—1—

<u>Reminder</u>: Given a set X

$\rightsquigarrow$ notion of equivalence relation on X, $R \subseteq X \times X$

$\rightsquigarrow$ given X & R $\rightsquigarrow$ notion of quotient of X w.r.t. R:

that is: a pair (Y,π) with
- $Y =$ a set
- $\pi : X \longrightarrow Y$ surjective map

such that: $(x,x') \in R \iff \pi(x) = \pi(x')$

$\rightsquigarrow$ <u>the abstract quotient</u>: use the R-orbits $R(x) = \{ y \in X \,/\, (x,y) \in R \}$

$$\begin{cases} X/R = \{ R(x) : x \in X \} \\ \pi_R : X \longrightarrow X/R, \quad \pi_R(x) = R(x). \end{cases}$$

<u>Rk</u>: Hence each element of X/R is an R-orbit $R(x)$, for some $x \in X$

—1'—

In particular, having a map into another Z:

$$X/R \xrightarrow{f} Z$$

$\iff$ having a map $\tilde{f} : X \longrightarrow Z$ such that $\tilde{f}(x) = \tilde{f}(x')$ whenever $x \sim_R x'$

<u>Proposition</u>: $(\forall)$ other quotient (Y,π) w.r.t. R is $\cong (X/R, \pi_R)$:

$$\begin{tikzcd} & X \arrow[dl, "\pi_R"'] \arrow[dr, "\pi"] & \\ X/R \arrow[rr, dashed, "?"] & & Y \end{tikzcd}$$

<u>Example</u>: $X = \mathbb{R}^{n+1} \setminus \{0\}$, R given by: $x \sim_R x' \iff x' = \lambda x$ for some $\lambda \in \mathbb{R}^*$

Then $X/R \cong \mathbb{P}^n$

<u>Example</u>: $X = \mathbb{R}^2$, R given by: $(x,y) \sim_R (x',y') \iff x' = x + t$ for some $t \in \mathbb{R}$

Then $X/R \cong$

— 1 —

Reminder: Given a set X / or a space $(X,\mathcal{T})$ $\quad x \sim_R x'$

$\rightsquigarrow$ notion of equivalence relation on X, $\underset{(x,x')\in}{R} \subseteq X \times X$

$\rightsquigarrow$ given X & R $\rightsquigarrow$ notion of topological quotient of $\underset{(X,\mathcal{T})}{X}$ w.r.t. R:

that is: a pair (Y,π) with — Y = a ~~set~~ space; $\pi : X \to Y$ ~~surjective~~ topological quotient map

such that: $\boxed{(x,x')\in R \iff \pi(x)=\pi(x')}$

$\rightsquigarrow$ the abstract topological quotient: use the R-orbits $R(x)=\{y\in X \,/\, (x,y)\in R\}$, $x\in X$

$$\begin{cases} X/R = \{R(x) : x\in X\} \\ \pi_R : X \to X/R, \quad \pi_R(x)=R(x). \end{cases}$$

endowed with the quotient top. induced by π_R.

Rk: Hence
- each element of X/R is an R-orbit $R(x)$, for some $x\in X$
- two such are the same; i.e. $R(x)=R(x')$ IFF $x\sim_R x'$.

— 1' —

In particular, having a continuous map into another space Z:

$$X/R \xrightarrow{f} Z, \quad R(x) \longmapsto \tilde{f}(x) \quad (f(R(x))=\tilde{f}(x))$$

$\iff$ having a continuous map $\tilde{f} : X \to Z$ such that $\tilde{f}(x)=\tilde{f}(x')$ whenever $x\sim_R x'$.

Proposition: $(\forall)$ other topological quotient (Y,π) w.r.t. R is $\underset{\text{homeomorphic}}{\cong}$ $(X/R,\pi_R)$:

$$\begin{tikzcd} & X \arrow[dl, "\pi_R"'] \arrow[dr, "\pi"] & \\ X/R \arrow[rr, dashed, "?"] & & Y \end{tikzcd}$$

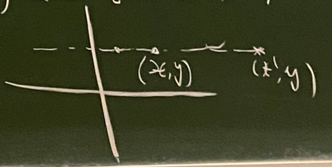
$\exists\, \iota$ s.t. $\iota\circ\pi_R=\pi$; explicit: $\iota(R(x))=\pi(x)$

& ι is a ~~bijection~~ homeomorphism.

Example: $X=\mathbb{R}^{n+1}\setminus\{0\}$, R given by: $x\sim_R x' \iff x'=\lambda x$ for some $\lambda\in\mathbb{R}^*$.

Then $X/R \cong \mathbb{P}^n$

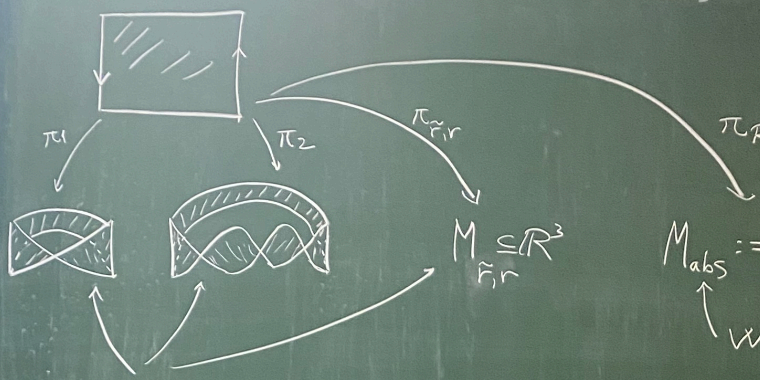
Example: $X=\mathbb{R}^2$, R given by: $(x,y)\sim_R(x',y') \iff \begin{cases} y'=y \\ x'=x+t \text{ for some } t\in\mathbb{R}\end{cases}$



Then $X/R \cong \mathbb{R}$, $\quad R(x,y)\longmapsto y$,

$\mathbb{R}^2 \xrightarrow{\pi_R} X/R$, $\quad X \xleftarrow{(2023,y)} y$, $\quad R(2023,y) \longleftarrow y$

Example $\boxed{-2-}$ $X = [0,1]\times[0,1]$, $R = R_{\text{Moebius}}$ (so that $(0,t)\sim(1,1-t)$)



$M_{abs} := [0,1]\times[0,1] \big/ R_{\text{Moebius}}$

$\uparrow$ with quotient topology

what topology:
- the quotient one (induced by π's)?
- the Euclidean one (induced from $\subseteq \mathbb{R}^3$)?

$\boxed{-2'-}$

0.25

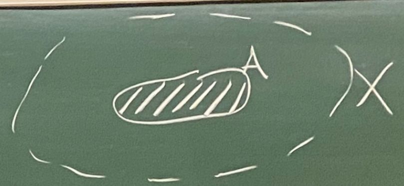
Proposition: Assume:
- $X \subseteq \mathbb{R}^k$ endowed with the Euclidean topology
- R = equivalence relation on X
- (Y,π) = a topological quotient of X modulo R $(\pi : X \longrightarrow Y)$
- $Y \subseteq \mathbb{R}^n$ (as subset).

If:
1. X = closed and bounded in $\mathbb{R}^k$
2. π is continuous as a map $X \longrightarrow \mathbb{R}^n$

then the topology on Y coincides with the Euclidean topology inherited from $Y \subseteq \mathbb{R}^n$

Special classes of quotients

-3-



Given $(X, \mathcal{T})$, $A \subseteq X$, define

$X /\!\!/ A :=$ "the space obtained from X by $\underbrace{\text{collapsing } A \text{ to a point}}_{\text{"gluing all elements of } A \text{ to each other"}}$"

$X /\!\!/ A = X / R_A$ the quotient modulo the equiv. rel.

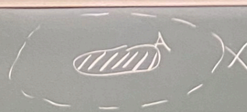
$R_A = \{ (x,y) \in X \times X \mid x = y \text{ OR } x, y \in A \}$ $\quad \left(a \sim_R b \ (\forall) a, b \in A \right)$.

Rk: Each $x \in X$ gives $\begin{cases} R_A(x) = \{x\} & \text{if } x \notin A \\ R_A(x) = A & \text{if } x \in A \end{cases} \in X/A$.

Fo

Special classes of quotients

[-3-]



Given $(X, \mathcal{J})$, $A \subseteq X$, define

$X /\!/ A$:= "the space obtained from X by collapsing A to a point"
"gluing all elements of A to each other"

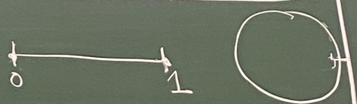
$X /\!/ A = X / R_A$ the quotient modulo the equiv. rel.

$R_A = \{(x,y) \in X \times X \;/\; x = y \text{ OR } x, y \in A\}$ $\quad (a \sim_R b \;(\forall)\, a, b \in A)$.

Rk: Each $x \in X$ gives $\begin{cases} R_A(x) = \{x\} & \text{if } x \notin A \\ R_A(x) = A & \text{if } x \in A \end{cases} \in X/A$.

Ex: $X = [0,1]$, $A = \{0, 1\}$.

Intuition $\Rightarrow$ quotient is S^1.



One way: $(Y = S^1, \pi)$ to define is a top quotient, $\pi : [0,1] \to S^1$, $\pi(t) = (\cos(2\pi t), \sin(2\pi t))$ still to check: a top. q. map.

Other way: $X /\!/ A \longleftrightarrow S^1$ cont.

$R_A(t) \longleftrightarrow (\cos 2\pi t, \sin 2\pi t)$

Proof that $X/A \cong S^1$ (homeomorphic) with the Euclidean topology.

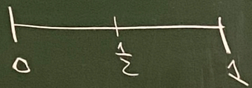
[-5-]

Take $\underset{\text{as set}}{Y = S^1}$, $\pi : [0,1] \to S^1$, $\pi(t) = (\cos 2\pi t, \sin 2\pi t)$

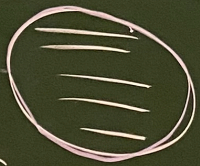
& endow S^1 with the resulting quotient topology.

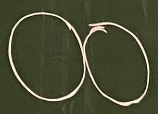
$\Rightarrow (Y, \pi)$ is a top. q. Use the prop

Ex: $X = [0,1]$, $A = \{0, \frac{1}{2}, 1\}$

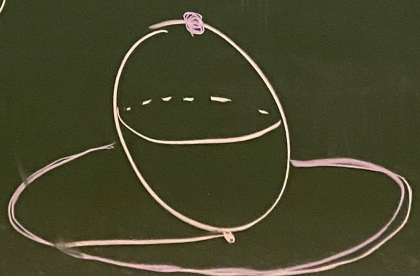


Ex: $X = D^2 = \{(x,y) \in \mathbb{R}^2 / x^2 + y^2 \le 1\}$
$\quad \cup\!\!\mid$
$A = S^1$





$\boxed{D^2 /\!/ S^1 \cong S^2}$



For
Ex
E
Group
Ex. of
• $(\mathbb{Z}, \ldots$
• $(\mathbb{R}^*, \ldots$
• S_n permutation
• for each set X
$(Bij(X) = \{f \ldots$
• for each top
$(Homeo(X) = \ldots$

)X

a point"

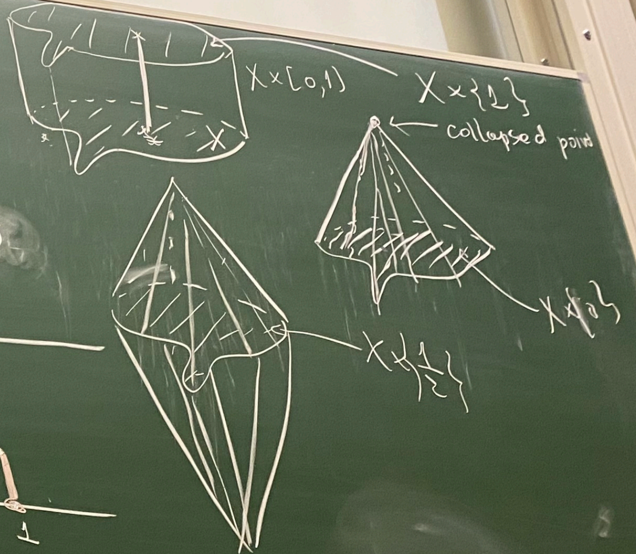
ents

other"

).

For any topological space $(X, \mathcal{J})$

- the cylinder of X : $Cyl(X) = X \times [0,1]$ $\quad \{x\}\times[0,1] \subset$
- the cone of X : $Cone(X) := X \times [0,1] \big/ X\times\{1\}$
- the suspension of X : $S(X) = Cone(X) \big/ X\times\{0\}$



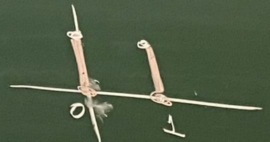
<u>Ex</u> : $X = \{0\}$

$Cyl(X) \cong [0,1]$

$Cone(X) = [0,1]$

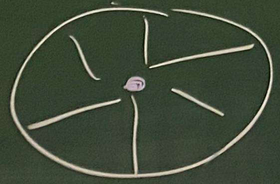
<u>Ex</u> : $X = \{0,1\}$

$Cyl(X) = \{0,1\}\times[0,1]$



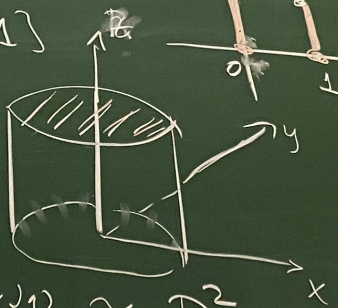
$S(X) \cong S^1$

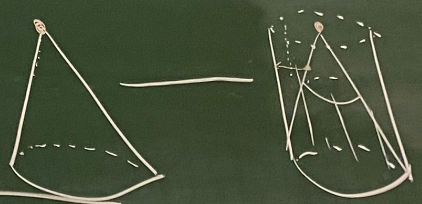
<u>Ex</u> : $X = S^1$



$Cyl(X) = S^1\times[0,1]$

$Cone(X) = S^1\times[0,1] \big/ S^1\times\{1\} \cong D^2$





$\boxed{S(S^1) = S^2}$ $\qquad$ $\boxed{S(S^2) = S^3}$

-6-

-7-

Def : An action of a group $(\Gamma, \cdot)$ on a space X is a group homomorph

...morphic)
...ogy.

...$(t) = (\cos 2\pi t, \sin 2\pi t)$

...quotient topology.

$$-6-$$

Group actions:

Ex. of groups:

- $(\mathbb{Z},+)$, $(\mathbb{Z}_n,+)$
- $(\mathbb{R}^*,\cdot)$, $(\{\pm 1\},\cdot)$
- S_n permutation group
- for each set X:
  $(\mathrm{Bij}(X) = \{f : X \to X \mid f = \text{bijective}\}, \circ)$
- for each top. space X:
  $(\mathrm{Homeo}(X) = \{f : X \to X \,/\, f = \text{homeomorphism}\}, \circ)$

Def: An...

Common to...

Γ...

$(g$...

Given an a...

Def: T...

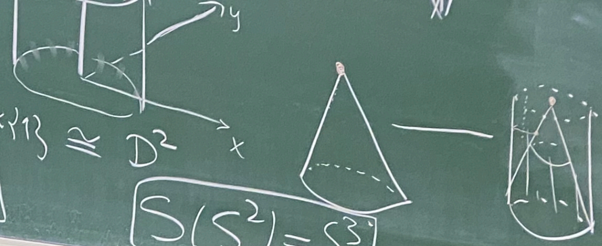
modulo...

Ex: $\Gamma = (\mathbb{R}$...

... a top. q. map.

$\mathrm{Cone}(X) = S^1\times[0,1] / S^1\times\{1\} \cong D^2$

$\boxed{S(S^1) = S^2}$



$\boxed{S(S^2) = S^3}$

... $t, \sin 2\pi t)$

... topology.

-6-

Group actions:

Ex. of groups:

- $(\mathbb{Z},+)$, $(\mathbb{Z}_n,+)$
- $(\mathbb{R}^*,\cdot)$, $(\{\pm 1\},\cdot)$
- S_n permutation group
- for each set X:
  $(\mathrm{Bij}(X) = \{f: X\to X \,/\, f = \text{bijective}\}, \circ)$
- for each top. space X:
  $(\mathrm{Homeo}(X) = \{f: X\to X \,/\, f = \text{homeomorphism}\}, \circ)$

-7-

Def: An action of a group $(\Gamma,\cdot)$ on a space X is a group homomorphism

$$\varphi: \Gamma \longrightarrow \mathrm{Homeo}(X), \quad \gamma \longmapsto \varphi_\gamma$$

$\varphi_{\gamma\cdot\gamma'} = \varphi_\gamma \circ \varphi_{\gamma'}$

$\varphi_{1_\Gamma} = \mathrm{Id}_X$

$\varphi_{\gamma^{-1}} = \varphi_\gamma^{-1}$

Common to re-pack this data in a map

$$\Gamma \times X \longrightarrow X$$

$$(\gamma, x) \longmapsto \varphi_\gamma(x) \overset{\text{notation}}{=\!=} : \gamma\cdot x.$$

Given an action $\Rightarrow$ an equiv. rel R_Γ on X:

$$x \sim_{R_\Gamma} y \iff y = \gamma\cdot x \text{ for some } \gamma\in\Gamma.$$

$$R_\Gamma = \{(x,y)\in X\times X \,/\, y = \gamma x \text{ —"— }\}$$

Def: The quotient X/R_Γ is called the quotient of X modulo the action of Γ and denoted ... $X/\!\!/\Gamma$

Ex: $\Gamma = (\mathbb{R}^*,\cdot)$ acting on $\mathbb{R}^{n+1}\setminus\{0\}$

$\cong (X/R, \pi_R)$

homeomorphic

$$\begin{tikzcd} & X \arrow[dl, "\pi_R"'] \arrow[dr, "\pi"] & \\ X/R \arrow[rr, dashed, "\iota"] & & Y \end{tikzcd}$$

$\exists \iota$ s.t. $\iota \circ \pi_R = \pi$; explicitly:

$\iota(R(x)) = \pi(x)$

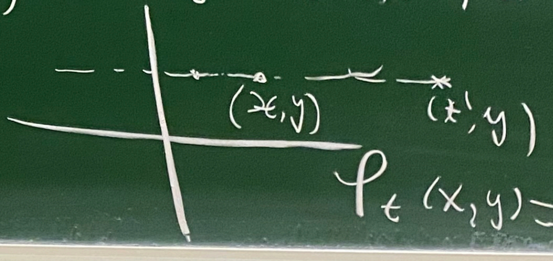
& ι is a ~~bijection~~ homeomorphism.

Example: $\boxed{X = \mathbb{R}^{n+1} \setminus \{0\}}$, R given by: $x \sim_R x' \iff x' = \lambda x$ for some $\lambda \in \mathbb{R}^*$

Γ

Then $X/R \cong \mathbb{P}^n$

Example: $X = \mathbb{R}^2$, R given by: $(x,y) \sim_R (x',y') \iff \begin{cases} y' = y \\ x' = x + t \end{cases}$ for some $t \in \mathbb{R}$.



$\Gamma = (\mathbb{R}, +)$

$\varphi_t(x,y) = (x+t, y)$

Then $X/R \cong \mathbb{R}$, $R(x,y) \longmapsto y$

$R(2023, y) \longleftarrow\!\shortmid\; y$

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r] & X/R \\ & X \arrow[u, "\pi_R"] & (2023, y) \arrow[l, dashed, mapsto] & y \arrow[l, mapsto] \end{tikzcd}$$