Actions : $(\Gamma, \cdot)$ a group (e.g. $(\mathbb{Z}, +)$, $(\mathbb{R}^*, \cdot)$, $(\mathbb{R}_{>0}, \cdot)$ etc)

→ action of Γ on a set X: a map

$$\Gamma \times X \longrightarrow X, \quad (\gamma, x) \longmapsto \gamma \cdot x \quad \text{satisfying} \begin{cases} \gamma_1(\gamma_2 x) = (\gamma_1 \gamma_2) x \\ e\, x = x \end{cases}$$

→ equivalently: group homomorphism

$$\varphi : \Gamma \longrightarrow Bij(X), \quad \gamma \longmapsto \varphi_\gamma$$

$\varphi_\gamma(x) = \gamma \cdot x$

→ action of Γ on a SPACE X : require $\varphi_\gamma \in Homeo(X)$ $(\forall)\, \gamma \in \Gamma$

→ action ⇒ an equivalence relation R_Γ on X : $\boxed{x \sim_{R_\Gamma} y \Leftrightarrow \exists\, \gamma \in \Gamma \text{ s.t. } y = \gamma \cdot x \quad (y = \varphi_\gamma(x))}$

Notice: $R_\Gamma(x) = \{\gamma \cdot x \,/\, \gamma \in \Gamma\}$ also – denoted $\Gamma \cdot x$

– called the ORBIT OF THE ACTION through x

→ $X/\Gamma := X/R_\Gamma$, hence: $X/\Gamma = \{\Gamma \cdot x \,/\, x \in X\}$, $\pi_R : X \longrightarrow X/R$, $\pi_R(x) = \Gamma \cdot x$

with QUOTIENT TOPOLOGY (VIA π_R)

Actions: $(\Gamma, \bullet)$ a group (e.g. $(\mathbb{Z},+)$, $(\mathbb{R}^*,\cdot)$, $(\mathbb{R}_{>0},\cdot)$ etc)

→ action of Γ on a set X: a map

$\Gamma\times X \longrightarrow X$, $(\gamma,x)\longmapsto \gamma\bullet x$ satisfying $\begin{cases} \gamma_1\bullet(\gamma_2\bullet x) = (\gamma_1\cdot\gamma_2)\bullet x \\ e\bullet x = x \end{cases}$

→ equivalently: group homomorphism

$\varphi:\Gamma\longrightarrow Bij(X)$, $\gamma\longmapsto\varphi_\gamma$

$\varphi_\gamma(x)=\gamma\bullet x$ (Hence $\varphi_\gamma: X\to X$)

→ action of Γ on a $\boxed{\text{SPACE}}$ X: require $\varphi_\gamma\in Homeo(X)$ $(\forall)\gamma\in\Gamma$ ($\Leftrightarrow$ each φ_γ is continuous)

→ action $\Rightarrow$ an equivalence relation R_Γ on X: $\boxed{x \sim_{R_\Gamma} y \Leftrightarrow \exists\, \gamma\in\Gamma \text{ s.t. } y=\gamma\bullet x \quad (y=\varphi_\gamma(x))}$

$x\in X$: Notice. $R_\Gamma(x)=\{\gamma\bullet x / \gamma\in\Gamma\}$ also — denoted $\Gamma\bullet x$ $(\subseteq X)$
— called the ORBIT OF THE ACTION through x

→ $\boxed{X/\Gamma} = X/R_\Gamma$, hence:

$X/\Gamma = \{\Gamma\bullet x / x\in X\}$, $\pi_R: X\longrightarrow X/R$, $\pi_R(x)=\Gamma\bullet x$

The quotient of X modulo Γ
(modulo the action of Γ)

with QUOTIENT TOPOLOGY (VIA π_R)

Example 1: $\Gamma = (\mathbb{R}^*, \cdot)$, $X = \mathbb{R}^{n+1}\setminus\{0\}$, action: $\varphi_\lambda(x) = \lambda \cdot x := \boxed{\lambda x}$ (usual multiplication by scalars $\Rightarrow$ we get the projective space $\boxed{\mathbb{P}^n \cong (\mathbb{R}^{n+1}\setminus\{0\})/\mathbb{R}^*}$

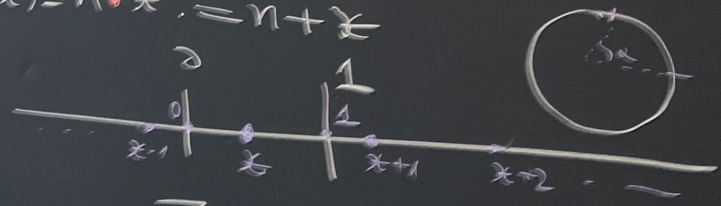
Example 2: $\Gamma = (\mathbb{Z}_2, +) \cong (\{\pm 1\}, \cdot)$, $X = S^n$, action: $\varphi_\varepsilon(x) = \varepsilon \cdot x := \varepsilon x = \begin{cases} x & \varepsilon = 1 \\ -x & \varepsilon = -1 \end{cases}$

$\Rightarrow$ get again $\boxed{\mathbb{P}^n \cong S^n/\mathbb{Z}_2}$

Example 3: $\Gamma = (\mathbb{R}_{>0}, \cdot)$, $X = \mathbb{R}^{n+1}\setminus\{0\}$, $\varphi_\lambda(x) = \lambda x \Rightarrow$ we get $(\mathbb{R}^{n+1}\setminus\{0\})/\mathbb{R}_{>0} \cong S^n$

Example 4: $\Gamma = (\mathbb{Z}, +)$, $X = \mathbb{R}$, action $\varphi_n(x) = n \cdot x := n + x$

Each element in $\mathbb{R}/\mathbb{Z}$ is represented by a number $x \in [0,1]$, uniquely, except for 0 and 1. Intuitively $\Rightarrow S^1$.



Indeed: $X/R \longrightarrow S^1$, $\mathbb{Z} + x \longmapsto (\cos 2\pi x, \sin 2\pi x)$

$\mathbb{Z} \cdot x = \mathbb{Z} + x = \{n + x : n \in \mathbb{Z}\}$

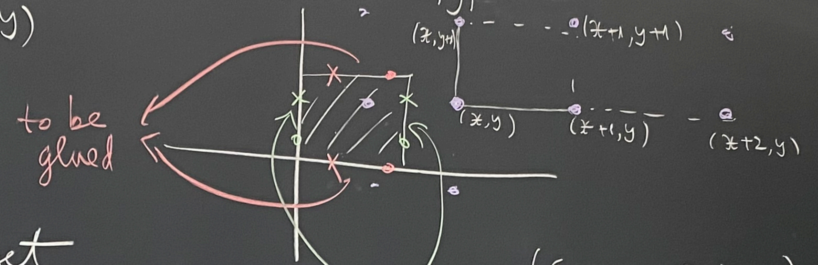
Hence $\boxed{S^1 \cong \mathbb{R}/\mathbb{Z}}$

Example: $\Gamma = (\mathbb{Z}^2, +)$, $X = \mathbb{R}^2$, action $(n,m)\cdot(x,y)$

$(n,m) \in \mathbb{Z}^2$, $(x,y) \in \mathbb{R}^2$

It is the torus!

$\mathbb{R}^2/\mathbb{Z}^2 \longrightarrow$ torus T^2 ...



Other actions on $\mathbb{R}^2 \Rightarrow$ can get Klein bottle, Moebius band, ... (see exercises)

Examples.

Be aware: these are nice quotients but there are also very "nasty" examples (e.g. look at $\Gamma = \mathbb{Z} + \mathbb{Z}\sqrt{2} = \{m + n\sqrt{2} : m, n \in \mathbb{Z}\} \subseteq (\mathbb{R}, +)$ acting on $\mathbb{R}$ by addition

OR: $\Gamma = (\mathbb{Q}, +)$ " ———)

$\Longrightarrow$ $\boxed{\text{quotients may fail to be even Hausdorff.}}$

FACT: If Γ = finite group & X = Hausdorff $\Rightarrow$ X/Γ = Hausdorff.

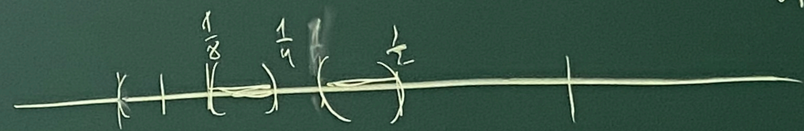
4T

Corollary: $f: X \to \mathbb{R}$ continuous, X = connected $\Rightarrow$ Im f must be an interval

Corollary: If X = connected space $\Rightarrow$ any quotient Y of X is connected.

$\pi: X \to Y$ cont. surj

U, V



—4—

CONNECTEDNESS

Def. A topological space X is called:

- CONNECTED : if it cannot be written as $X = U \cup V$ with $\begin{cases} U, V \subseteq X \text{ open} \\ U \cap V = \emptyset \\ U \neq \emptyset, V \neq \emptyset \end{cases}$
- PATH CONNECTED : if $(\forall)\, x, y \in X$, $\exists\, \gamma : [0,1] \to X$ continuous s.t. $\begin{cases} \gamma(0) = x \\ \gamma(1) = y \end{cases}$

Theorem. PATH CONNECTED $\Rightarrow$ CONNECTED.

Lemma: $[0,1]$ is connected. ☺

Prop. (i) $f : X \to Y$ continuous, X = connected $\Rightarrow$ $f(X)$ is connected (with induced topology)

(ii) Given space X: if $(\forall)\, x, y \in X$, $\exists\, \Gamma \subseteq X$ connected s.t. $x, y \in \Gamma$ then X = connected.

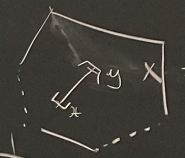
—5—

Examples: ① any interval $\underset{x,y}{\overset{\in}{I}} \subseteq \mathbb{R}$ is path connected: use $\gamma : [0,1] \to I$, $\gamma(t) = (1-t)x + ty$ ☺

$\gamma(0) = x$, $\gamma(1) = y$

② Any $\underbrace{\text{convex subset } X \subseteq \mathbb{R}^k}$ is path connected: use $\gamma(t) = (1-t)x + ty$.

☺ $(\forall)\, x, y \in X \Rightarrow [x,y] \subseteq X$

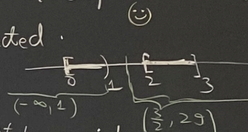


$[x,y] = \{(1-t)x + ty \,/\, t \in [0,1]\}$

③ $X = \mathbb{R}^2 \setminus \{0\}$ not convex but ... still path connected ☺

④ $X = \underbrace{[0,1)} \cup \underbrace{[2,3]}$ is not connected.

$(-\infty,1) \cap X = U \qquad V = (\tfrac{3}{2}, 29) \cap X$ ☹

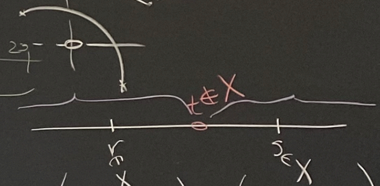


Actually: if X = connected $\Rightarrow X$ must be an interval

⑤ $X = \mathbb{R}$ with $\mathcal{T}_e$ ... not connected ☹

$\mathbb{R} = \underbrace{(-\infty, 0)}_{U} \cup \underbrace{[0, \infty)}_{V}$

$\boxed{\text{Cor: } X \subseteq \mathbb{R} \text{ connected} \Leftrightarrow X = \text{an interval}}$



$X = \left((-\infty, t) \cap X \right) \cup \left((t, \infty) \cap X \right)$

$\Rightarrow$ would not be connected

—6—

proof of Lemma & Prop $\Rightarrow$ Theorem:

Assume X = path connected $\Rightarrow (\forall)\, x, y \in X$, $\exists\, \gamma : [0,1] \to X$ s.t. $\gamma(0) = x$, $\gamma(1) = y$.

Look at $\Gamma := \mathrm{Im}(\gamma) \subseteq X$

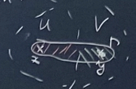
$\left.\begin{array}{l} \gamma : \underbrace{[0,1]}_{\text{connected}} \to X \text{ continuous} \end{array}\right\} \xRightarrow[\text{of Prop}]{(i)}$ Γ is connected, $x, y \in \Gamma$ $\xRightarrow{\text{(ii) of Prop}}$ ☺

Proof of Prop, part (ii): By contrad: assume $X \neq$ connected $\Rightarrow$

$\Rightarrow$ can write $X = U \cup V$, $U, V \subseteq X$ open, $U \cap V = \emptyset$, $U, V \neq \emptyset$

Choose $x \in U$, $y \in V$. Apply hypothesis $\Rightarrow$ find Γ connected with $x \in \Gamma$, $y \in \Gamma$

Notice $\Gamma = \underbrace{(\Gamma \cap U)}_{\text{open in } \Gamma} \cup \underbrace{(\Gamma \cap V)}_{\text{open in } \Gamma}$ contrad. with

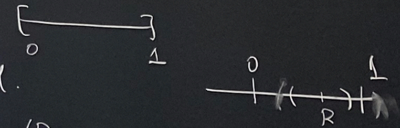


—7—

Proof of Lemma: Assume $[0,1] = U \cup V$, $U, V \subseteq [0,1]$ opens, $U \cap V = \emptyset$

Notice: U = the complement in $[0,1]$ of V hence closed as well.

Consider $R := \sup U$ is a limit of a sequence of numbers from U. Since U is closed $\Rightarrow R \in U$.



Claim $R = 1$. If not, since $\left.\begin{array}{l} R \in U \\ U = \text{open} \end{array}\right\} \Rightarrow \exists\, \varepsilon > 0$ s.t. $(R - \varepsilon, R + \varepsilon) \cap [0,1] \subseteq U$

$\Rightarrow R + \frac{\varepsilon}{2} \in U$ impossible since $> R$.

All together: $1 \in U$.

Same reasoning to $V \Rightarrow 1 \in V$ ↯

—4—

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Exam

②

③

④

$(-\infty, 1)$

A c

⑤

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(-5-)

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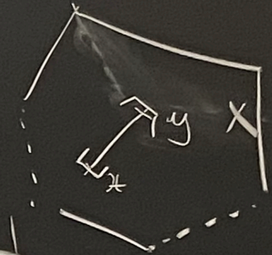

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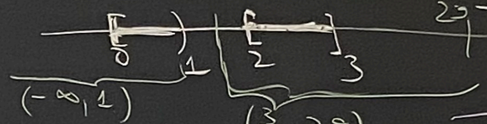

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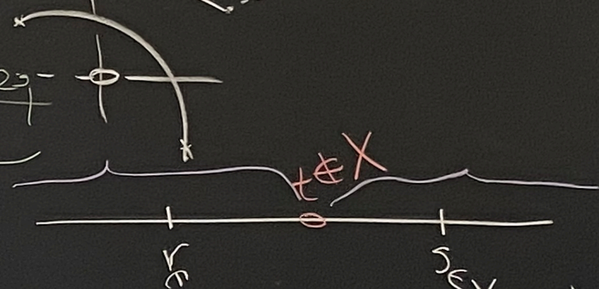

$r \in X$, $s \in X$

$X = \left((-\infty, t) \cap X\right) \cup \left((t, \infty) \cap X\right)$

$\Rightarrow$ would not be connected

⑤ $X = \mathbb{R}$ with J_e ... not connected

$\mathbb{R} = \underbrace{(-\infty, 0)}_{U} \cup \underbrace{[0,\infty)}_{V}$

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(-7-)

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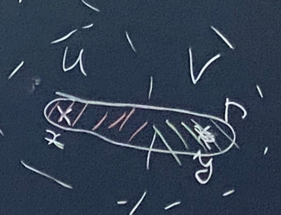

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All together:

Same reasoni...

induced topology

connected

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$t \notin X$

$r \in X$ $\quad s \in X$

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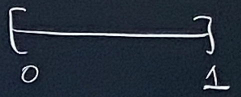

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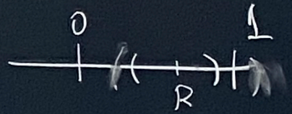

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