Reminder: arbitrary topological space X... called COMPACT if:

(∀) family $\mathcal{U} = \{U_i : i \in I\}$ of opens in X s.t. $X = \bigcup_{i \in I} U_i$ ($\mathcal{U}$ = open cover of X)

(∃) $i_1, \dots, i_k \in I$ such that $X = U_{i_1} \cup \dots \cup U_{i_k}$ ($\{U_{i_1}, \dots, U_{i_k}\}$ open subcover)

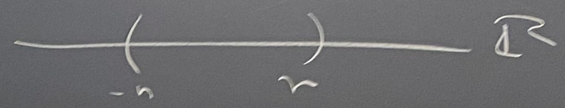
Remark: For $A \subseteq X$ ← space, endowing A with the induced topology (opens in A: $A \cap U$ with U-open in X)

then the topological space A is compact iff

(∀) family $\mathcal{U} = \{U_i : i \in I\}$ of opens in X s.t. $A \subseteq \bigcup_{i \in I} U_i$ — $A = \bigcup_{i \in I} (A \cap U_i)$

(∃) $i_1, \dots, i_k \in I$ such that $A \subseteq U_{i_1} \cup \dots \cup U_{i_k}$.

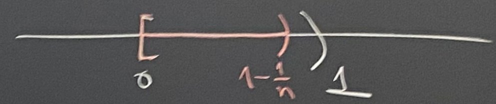
Ex of non-compact



- $\boxed{X=\mathbb{R}} = \bigcup_{n\in\mathbb{N}} (-n,n)$

Same argument $\Rightarrow$ any compact $X \subseteq \mathbb{R}^n$ must be bounded.

- $\boxed{X=[0,1)} = \bigcup_{n\in\mathbb{N}} [0, 1-\frac{1}{n})$



Similar reasoning $\Rightarrow$ any compact $X \subseteq \mathbb{R}^n$ must be closed in X

Ex of compact:

$\boxed{X=[0,1]}$ (with Euclidean topology)

proof: Assume $[0,1] = \bigcup_{i\in I} U_i$ with U_i opens in $[0,1]$

Let:

$A = \{ a \in [0,1] \,/\, [0,a]$ can be covered by a finite number of U_i's $\}$; $[0,b] \supseteq [0,a]$

To PROVE: $1 \in A$.

Remarks
- $0 \in$
$\Rightarrow \exists$
Sin
$\Rightarrow$
$\Rightarrow$
- no
- if
$A =$

$\boxed{2}$

$\mathbb{R}$

t be bounded.

be closed in X

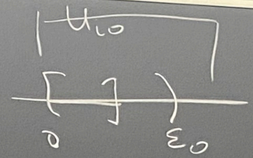
U_i: opens in $[0,1]$

}

To PROVE:
$1 \in A$.

Remarks:

$\boxed{3}$



- $0 \in A$: $0 \in [0,1] = \cup U_i \Rightarrow$

$\Rightarrow \exists\, i_0 \in I$ s.t. $0 \in U_{i_0}$
Since U_{i_0} = open in $[0,1]$ $\Big\} \Rightarrow$

$\Rightarrow \underbrace{[0,1] \cap (-\varepsilon_0, \varepsilon_0)}_{[0,\varepsilon_0)} \subseteq U_{i_0}$ for some $\varepsilon_0 > 0$

$\Rightarrow [0, \frac{\varepsilon_0}{2}] \subseteq A$

- now you could prove that A is $\left\{ \begin{array}{l} \text{open} \\ \text{closed} \end{array} \right.$ in $[0,1]$

connectedness of $[0,1] \Rightarrow A = [0,1]$

- if $a \in A \Rightarrow$ any $b \leq a$ is in A.

$A = [0,s]$ or $A = [0,s)$

$\boxed{4}$

- s must be in A.

$s \in [0,1] \Rightarrow$

$\Rightarrow \exists\, i_s$ s.t. $s \in \overset{\text{open}}{U_{i_s}}$ =

$[0, s - \frac{\varepsilon}{2}]$ is covere

$[s - \frac{\varepsilon}{2}, s]$ is covered

$\Rightarrow [0,s]$ is covere

- $s = 1$. If not $\Rightarrow \exists\, \varepsilon$

but s

$\Rightarrow$ [

$\Rightarrow$ s

2

— 1R

t be bounded.

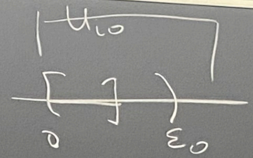
be closed in X

U_i: opens in [0,1]

} To PROVE: $1 \in A$.

Remarks

3



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- if $a \in A \Rightarrow$ any $b \leq a$ is in A.
  $A = [0,s]$ or $A = [0,s)$

4

- S must be in A.
  $s \in [0,1] \Rightarrow$
  $\Rightarrow \exists\, i_s$ s.t. $s \in U_{i_s}$ (open) =
  $[0, s - \frac{\varepsilon}{2}]$ is covere
  $[s - \frac{\varepsilon}{2}, s]$ is covered
  $\Rightarrow [0,s]$ is covere
- $s = 1$. If not $\Rightarrow \exists\, \varepsilon$
  but s
  $\Rightarrow$ [0
  $\Rightarrow$ s

Prop 4.18: Closed inside compact is compact: [-5-]

If $(X,\mathcal{T})$ = compact space, A - closed in $X \Rightarrow A$ is compact.

Prop 4.19: Compact inside Hausdorff is closed:

If $(X,\mathcal{T})$ = Hausdorff, $A \subseteq X$, A - compact $\Rightarrow A$ is closed in X.

Prop 4.20: In a Hausdorff space, any two disjoint compacts can be separated topologically

If $(X,\mathcal{T})$ = Hausdorff, $A, B \subseteq X$, $A \cap B = \emptyset$, A & B compact $\Rightarrow$

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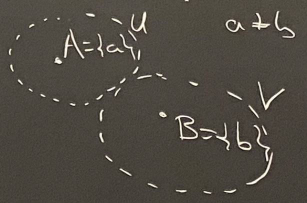


Theorem 4.23: X, Y - compact $\Rightarrow X \times Y$ = compact.

Lemma 4.17: $[0,1]$ is compact.

Corollary: F

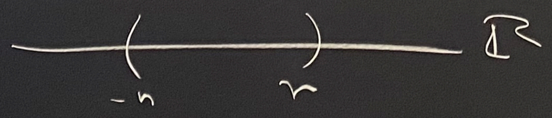
Theorem 4.2

Theorem 4.2

Corollary

Theore

Ex of non-compact [-2-]



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- $\boxed{X=[0,1)} = \bigcup [0, 1-\frac{1}{n})$

Remarks: [-3-]

- $0 \in A$: $0 \in [0,1] = \bigcup U_i \Rightarrow$

$\Rightarrow \exists\, i_0 \in I$ s.t. $0 \in U_{i_0}$

Since U_{i_0} = open in $[0,1]$

$\Rightarrow [0,1] \cap (-\varepsilon_0, \varepsilon_0) \subseteq U$

-6-

Corollary: For $A \subseteq \mathbb{R}^n$ (with Eucl. top): $A = \text{compact} \iff A$ is closed & bounded in $\mathbb{R}^n$.

Theorem 4.26: $\left.\begin{array}{l} f: X \to Y \text{ continuous} \\ A \subseteq X \text{ compact} \end{array}\right\} \Rightarrow f(A)$ is compact | Image of a compact by a continuous fct. is compact

Theorem 4.26: $\left.\begin{array}{l} f: X \to Y \text{ continuous bijection} \\ X = \text{compact},\ Y = \text{Hausdorff} \end{array}\right\} \Longrightarrow f$ is a homeomorphism.

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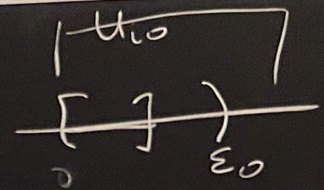
Theorem: Relationship to sequential compactness

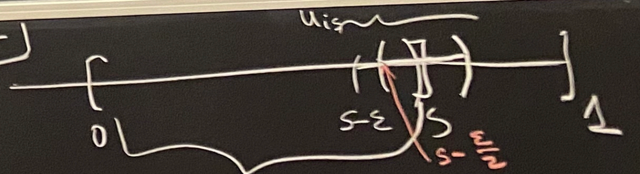


$\Rightarrow$

$\}\Rightarrow$

-4-

- s must be in A.



$s \in [0,1] \Rightarrow$

$\Rightarrow \exists\, i_s$ s.t. $s \in U_{i_s} \Rightarrow (s-\varepsilon, s] \subseteq U_{i_s}$ for $\varepsilon > 0$

$[0, s-\frac{\varepsilon}{2}]$ is covered by a finite no of $U_i \cap ($ $\Longrightarrow$

—2—

Ex of non-compact

- $\boxed{X=\mathbb{R}} = \bigcup_{n\in\mathbb{N}} (-n,n)$

Same argument $\Rightarrow$ any compact $X \subseteq \mathbb{R}^n$ must be bounded.

- $\boxed{X=[0,1)} = \bigcup_{n\in\mathbb{N}} [0, 1-\frac{1}{n})$

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Ex of compact:

$\boxed{X=[0,1]}$ (with Euclidean topology)

proof: Assume $[0,1] = \bigcup_{i\in I} U_i$ with U_i opens in $[0,1]$

Let:

$A = \left\{ a\in[0,1] \;/\; \begin{array}{l}[0,b] \\ \cap\!| \\ [0,a] \text{ can be covered by} \\ \text{a finite number of } U_i\text{'s}\end{array} \right\}$ To PROVE: $1\in A$

—3—

Remarks

- $0\in A$: $0\in[0,1] = \cup U_i \Rightarrow$
  $\Rightarrow \exists\, i_0 \in I$ s.t. $0\in U_{i_0}$ $\}\Rightarrow$
  Since U_{i_0} = open in $[0,1]$
  $\Rightarrow \underbrace{[0,1]\cap(-\varepsilon_0,\varepsilon_0)}_{[0,\varepsilon_0)} \subseteq U_{i_0}$ for some $\varepsilon_0>0$
  $\Rightarrow [0,\frac{\varepsilon_0}{2}] \subseteq A$
- now you could prove that A is open / closed in $[0,1]$ $\Rightarrow$ connectedness of $[0,1] \Rightarrow A=[0,1]$
- if $a\in A \Rightarrow$ any $b\le a$ is in A
  $A=[0,s]$ or $A=[0,s)$

—4—

- s must be in A.
  $s\in[0,1] \Rightarrow$
  $\Rightarrow \exists\, i_s$ s.t. $s\in U_{i_s}$ (open) $\Rightarrow (s-\varepsilon, s] \subseteq U_{i_s}$ for $\varepsilon>0$
  $[0, s-\frac{\varepsilon}{2}]$ is covered by a finite no of U_i's
  $[s-\frac{\varepsilon}{2}, s]$ is covered by U_{i_s} $\Bigg\}\Rightarrow$
  $\Rightarrow [0,s]$ is covered by a finite $\Rightarrow s\in A$
- $s=1$. If not $\Rightarrow \exists\,\varepsilon>0$ s.t. $(s-\varepsilon, s+\varepsilon)\subseteq U_{i_s}$
  but $s\in A \Rightarrow [0,s]$ covered by a finite $\Bigg\}\Rightarrow$
  $\Rightarrow [0, s+\frac{\varepsilon}{2}]$ still — " —
  $\Rightarrow s+\frac{\varepsilon}{2}\in A$ contradiction.

—5—

Prop 4.18: Closed inside compact is compact:
If $(X,\mathcal{T})$ = compact space, A - closed in X $\Rightarrow A$ is compact.

Prop 4.19: Compact inside Hausdorff is closed
If $(X,\mathcal{T})$ = Hausdorff, $A\subseteq X$, A - compact $\Rightarrow A$ is closed in X.

Prop 4.20: In a Hausdorff space, any two disjoint compacts can be separated topologically
If $(X,\mathcal{T})$ = Hausdorff, $A,B\subseteq X$, $A\cap B=\emptyset$, A & B compact $\Rightarrow$
$\Rightarrow \exists$ opens $U,V\subseteq X$ s.t. $A\subseteq U$, $B\subseteq V$ and $U\cap V=\emptyset$

Theorem 4.23: X,Y - compact $\Rightarrow X\times Y$ = compact

$A=\{a\}$, $a\neq b$, $B=\{b\}$

Lemma 4.17: $[0,1]$ is compact.

—6—

Corollary: For $A\subseteq\mathbb{R}^n$ (with Eucl. top): A = compact $\Leftrightarrow$ A is closed & bounded in $\mathbb{R}^n$.

Theorem 4.26: $\left.\begin{array}{l} f: X\to Y \text{ continuous} \\ A\subseteq X \text{ compact}\end{array}\right\} \Rightarrow f(A)$ is compact | Image of a compact by a continuous fct. is compact

Theorem 4.26: $\left.\begin{array}{l} f: X\to Y \text{ continuous bijection} \\ X = \text{compact}, Y = \text{Hausdorff}\end{array}\right\} \Rightarrow f$ is a homeomorphism

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Theorem: Relationship to sequential compactness

-5-

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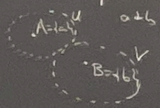

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-6-

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Theorem: Relationship to sequential compactness

Prop 4.19: Compact inside Hausdorff is closed:

5

If $(X,\mathcal{T})$ = Hausdorff, $A \subseteq X$, A-compact $\Rightarrow$ A is closed in X.

Prop 4.20: In a Hausdorff space, any two disjoint ...

---

proof of corollary | assuming Prop 4.19, Lemma, Thm 4.26, Thm 4.23, Prop 4.18.

"$\Longrightarrow$" Assume A = compact. Have seen: A-bounded.

$A \subseteq \mathbb{R}^n$ Hausd. $\xrightarrow[4.19]{\text{prop}}$ A - closed in $\mathbb{R}^n$.

"$\Longleftarrow$" Assume A - closed and bounded.

On the other hand: $[0,1]$ = cpt $\Rightarrow$ any interval $[a,b]$ is compact

$(f:[0,1]\to[a,b],\ f(t)=(1-t)a+tb$ continuous, & use Thm 4.26)

Using now thm 4.23 $\Rightarrow$ $\underbrace{[a,b]\times\dots\times[a,b]}_{n}$ is compact

$= [a,b]^n$

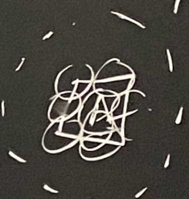
Since A-bounded $\Rightarrow \exists\, a,b$ s.t. $A \subseteq [a,b]^n \xrightarrow[4.18]{\text{Prop}} A$ = cpt $\square$

proof of 4.18: Start with arbitrary $\mathcal{U}=\{U_i : i\in I\}$ opens in X s.t. $A \subseteq \bigcup_{i\in I} U_i$.

To prove ...

Know: X = cpt

A = closed



X N...

$\Rightarrow$ find $i_1,\dots,i$...

$\Rightarrow A \subseteq U_{i_1}\cup\dots$

proof of 4.26: f...

To prove: $f(A$...

Start with ar...

s.t. $f(A) \subseteq \bigcup$...

Remark: $\{f^{-1}(U_i)$...

compact $\Rightarrow A$ is closed in X.

Lemma, Thm 4.26, Thm 4.23, Prop 4.18.

A - bounded.

closed in $\mathbb{R}^n$.

ed.

interval $[a,b]$ compact

, $f(t)=(1-t)a+tb$

se Thm 4.26)

$b]$ is compact

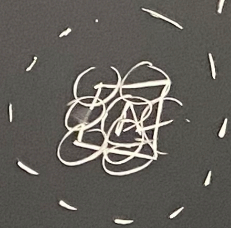
$,b]^n \xrightarrow{\text{Prop}} A = \text{cpt}$ ☐

$\mathcal{U}=\{U_i : i\in I\}$

To prove $\exists i_1,\dots,i_k$ s.t. $A\subseteq U_{i_1}\cup\dots\cup U_{i_k}$. 8

Know. $X=$ cpt

$A=$ closed in $X \Rightarrow X\setminus A$ is open $\Rightarrow$



Notice: $X=(X\setminus A)\cup\bigsqcup_{i\in I} U_i$

$\Rightarrow$ find $i_1,\dots,i_k$ s.t. $X=(X\setminus A)\cup U_{i_1}\cup\dots\cup U_{i_k}$.

$\Rightarrow A\subseteq U_{i_1}\cup\dots\cup U_{i_k}$ ☺

$\boxed{\text{proof of 4.26}}$: $f:X\to Y$ cont, $A\subseteq X$ cpt.

To prove: $f(A)$ is compact.

Start with arbitrary $\{U_i : i\in I\}$ opens in Y

s.t. $f(A)\subseteq\bigsqcup_{i\in I} U_i$

$$\begin{array}{ccc} f: X & \longrightarrow & Y \supseteq U_i \\ \cup & & \cup \\ A & & f(A) \end{array}$$

Remark: $\{f^{-1}(U_i) : i\in I\}$ opens in X

and $A\subseteq$ (cpt)

$\Rightarrow \exists i_1,\dots,i_k$

Proof of 4.2

i.e. $U\subseteq$

Notice this

proving. A

$A \subseteq X$ compact | by a continuous fct is compact

-8-

k

and $\underset{\substack{\uparrow \\ \text{cpct}}}{A} \subseteq \bigcup_{i\in I} f^{-1}(U_i)$ ($a \in A \Rightarrow f(a) \in f(A) \Rightarrow \exists\, i \in I$ s.t. $f(a) \in U_i \Rightarrow a \in f^{-1}(U_i)$)

-9-

$\Rightarrow \exists\, i_1, \ldots, i_k \in I$ s.t. $A \subseteq f^{-1}(U_{i_1}) \cup \ldots \cup f^{-1}(U_{i_k}) \Rightarrow \underset{\substack{\psi \\ f(A)}}{f(A) \subseteq} U_{i_1} \cup \ldots \cup U_{i_k}$

Proof of 4.27: Still to show: f^{-1} = continuous:

i.e. $U \subseteq X$ open $\Longrightarrow$ $\underbrace{(f^{-1})^{-1}(U)}_{f(U)}$ - open in Y.

$* \; f^{-1}(x) \in U$

$x \in f(U)$

Notice this is equivalent (since $f(X \setminus U) = Y \setminus f(U)$) to proving: A - closed in X $\Longrightarrow$ $f(A)$ - closed in Y.

But X = compact $\overset{4.18}{\Longrightarrow}$ A = compact $\overset{4.26}{\Longrightarrow}$ $f(A)$ = compact, but Y = Hausdorff $\overset{4.19}{\Longrightarrow}$ $f(A)$ - closed in Y.

Y

$Y \ni$

$\cup$

$f(A)$

U_i

If (X, τ) = Hausdorff, $A \subseteq X$, A - compact $\Rightarrow$ A is closed in X.

Prop 4.20: In a Hausdorff space, any two disjoint compacts can be

Proof of 4.20: Write $Y | Z$ if $\begin{cases} \exists \text{ open } U \supset Y \\ \exists \text{ open } V \supset Z \end{cases}$ s.t. $U \cap V = \emptyset$ for $Y, Z \subseteq X$

Claim: if $\boxed{\begin{matrix} Y | \{z\} \text{ for all } z \in Z \\ Z = \text{compact} \end{matrix}} \Rightarrow Y | Z$.

This would imply the ~~result~~: for A, B as above:

$(\forall) a \in A$, $(\forall) b \in B$ $\to$ $\{a\} | \{b\}$

$A \cap B = \emptyset \Rightarrow a \neq b$ $\xrightarrow{\text{Hausdorff}}$ Can use claim for $Y = \{a\}$, $Z = B$

$\Rightarrow \{a\} | B \Rightarrow B | \{a\}$ $(\forall) a \in A$. Claim again $\Rightarrow B | A$ ∎

proof of claim:

For any $\boxed{z \in Z}$: find opens U_z, V_z s.t.:

$Y \subseteq U_z$, $\boxed{z \in V_z}$, $U_z \cap V_z = \emptyset$ (*)

Remark: $\{V_z : z \in Z\}$ cover Z $\left(Z \subseteq \bigcup_{z \in Z} V_z \right)$

But Z = cp

s.t. $Z \subseteq$

$Y \subseteq$

$\to$ indeed: if $x \in$

$\Rightarrow$ $x \in V_{z_i}$ for

$x \in U_{z_i}$ for

Proof of 4.19:

$\cap U_z$

...wo disjoint compacts can be separated

Theorem 4.27: $f: X \to Y$ continuous bij...

$\supset Y$

$\supset Z$

...bove:

$, Z = B$

$\Rightarrow B | A$ $\boxtimes$

$= \emptyset)$ $(*)$

$= \bigcup_{z \in Z} V_z)$

s.t. $U \cap V = \emptyset$. But Z = cpct $\Rightarrow \exists\, z_1, \dots, z_k \in Z$ -11-

s.t. $Z \subseteq \underbrace{V_{z_1} \cup \dots \cup V_{z_k}}_{\text{open } V \text{ in } X}$

$x \in Y \subseteq \underbrace{U_{z_1} \cap \dots \cap U_{z_k}}_{\text{open } U \text{ in } X}$.

$U \cap V = \emptyset$

$\to$ indeed: if $x \in U \cap V$ $<$ $x \in U \Rightarrow$; $x \in V$

$\Rightarrow$ $x \in V_{z_{i_0}}$ for some $i \in \{1, \dots, k\}$; $x \in U_{z_j}$ for all $j \in \{1, \dots, k\}$

For that i_0: $x \in V_{z_{i_0}} \cap U_{z_{i_0}}$ $\lightning (*)$ $\boxtimes$

Proof of 4.19:

$\bigcap U_z$

$\bigcup V_z$

4.19: For $\boxed{x \notin A}$...

$A = A$, $B = \{x$...

The fact that x...

s.t.

$\Leftrightarrow x \notin \bar{A}$.

Hence $x \notin A \Rightarrow$...

OR: $x \in \bar{A} \Rightarrow$...

IE: $\bar{A} \subseteq A$

But $A \subseteq \bar{A}$ allw...

Thm 4.27: $f: X \to Y$ continuous bijection)

$\boxed{-11-}$

$U \cap V = \phi$

...hat is: $\overline{\cap U_{z_{i_0}}}$ $\not\downarrow(x)$ $\boxed{\Xi}$

$\cup V_z$

4.19: For $\boxed{x \notin A}$ can apply Prop 4.20 to

$A = A$, $B = \{x\}$ $\Rightarrow$ $\exists$ U, V-opens s.t. $\left\{ \begin{array}{l} x \in U \\ A \subseteq V \quad U \cap V = \phi \end{array} \right.$

$\exists U \in \mathcal{T}(x)$ s.t.

in part: $U \cap A = \phi$.

The fact that x admits $U \in \mathcal{T}(x)$ s.t. $U \cap A = \phi$

$\Longleftrightarrow$ $x \notin \bar{A}$.

Hence $x \notin A \Rightarrow x \notin \bar{A}$

OR: $x \in \bar{A} \Rightarrow x \in A$.

IE: $\bar{A} \subseteq A$

But $A \subseteq \bar{A}$ always

$\Rightarrow A = \bar{A} \Rightarrow A = \text{closed}$ $\boxed{\checkmark}$

-5-

Prop 4.18: Closed inside compact is compact:

(2) If $(X,\mathcal{J})$ = compact space, A - closed in X $\Rightarrow A$ is compact

Prop 4.19: Compact inside Hausdorff is closed

(5) If $(X,\mathcal{J})$ = Hausdorff, $A \subseteq X$, A - compact $\Rightarrow A$ is closed in X

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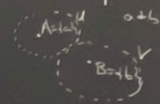
(4) If $(X,\mathcal{J})$ = Hausdorff, $A,B \subseteq X$, $A \cap B = \emptyset$, A & B compact $\Rightarrow$

$\Rightarrow \exists$ opens $U,V \subseteq X$ s.t. $A \subseteq U$, $B \subseteq V$ and $U \cap V = \emptyset$



Theorem 4.23: X,Y - compact $\Rightarrow X \times Y$ = compact

Lemma 4.17: $[0,1]$ is compact

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Corollary: For $A \subseteq \mathbb{R}^n$ (with Eucl top): A = compact $\Leftrightarrow A$ is closed & bounded in $\mathbb{R}^n$

(3) **Theorem 4.26**: $\left.\begin{array}{l} f: X \to Y \text{ continuous} \\ A \subseteq X \text{ compact} \end{array}\right\} \Rightarrow f(A)$ is compact | Image of a compact by a continuous fct is compact

(6) **Theorem 4.27**: $\left.\begin{array}{l} f: X \to Y \text{ continuous bijection} \\ \underline{X = \text{compact}},\ \underline{Y = \text{Hausdorff}} \end{array}\right\} \Rightarrow f$ is a homeomorphism

Corollary 4.28: $\left.\begin{array}{l} f: X \to Y \text{ continuous, injective} \\ X = \text{compact},\ Y = \text{Hausdorff} \end{array}\right\} \Rightarrow f$ is an embedding

Theorem: Relationship to sequential compactness

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Proof of 4.20 Write $Y | Z$ if $\left\{\begin{array}{l} \exists \text{ open } U \supset Y \\ \exists \text{ open } V \supset Z \end{array}\right.$ s.t. $U \cap V = \emptyset$ for $Y, Z \subseteq X$

Claim: if $\boxed{\begin{array}{l} Y | \{z\} \text{ for all } z \in Z \\ Z = \text{compact} \end{array}}$ $\Rightarrow Y | Z$

This would imply the ~~result~~ for A, B as above

$(\forall) a \in A$, $(\forall) b \in B$ $\quad \{a\} | \{b\}$

$A \cap B = \emptyset \Rightarrow a \neq b$ Hausdorff. Can use claim for $\underline{Y = \{a\}}$, $\underline{Z = B}$

$\Rightarrow \{a\} | B \Rightarrow B | \{a\}$ $(\forall) a \in A$. Claim again $\Rightarrow B | A$ ∎

proof of claim

For any $\boxed{z \in Z}$ find opens U_z, V_z s.t.

$Y \subseteq U_z$, $\boxed{z \in V_z}$, $U_z \cap V_z = \emptyset$ (*)

Remark: $\{V_z : z \in Z\}$ cover Z $\left(Z \subseteq \bigcup_{z \in Z} V_z\right)$

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But Z = cpt $\Rightarrow \exists z_1, \dots, z_k \in Z$

s.t. $Z \subseteq \underbrace{V_{z_1} \cup \dots \cup V_{z_k}}_{\text{open } V \text{ in } X}$

$Y \subseteq \underbrace{U_{z_1} \cap \dots \cap U_{z_k}}_{\text{open } U \text{ in } X}$ $\quad U \cap V = \emptyset$

$\Rightarrow$ indeed: if $\exists x \in U \cap V$ $<$ $x \in U \Rightarrow$ / $x \in V$

$\Rightarrow$ $<$ $x \in V_{z_{j_0}}$ for some $j_0 \in \{1, \dots, k\}$ / $x \in U_{z_j}$ for all $j \in \{1, \dots, k\}$. For that j_0: $x \in V_{z_{j_0}} \cap U_{z_{j_0}}$ $\lightning$ (*) ∎

Proof of 4.19

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4.19: For $\boxed{x \notin A}$ can apply Prop 4.20 to $\exists U \in \mathcal{J}(x)$ s.t.

$A = A$, $B = \{x\}$ $\Rightarrow \exists U, V$ - opens s.t. $x \in U$, $A \subseteq V$, $U \cap V = \emptyset$

in part. $U \cap A = \emptyset$

The fact that x admits $U \in \mathcal{J}(x)$ s.t. $U \cap A = \emptyset$

$\Leftrightarrow \underline{x \notin \bar{A}}$

Hence $x \notin A \Rightarrow x \notin \bar{A}$

OR: $x \in \bar{A} \Rightarrow x \in A$

IE: $\left.\begin{array}{l} \bar{A} \subseteq A \\ \text{But } A \subseteq \bar{A} \text{ always} \end{array}\right\} \Rightarrow A = \bar{A} \Rightarrow A$ = closed ∎