Prop 4.18: $\boxed{\text{Closed inside compact is compact}}$:

$(X,\mathcal{T})$ = cpct space, A - closed in X $\Rightarrow$ A (with induced topology) is compact.

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Theorem 4.23: $\boxed{\text{Product of two compacts is compact}}$:

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Corollary: For any $\boxed{A \subseteq \mathbb{R}^n}$ (with the Euclidean topology):

$\boxed{A = \text{compact} \Leftrightarrow A = \text{bounded and closed in } \mathbb{R}^n}$

Theorem 4.26: $\boxed{\text{Continuous functions send compacts to compacts}}$:

$f: X \longrightarrow Y$ continuous, $A \subseteq X$ compact $\Rightarrow$ $f(A)$ - is compact

Theorem 4.27 (or: 4.28): $\boxed{\text{Continuous bijection (or: injection) from compact to Hausdorff} \Rightarrow \text{homeomorphism (or: embedding)}}$

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Theorem: (1)

(2)

<u>Reminder</u>: X = compact if

$(\forall)$ family $\mathcal{U} = \{U_i : i \in I\}$ of opens in X s.t. $X = \bigcup_{i \in I} U_i$

$(\exists)\ i_1, \dots, i_k \in I$ such that $X = U_{i_1} \cup U_{i_2} \cup \dots \cup U_{i_k}$.

<u>Rk</u>: for $A \subseteq X$ with induced topology, compactness of $A \iff$

$(\forall)$ ——— " ——— $A \subseteq \bigcup_{i \in I} U_i$

$(\exists)$ ——— " ——— $A \subseteq U_{i_1} \cup U_{i_2} \cup \dots \cup U_{i_k}$.

<u>Def</u>. A space X is called SEQUENTIALLY COMPACT if:

$(\forall)\ (x_n)_{n \geq 1}$ a sequence in X

$(\exists)$ subsequence $x_{n_1}, x_{n_2}, x_{n_3}, \dots$ such that $(x_{n_k})_{k \geq 1}$ is convergent (in X !!)

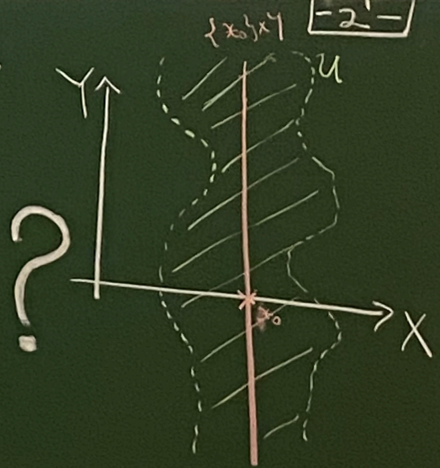
(hence $n_1 < n_2 < n_3 < \dots$)

<u>Lemma (the tube lemma)</u>: X, Y - spaces, $x_0 \in X$

$\mathcal{U} \subseteq X \times Y$ open s.t. $\{x_0\} \times Y \subseteq \mathcal{U}$

Can one find

W = open neighbd. of x_0 in X such that



Lemma says: yes <u>if</u>

Reminder: X = compact if

some indexing set

(∀) family $\mathcal{U} = \{U_i : i \in I\}$ of opens in X s.t. $X = \bigcup_{i \in I} U_i$

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Def: A space X is called SEQUENTIALLY COMPACT if:

(∀) $(x_n)_{n \geq 1}$ a sequence in X ($x_n \in X$ (∀) n)

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(hence $n_1 < n_2 < n_3 < \dots$)

(in X !!)

Ex: $X = \mathbb{R}$ not: $x_n = n$

$X = [0, 1)$ not: $x_n = 1 - \frac{1}{n}$

Lemma (the tube lemma): X, Y - spaces, $x_0 \in X$

$\mathcal{U} \subseteq X \times Y$ $\boxed{\text{open}}$ s.t. $\boxed{\{x_0\} \times Y \subseteq \mathcal{U}}$

Can one find

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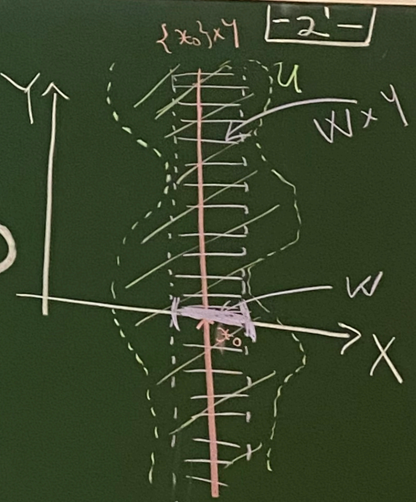
$W \times Y \subseteq \mathcal{U}$?

Lemma says: yes if Y = compact



— 1 —

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— 1' —

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$$\left.\begin{array}{l} f: X \to Y \text{ continuous bijective} \\ X = \text{compact}, Y = \text{Hausdorff} \end{array}\right\} \Rightarrow f \text{ is a homeomorphism.}$$

Theorem: (1) If X = 1st countable: $(X = \text{cpct}) \Rightarrow (X = \text{sequentially compact})$

(2) If X = metric space, then "$\Leftrightarrow$" holds.

proof of the Tube Lemma:

We know: ① $\mathcal{U} \subseteq X \times Y$ open in $\underbrace{X \times Y}_{\text{with the prod topology}}$ i.e.

① $(\forall)(a,b) \in \mathcal{U}$ $\exists$ $W \in \mathcal{T}_X(a)$, $V \in \mathcal{T}_Y(b)$ s.t. $W \times V \subseteq \mathcal{U}$

② $\{x_0\} \times Y \subseteq \mathcal{U}$ i.e. $(x_0, y) \in \mathcal{U}$ for all $y \in Y$

Hence, for all $y \in Y$: we find open neighbhds $\begin{cases} W_y \in \mathcal{T}(x_0) \\ V_y \in \mathcal{T}(y) \end{cases}$ such that $\boxed{W_y \times V_y \subseteq \mathcal{U}}$

$W = \cancel{\bigcap_{y \in Y}} W_y$

Also know: Y = compact.

Notice that $\{V_y \mid y \in Y\}$ is an open cover of Y

$\Rightarrow \exists\, y_1, \dots, y_k \in Y$ s.t. $\boxed{Y = V_{y_1} \cup V_{y_2} \cup \dots \cup V_{y_k}}$

Remark: $W = W_{y_1} \cap \dots \cap W_{y_k}$ does the job!

Indeed: for $(x_0, y) \in W \times Y$ since $y \in Y = V_{y_1} \cup \dots \cup V_{y_k} \Rightarrow$

$\Rightarrow \exists\, i$ s.t. $\boxed{y \in V_{y_i}}$. But $x_0 \in W = W_{y_1} \cap \dots \cap W_{y_k} \Rightarrow \boxed{x_0 \in W_{y_i}}$. Hence $(x_0, y) \in W_{y_i} \times V_{y_i} \subseteq \mathcal{U}$ ☐

proof of Thm 4.23: To prove $X \times Y$ = cpct: start with an arbitrary open cover

$$X \times Y = \bigcup_{i \in I} U_i \qquad U_i \subseteq X \times Y \text{ open.}$$

For each $\boxed{x \in X}$, look at $\{x\} \times Y$ — this is compact (homeomorphic to Y: $\{x\} \times Y \longleftrightarrow Y$, $(x, y) \longleftrightarrow y$)

$$\underbrace{\{x\} \times Y}_{A} \subseteq X \times Y = \bigcup_{i \in I} U_i \Rightarrow$$

$\Rightarrow$ Find $I_x \subseteq I$ finite subset s.t. $\{x\} \times Y \subseteq \underbrace{\bigcup_{j \in I_x} U_j}_{\mathcal{U}}$ (I_x - finite)

Use the tube lemma $\Rightarrow$

$\Rightarrow \boxed{W_x \in \mathcal{T}_X(x)}$ s.t. $W_x \times Y \subseteq \bigcup_{j \in I_x} U_j$

Now $\{W_x : x \in X\}$ open cover of $X \Rightarrow \exists\, x_1, \dots, x_\ell$ s.t. $X = W_{x_1} \cup \dots \cup W_{x_\ell}$

Finally, we have

$$X \times Y \subseteq (W_{x_1} \cup \dots \cup W_{x_\ell}) \times Y \subseteq \bigcup_{p=1}^{\ell} \bigcup_{j \in I_{x_p}} U_j. \quad ☐$$

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We know: ① $\mathcal{U} \subseteq X \times Y$ open in $X \times Y$ ← with the prod topology i.e.:

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Use the tube lemma $\Rightarrow$

$$\Rightarrow \boxed{W_x \in \mathcal{T}_X(x)} \text{ s.t. } \underline{W_x \times Y \subseteq \bigcup_{j \in I_x} U_j}.$$

Now $\{W_x : x \in X\}$ open cover of $X \Rightarrow \exists\, x_1, \dots, x_\ell$ s.t. $X = W_{x_1} \cup \dots \cup W_{x_\ell}$

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$V_{y_i} \times V_{y_i} \subseteq \mathcal{U}$ $\blacksquare$

proof of the Tube Lemma: —3—

We know: ① $\mathcal{U} \subseteq X\times Y$ open in $X\times Y$ (with the prod topology) i.e.:

① $(\forall)(a,b)\in\mathcal{U}\ \exists\ W\in\mathcal{J}_X(a),\ V\in\mathcal{J}_Y(b)$ s.t. $W\times V\subseteq\mathcal{U}$

② $\{x_0\}\times Y\subseteq\mathcal{U}$ i.e. $(x_0,y)\in\mathcal{U}$ for all $y\in Y$

Hence, for all $y\in Y$: we find open neighbhds $\left\{\begin{array}{l} W_y\in\mathcal{J}(x_0)\\ V_y\in\mathcal{J}(y)\end{array}\right.$ such that $\boxed{W_y\times V_y\subseteq\mathcal{U}}$

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Proof of Thm 4.23: To prove $X\times Y$ = cpt: start with an arbitrary open cover —4—

$$X\times Y = \bigcup_{i\in I} U_i \qquad U_i\subseteq X\times Y \text{ open.}$$

For each $\boxed{x\in X}$, look at $\{x\}\times Y$ — this is compact (homeomorphic to Y: $\{x\}\times Y\longleftrightarrow Y$, $(x,y)\longleftrightarrow y$)

$\underbrace{\{x\}\times Y}_{A}\subseteq X\times Y = \bigcup_{i\in I} U_i \Rightarrow$

$\Rightarrow$ Find $I_x\subseteq I$ finite subset s.t. $\{x\}\times Y\subseteq \underbrace{\bigcup_{j\in I_x} U_j}_{\mathcal{U}}$.

Use the tube lemma $\rightarrow$

$\Rightarrow \boxed{W_x\in\mathcal{J}_X(x)}$ s.t. $\underline{W_x\times Y\subseteq\bigcup_{j\in I_x} U_j}$. I_x - finite

Now $\{W_x : x\in X\}$ open cover of $X\Rightarrow\exists\ x_1,\dots,x_\ell$ s.t. $X = W_{x_1}\cup\dots\cup W_{x_\ell}$.

Finally, we have

$$X\times Y = (W_{x_1}\cup\dots\cup W_{x_\ell})\times Y\subseteq\bigcup_{p=1}^{\ell}\bigcup_{j\in I_{x_p}} U_j. \ ☒$$

proof of part (1) of Theorem: —5—

Hence: assume $\boxed{X = 1^{st}\text{ countable \& compact}}$

To prove: X = sequentially compact.

Hence: let $\boxed{(x_n)_{n\geq1}\text{ be an arbitrary sequence in } X}$

To prove: $\exists$ subsequence of $(x_n)_{n\geq1}$ convergent in X.

Step 1: We claim $\exists\ x\in X$ s.t. $(\forall)V\in\mathcal{J}(x)$ contains infinitely many elements of the sequence

pf: By contrad: assume $\nexists\ z$ as above

$\Rightarrow (\forall)z\in X\ \exists\ V_z\in\mathcal{J}(z)$ which only contains a finite number of the terms

Now $\{V_z : z\in X\}$ = open cover of $X\Rightarrow\exists\ z_1,\dots,z_k\in X$ s.t.

$X = V_{z_1}\cup\dots\cup V_{z_k}$. impossible. $x_1, x_2, x_3, \dots$

With x from step 1, choose a countable basis of neighbhds of x —6—

$B_1\supset B_2\supset B_3\supset\dots\ \in\mathcal{J}(x)$

$\left.\begin{array}{l}\exists\ n_1\in\mathbb{N}\text{ s.t. } x_{n_1}\in B_1\\ \exists\ n_2>n_1\text{ s.t. } x_{n_2}\in B_2\\ \exists\ n_3>n_2\text{ s.t. } x_{n_3}\in B_3\\ \dots\end{array}\right\}\Rightarrow (x_{n_k})_{k\geq1}$ with $x_{n_k}\in B_k$ $\Downarrow$ $\lim_{k\to\infty} x_{n_k} = x$

Hence we found a conv. subsequence.

$I_z = \{i : x_i\in V_z\}$ is finite.

proof of Part 2 of Thm:

Hence: still to prove if (X,d) = metric space, X = seq. compact | to prove X = cpt.

Step 2: similarly by contradiction

Assume: $X = \bigcup_{i\in I} U_i$ $\quad U_i\subseteq X$ opens —7—

Step 1: $(\forall)\delta>0$: $\exists\ x_1,\dots,x_k\in X$ s.t. $X = B(x_1,\delta)\cup\dots\cup B(x_k,\delta)$

Step 2: $\exists\ \delta>0$ s.t. $(\forall)x\in X$, $B(x,\delta)\subseteq U_i$ for some $i\in I$.

pf of Step 1: By contrad $\Rightarrow\exists\ \delta>0$ s.t. X is not a finite union of δ-balls.

Start with $x_1\in X$

Choose $x_2\in X\setminus B(x_1,\delta)$

$x_3\in X\setminus(B(x_1,\delta)\cup B(x_2,\delta))$

$x_4\in X\setminus(B(x_1,\delta)\cup B(x_2,\delta)\cup B(x_3,\delta))$

etc $\Rightarrow$ sequence $(x_k)_{k\geq1}$ with $\underline{d(x_i,x_j)\geq\delta}$ $(\forall)i\neq j$.

By hypothesis $\Rightarrow$ we find a subsequence

$x_{k_1}, x_{k_2},\dots\longrightarrow x\in X$. In part: $\exists\ \ell_0$ s.t.

$d(x_{k_\ell},x)<\frac{\delta}{2}$ $(\forall)\ell\geq\ell_0\Rightarrow d(x_{k_{\ell+1}},x_{k_\ell})<\frac{\delta}{2}+\frac{\delta}{2}=\delta$ ↯

Theorem 4.23: $\boxed{\text{Product of two compacts is compact}}$: $U \cap V = \emptyset$.

-8-

$\boxed{\textbf{Def 4.49}: X = \text{space. A one-point compactification of } X \text{ is a Hausdorff compact space } \tilde{X} \text{ together with an embedding } i: X \longrightarrow \tilde{X} \text{ s.t. } \tilde{X} \setminus i(X) \text{ is just a point.}}$

Q: Given X: can one find $\tilde{X}$? $\underbrace{\text{How many?}}_{\text{unique !!}}$

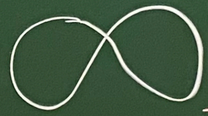
How to find $\tilde{X}$

Ex: $X = [0,1)$ $\qquad \tilde{X} = [0,1]$

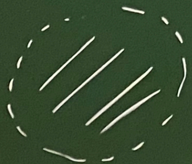
$X = (0,1)$ $\qquad \tilde{X} = S^1$ $\qquad i: (0,1) \longrightarrow S^1$

$i(t) = (\cos 2\pi t, \sin 2\pi t)$

$S^1 \setminus i((0,1)) = \{(1,0)\}$

$X = (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$ 

$\to$ Local compactness

$X =$  $\qquad S^2$

$\to$ Stone-Weierstrass

$\dashrightarrow$ Gelfand-Naimark

$\dashrightarrow$ Finite partitions of unity

$X =$ 

Prop 4.19: $\boxed{\text{Compact inside Hausdorff is closed}}$:

$(X,\mathcal{J})$ = Hausdorff space, $A \subseteq X$, A-cpct $\Rightarrow A$ is closed in X

Prop 4.20: $\boxed{\text{In Hausdorff spaces, disjoint compacts can be separated topologically}}$:

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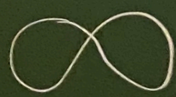
$X = (0,\frac{1}{2}) \cup (\frac{1}{2},1)$ 

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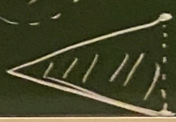
$X =$ 

→ Local compactness

→ Stone-Weierstrass

--→ Gelfand-Naimark

--→ Finite partitions of unity