $x, y \in X$:

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or, equivalently: $\boxed{x \sim_R x' \iff x' = \lambda x \text{ for some } \lambda \in \mathbb{R}^*}$.

Then $(\mathbb{P}^n, \pi)$ is a quotient of X modulo R.

Rk: $(X/R, \pi_R)$ is a quotient of X modulo R because: $R(x) = R(x') \iff x \sim_R x'$.

(0.18)
Proposition: Given R on a set X: $\boxed{\text{any quotient of } X \text{ modulo } R \text{ is is isomorphic to the abstract one}}$

More precisely:

① $(\forall)$ quotient (Y, π) of X modulo R the map

$$\tau : X/R \to Y, \quad R(x) \longmapsto \pi(x)$$

defines a bijection. τ can also be described as "the unique map s.t. $\boxed{\tau \circ \pi_R = \pi}$"

$$\tau(R(x)) = \pi(x)$$

$$\begin{tikzcd} & X \arrow[dl, "\pi_R"'] \arrow[dr, "\pi"] & \\ X/R \arrow[rr, dashed, "\tau"'] & & Y \end{tikzcd}$$

② Similarly in the topological context: start with $(X, \mathcal{T})$ ... $\Rightarrow$ τ is a homeomorphism.

through
$\mathcal{X}$
ical space

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$(Z, \mathcal{J}_Z)$

Pf: $\tilde{\imath} \circ \pi_R = \pi \iff \tilde{\imath}$ is given by $(*)$ [-9-]

But we still have to show that $\tilde{\imath}$ is well defined i.e.:

$\pi_*(\mathcal{J}) := \{V \subseteq Y \mid \pi^{-1}(V) \in \mathcal{J}_X\}$

$R(x) = R(x') \Longrightarrow \pi(x) = \pi(x')$

$(**) \Updownarrow$ $(x, x') \in R$ $\Uparrow$ since (Y, π) is a quotient.

injectivity of $\tilde{\imath}$

Prop applied to the top q. map π_R

Q use Prop

$\tilde{\imath} \circ \pi_R = \pi$ cont $\Leftarrow$ $\tilde{\imath}$ continuous

$\tilde{\imath}^{-1} \circ \pi = \pi_R$ cont $\Longrightarrow$ $\tilde{\imath}^{-1}$ continuous. — " — π

Conclusions: THE Moebius band:

$$M_{Moebius} = [0,1] \times [0,1] / R_{Moebius}.$$

Def: Given R on X, a ~~quotient~~ topological quotient of X modulo R is a pair (Y, π) consisting of: [-6-]

a ~~set Y~~ a space $(Y, \mathcal{J}_Y)$

a ~~surjective map~~ topological quotient map