—1—

Reminder: a space X is called:

- CONNECTED: if X cannot be written as $X = U \cup V$ with U, V - disjoint, non-empty opens

$\iff$ $\boxed{\text{the only } U \subseteq X \text{ that are both open and closed in } X \text{ are } \emptyset \text{ and } X.}$

- PATH CONNECTED: if $(\forall)\, x, y \in X$, $\exists\, \gamma : [0,1] \to X$ continuous with $\begin{cases} \gamma(0) = x \\ \gamma(1) = y. \end{cases}$

THEOREM: PATH CONNECTED $\Rightarrow$ CONNECTED.

PROPOSITION: (i) $\left.\begin{array}{l} f: X \to Y \text{ continuous} \\ X = \text{connected} \end{array}\right\} \Rightarrow f(X)$ is connected

(ii) if $(\forall)\, x, y \in X$, $\exists\, \Gamma \subseteq X$ connected with $\begin{cases} x \in \Gamma \\ y \in \Gamma \end{cases}$ then X = connected.

—1'—

Lemma: $[0,1]$ is connected.

Exercise 1: $\left.\begin{array}{l} A, B \subseteq X,\ A, B \text{ - connected} \\ A \cap B \neq \emptyset \end{array}\right\} \Rightarrow A \cup B$ is connected.

proof: Let $Y = A \cup B$. Assume not connected $\Rightarrow$

$\Rightarrow$ $Y = \underline{U} \cup V$, U, V - opens in X, nonempty, $U \cap V = \emptyset$

$\Rightarrow$ $A = (A \cap U) \cup (A \cap V) \xrightarrow[\text{connected}]{A} \underbrace{A \cap U = \emptyset}_{A \subseteq V}$ or $\underbrace{A \cap V = \emptyset}_{A \subseteq U}$.

Similarly for $B \Rightarrow$ $B \subseteq V$ or $B \subseteq U$ — IMPOSSIBLE. $\boxtimes$

$\left.\begin{array}{l} X = U \cup V \\ U \cap V = \emptyset \end{array}\right\} \rightsquigarrow V = X \setminus U$

And V = open $\iff U$ = closed

Exercise: If $A \subseteq X$ is connected $\Rightarrow \bar{A}$ is connected.

-2-

COROLLARY: For $X \subseteq \mathbb{R}$: X = connected $\Longleftrightarrow$ X = an interval.

COROLLARY: If $\left.\begin{array}{l} f: X \to \mathbb{R} \text{ continuous} \\ X = \text{connected} \end{array}\right\}$ then $\mathrm{Im}(f)$ is an interval.

COROLLARY: Any quotient of a connected space is automatically connected.

& combine with "THE REMOVE 1-POINT TRICK":

if X and Y were homeomorphic $\qquad f: X \to Y$

then $(\forall)\, x \in X$, $\exists\, y \in Y$ $\qquad x \mapsto f(x) = y$

such that $X \setminus \{x\}$ and $Y \setminus \{y\}$ are homeomorphic as well.

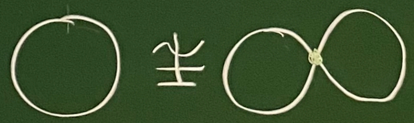
-2'-

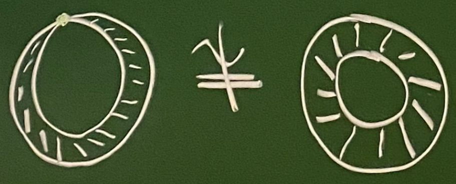
For instane: one can now prove

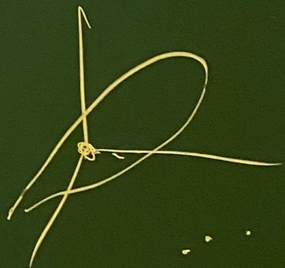
① $\mathbb{R} \not\cong \mathbb{R}^2 \qquad \mathbb{R}^2 \quad \mathbb{R}^3$

② $\mathbb{R} \not\cong S^1$

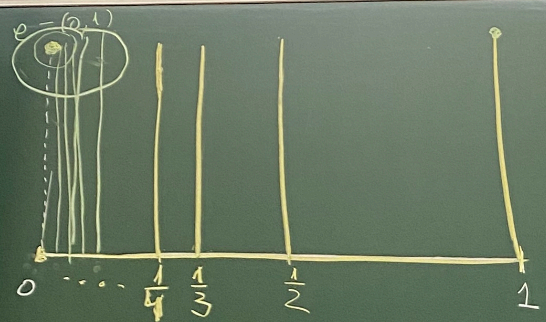
③ $[0,1) \not\cong (0,1)$

④ 

⑤ 



Algebraic Topology



$X = \{e\} \cup X'$
connected
but
NOT
PATH CONNECTED

proof of X = connected:

Assume it is not $\Rightarrow X = U \cup V$ with $U, V \dots\dots$

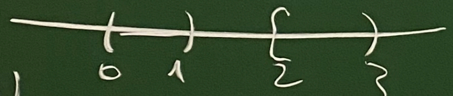
$\Rightarrow X' = (X' \cap U) \cup (X' \cap V) \Rightarrow X' \cap U = \emptyset$ or $X' \cap V = \emptyset$

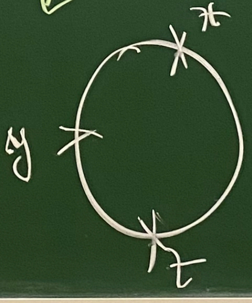
where we can use that X' is connected (because path connected)

$\Rightarrow U \subseteq \{e\}$ or $V \subseteq \{e\} \Rightarrow U = \{e\}$ or $V = \{e\}$.

But $\{e\}$ is not open in X ↯ $\Rightarrow X$ = connected.

$X \subseteq \mathbb{R}$
$\parallel$
$(0,1] \cup [2,3)$





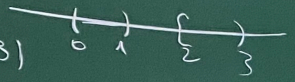
$S^1 \setminus \{x, y, z\}$

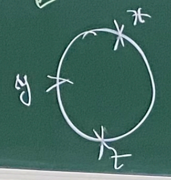
ef: X = space. A connected component of X

any $C \subseteq X$ connected, which is not co

But $\{p\}$ is not open in X $\Rightarrow X =$ connected

$X \subseteq \mathbb{R}$
$(0,1] \cup [2,3)$





$S^1 \setminus \{x, y, z\}$

Def: $X =$ space. A connected component of X is any $C \subseteq X$ connected, which is $\underbrace{\text{not contained}}_{C \subsetneq C'}$ in any $C' \subseteq X$ connected. -4-

Prop: Any c.c. of X is closed in X, and the collection of all connected components of X forms a partition of X, i.e.:

(i) any $x \in X$ belongs to some c.c. C.

(ii) if C_1, C_2 are c.c. $\Rightarrow C_1 = C_2$ or $C_1 \cap C_2 = \emptyset$.

Conclusion: We can talk about the part. of X into connected components

$$X = \bigsqcup_{i \in I} C_i$$

proof: 1st part: use $\underbrace{\text{last exercise}}$. -5-

$C =$ conn. comp. & $\Rightarrow$ the inclusion $C \subseteq \overline{C}$ cannot be strict.

$\Rightarrow C = \overline{C} \Rightarrow C =$ closed.

For (ii): use Exercise 1:

$C_1, C_2 =$ c.c. $\begin{cases} C_1 \cap C_2 = \emptyset \ \checkmark \\ \left.\begin{array}{l} C_1 \cap C_2 \neq \emptyset \\ C_1, C_2 = \text{conn} \end{array}\right\} \Rightarrow C_1 \cup C_2 = \text{connected} \end{cases}$

$C_1 = C_1 \cup C_2 \supseteq C_2$

$\Rightarrow C_1 = C_2$

For (i): fix $x \in X$.

Define: $C(x) :=$ the union of all $C \subseteq X$ that are connected and contain x.

Claim: $C(x) =$ connected. We do it by applying (ii) of the prop.

Let $\underbrace{y, z \in C(x)}_{y \in C_1,\ z \in C_2}$. Look for $\Gamma \subseteq C(x)$ s.t. $\Gamma =$ connected, $y \in \Gamma$, $z \in \Gamma$

$y \in C_1$, $z \in C_2$ with $\underbrace{C_1, C_2\text{-connected containing } x}_{C_1 \cap C_2 \neq \emptyset}$. Take $\Gamma := C_1 \cup$

Remark: If
$\begin{cases} X = X_1 \cup \dots \cup \\ X_i = \text{connected} \end{cases}$
then
$\left\{\begin{array}{l} \text{all } X_i \\ \text{are closed} \end{array}\right\} \Longleftrightarrow \left\{\begin{array}{l} \text{all } X_i \\ \text{are open} \end{array}\right.$


Reminder a space X is called:
• CONNECTED: if X cannot be written as
X = U ∪ V with U, V - disjoint, non-empty opens
the only U ⊆ X that are both open and closed in X are ∅ and X
• PATH CONNECTED: if (∀) x, y ∈ X, ∃ γ: [0,1] → X continuous with
THEOREM PATH CONNECTED ⇒ CONNECTED
PROPOSITION (i) f: X → Y continuous, X = connected ⇒ f(X) is connected
(ii) if (∀) x, y ∈ X ∃ Γ ⊆ X connected with x ∈ Γ, y ∈ Γ
then X = connected
Lemma [0,1] is connected.
Exercise A, B ⊆ X, A, B - connected, A ∩ B ≠ ∅ ⇒ A ∪ B is connected
Similarly for B
X = U ∪ V, U ∩ V = ∅
V = X \ U
Exercise
X = {e} ∪ X'
connected
but
NOT
PATH CONNECTED
proof of X = connected
Assume it is not ⇒ X = U ∪ V with U, V ...
⇒ X' = (X' ∩ U) ∪ (X' ∩ V)
But {e} is not open in X ⇒ X = connected
Def: X = space. A connected component of X is any C ⊆ X connected, which is not contained in any C' ⊆ X connected
C ⊊ C'
Prop: Any c.c. of X is closed in X, and the collection of all connected components of X forms a partition of X, i.e.
(i) any x ∈ X belongs to some c.c.
(ii) if C₁, C₂ are c.c. ⇒ C₁ = C₂ or C₁ ∩ C₂ = ∅
Conclusion: We can talk about the partition of X into connected components
X = ⊔ Cᵢ
i ∈ I
proof 1st part: use last exercise
⇒ C = C̄ ⇒ C = closed
For (ii) use Exercise 1
For (i): fix x ∈ X
Define C(x) = the union of all C ⊆ X that are connected and contain x
Remark

xercise. -5-

clusion $C \subseteq \overline{C}$
t be strict.

$= \phi$ ✓

$2 \neq \phi$ } $\Rightarrow C_1 \cup C_2 = \text{connected}$
$= \text{conn}$

$\begin{matrix} \text{\rotatebox{90}{$\subseteq$}} & \text{\rotatebox{90}{$\supseteq$}} \\ C_1 & C_2 \end{matrix}$

$C_1 = C_2$

eunion of all $C \subseteq X$ that are connected and contain x.

We do it by applying (ii) of the prop.

Look for $\Gamma \in C(x)$ s.t. $\Gamma = \text{connected}$, $y \in \Gamma$, $z \in \Gamma$. Take $\Gamma := C_1 \cup C_2$ } $\overset{\text{Ex1}}{\Longrightarrow} \Gamma = \text{connected}$ ☒

with C_1, C_2-connected containing x, $C_1 \cap C_2 \neq \phi$

-6-

Remark: If

$$\begin{cases} X = \underbrace{X_1 \cup \dots \cup X_n} \\ X_i = \text{connected} \ \forall i \end{cases} \qquad X_i \cap X_j = \phi \quad (\forall)\, i \neq j.$$

then:

$$\begin{Bmatrix} \text{all } X_i \\ \text{are closed} \end{Bmatrix} \Longleftrightarrow \begin{Bmatrix} \text{all } X_i \\ \text{are open} \end{Bmatrix} \Longleftrightarrow \begin{Bmatrix} \text{this is the partition} \\ \text{by connected components} \end{Bmatrix}$$

-5-

$\overline{C}$

...t.

$\Rightarrow C_1 \cup C_2 = \text{connected}$

$\quad \shortparallel \qquad \cup\!|$

$\quad C_1 \qquad C_2$

of all $C \subseteq X$ that are connected and contain x.

it by applying (ii) of the prop.

$\Gamma \subseteq C(x)$ s.t. Γ = connected, $y \in \Gamma$, $z \in \Gamma$. Take $\Gamma := C_1 \cup C_2 \Big\} \xRightarrow{\text{Ex1}} \Gamma = \text{connected}$ $\square$

C_2-connected $\underbrace{\text{containing } x}_{C_1 \cap C_2 \neq \emptyset}$.

-6-

Remark: If

$$\begin{cases} X = X_1 \cup \underbrace{X_2 \cup \dots \cup X_n} & X_i \cap X_j = \emptyset \quad (\forall)\, i \neq j. \\ X_i = \text{connected} \quad \forall i \end{cases}$$

then:

$$\begin{Bmatrix}\text{all } X_i \\ \text{are closed}\end{Bmatrix} \Longleftrightarrow \begin{Bmatrix}\text{all } X_i \\ \text{are open}\end{Bmatrix} \underset{\boxed{\Longrightarrow}}{\Longleftrightarrow} \begin{Bmatrix}\text{this is the partition} \\ \text{by connected} \\ \text{components}\end{Bmatrix}$$

Pf: To prove: each X_i is a C.C.

Assume $X_1 \subseteq C'$ with C' connected.

Get: $C' = \underbrace{(C' \cap X_1)}_{\substack{\text{open in } C' \\ \neq \emptyset}} \cup \underbrace{(C' \cap (X_2 \cup \dots \cup X_n))}_{\text{open in } C'} \Longrightarrow C' \cap (X_2 \cup \dots \cup X_n) = \emptyset$

$\Downarrow$

$C' \subseteq X_1 \Rightarrow C' = X_1$ $\square$

-7-

Def A space X is called compact if:

$(\forall)$ $\boxed{\text{family } \mathcal{U}=\{U_i : i\in I\} \text{ of opens in } X \text{ s.t } X=\bigcup_{i\in I} U_i}$ — $\mathcal{U}$ called: open cover of X

$\exists$ $i_1,\dots,i_k \in I$ such that:

$$X = U_{i_1}\cup\dots\cup U_{i_k}.$$

$\{U_{i_1},\dots,U_{i_k}\}$ called subcover of $\mathcal{U}$

Rk: Often we are interested in $A \subseteq X$ with A endowed with the induced topology. Wonder if A = cpt $\Longrightarrow$ looking at equalities $A=\bigcup_{i\in I}\underbrace{(A\cap U_i)}_{\text{opens in } A} = A\cap\underbrace{\left(\bigcup_{i\in I} U_i\right)}$

Hence A (with the induced top) is compact iff:

$(\forall)$ family $\mathcal{U}=\{U_i : i\in I\}$ of opens in X s.t

$$A\subseteq \bigcup_{i\in I} U_i$$

$\exists$ $i_1,\dots,i_k$ s.t

$$A\subseteq U_{i_1}\cup\dots\cup U_{i_k}.$$

-8-

Examples/remarks

① $(X, \mathcal{J}$ ... $)$ compact

⑥ $A \subseteq \mathbb{R}^n$ is ...

Why? We s...

$A \neq \bar{A}$

$\forall\ \{U_i\}_{i \in I}$ of opens in X s.t.

$A \subseteq \bigcup_{i \in I} U_i$

$\exists\ i_1, \dots, i_k$ s.t.

$A \subseteq U_{i_1} \cup \dots \cup U_{i_k}$.

Examples / remarks

① $(X, \mathcal{T}_{tr})$ compact

ϕ, X

② $(X, \mathcal{T}_{dis})$: $X = \bigcup_{x \in X} \{x\}$ compact $\Rightarrow X =$ finite

③ Any $A \subseteq \mathbb{R}^n$ that is compact (as a top. space with the induced top!), since $A \subseteq \bigcup_{r>0} B_{Eucl}(0, r)$

$\Rightarrow$ can choose $r_1, \dots, r_k$ and then $R = \max\{r_1, \dots, r_k\} \Rightarrow A \subseteq B(0, R)$

$A \subseteq B(0,r_1) \cup \dots \cup B(0, r_k)$

A must be bounded.

④ $\mathbb{R}$ not compact : $\mathbb{R} = \bigcup_{k \in \mathbb{Z}} (-k, k)$ open cover $\nexists$ finite subcover

⑤ $[0, 1) \subseteq \bigcup_k (-\infty, 1 - \frac{1}{k})$

$d' = \dfrac{d_{Eucl}}{1 + d_{Eucl}}$

0 1

⑥ $A \subseteq \mathbb{R}^n$ is compact

Why? We show: A

~~$A = \overline{A}$~~ $A \supset \overline{A}$

For $x \in X \setminus A$:

$A \subseteq \bigcup_{r>0} X$

NE

ted.

-empty opens

sed

with $\begin{cases}\gamma(0)=x\\ \gamma(1)=y\end{cases}$

connected

th $\begin{cases}x\in\Gamma\\ y\in\Gamma\end{cases}$

is connected

ed $\Rightarrow$

$U\cap V=\emptyset$

$\emptyset$

IMPOSSIBLE

$\mathcal{U}$ $\boxtimes$

$A\subseteq X$ is

$\bar{A}$ is connected

Def A space X is called compact if:

$(\forall)$ family $\mathcal{U}=\{U_i : i\in I\}$ of opens in X s.t. $X=\bigcup_{i\in I} U_i$ — $\mathcal{U}$ called: open cover of X

$\exists\ i_1,\dots,i_k\in I$ such that $X=U_{i_1}\cup\dots\cup U_{i_k}$ — $\{U_{i_1},\dots,U_{i_k}\}$ called subcover of $\mathcal{U}$

Rk: Often we are interested in $A\subseteq X$ with A endowed with the induced topology. Wonder if $A=$ cpt $\Rightarrow$ looking at equalities $A=\bigcup_{i\in I}(A\cap U_i)=A\cap\underbrace{\left(\bigcup_{i\in I}U_i\right)}$ (open)

Hence A (with the induced top) is compact iff:

$(\forall)$ family $\mathcal{U}=\{U_i : i\in I\}$ of opens in X s.t. $A\subseteq\bigcup_{i\in I}U_i$

$\exists\ i_1,\dots,i_k$ s.t. $A\subseteq U_{i_1}\cup\dots\cup U_{i_k}$

Examples/remarks

① $(X, \mathcal{T}_{tr})$ compact ($\emptyset, X$)

② $(X,\mathcal{T}_{disc})$: $X=\bigcup_{x\in X}\{x\}$ compact $\Rightarrow X=$ finite

③ Any $A\subseteq\mathbb{R}^n$, that is compact (as a top. space with the induced top!), since $A\subseteq\bigcup_{r>0}B_{Eucl}(0,r)$

$\Rightarrow$ can choose $r_1,\dots,r_k$ and then $R=\max\{r_1,\dots,r_k\}\Rightarrow A\subseteq B(0,R)$

$A\subseteq B(0,r_1)\cup\dots\cup B(0,r_k)$

A must be bounded

④ $\mathbb{R}$ not compact $\mathbb{R}=\bigcup_{k\in\mathbb{Z}}(-k,k)$ open cover $\nexists$ finite subcover

⑤ $[0,1)\subseteq\bigcup_k(-\infty,1-\frac{1}{k})$ $d'=\dfrac{d_{Eucl}}{1+d_{Eucl}}$

⑥ $A\subseteq\mathbb{R}^n$ is compact

Why? We show

$\cancel{A=\bar{A}}$ $\cancel{A\supseteq\bar{A}}$

For $x\in X\setminus A$:

$A\subseteq\cancel{\bigcup_{r>0}}\ X$

NE

⑥ $A \subseteq \mathbb{R}^n$ is compact $\Rightarrow$ A must be closed in $\mathbb{R}^n$. —9—

Why? We show: A

$\cancel{A = \bar{A}}$ $A \supset \bar{A}$ $x \in A \Leftarrow x \in \bar{A}$ $\boxed{x \notin A \Rightarrow x \notin \bar{A}}$

For $x \in X \setminus A$:

$$A \subseteq \bigcup_{r>0} X \setminus \overline{B(x, \tfrac{1}{r})}$$

NEXT TIME

R)

nded

finite subcover

For instance one can ...

① $\mathbb{R} \not\cong \mathbb{R}^2$

Ex. $X = (0,1) \cup (2,3)$

connected components

Ex. $X = S^1 \setminus 3$ points

Ex. $X = \mathbb{R}^2 \setminus$ ...

Sep 3 connected components