-2-

1-point compactification

Def: X = a space. A 1-point compactification of X is a Hausdorff compact space $\boxed{\tilde{X}}$ together with an embedding $\boxed{i : X \longrightarrow \tilde{X}}$ s.t. $\tilde{X} \setminus i(X)$ has only one point.

Q: does $\tilde{X}$ exist? How many?

Theorem: If X is a $\boxed{\text{locally compact}}$ Hausdorff space then:

1. $\exists$ a 1-point compactification
2. any two —"— s are homeomorphic

proof: $X^\infty = X \cup \{\infty\}$ where ∞ is an element not in X.

Topology $\mathcal{T}^\infty$ on X^∞:

call $U \subseteq X^\infty$ open in X^∞ if

- $U \subseteq X$ and is open in X $\quad \mathcal{T}$

OR

- $U = X^\infty \setminus K$ with $K \subseteq X$ compact $\quad \mathcal{T}(\infty)$

$\mathcal{T}^\infty = \mathcal{T} \cup \mathcal{T}$

(T2): $U, V \in \mathcal{T}^\infty$

Four cases 1. $U \in \mathcal{T}$

2. $U \in \mathcal{T}$

3. ...

4. $U \in$

$X^\infty \setminus K$

(T3) $U_i \in \mathcal{T}^\infty$ $(\forall)$

This holds if

Left with:

$U \in \mathcal{T}$, $V \in \mathcal{T}($

$X^\infty \setminus K$

$K \subseteq X$ co

-5-

Claim 3: $(X^\infty, \mathcal{T}^\infty)$ is compact.

Claim 4: the

neighbd. of x ... ch is compact

any point $x\in X$

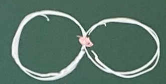
Examples:

— o —

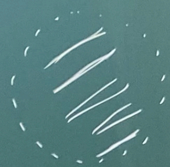
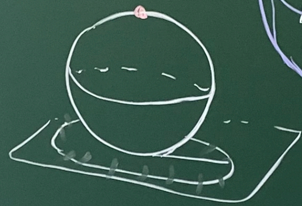
$U\cup V = U\cup(X^\infty\setminus K) = X^\infty\setminus K'$ where $K' = K\cap(X\setminus U)$.

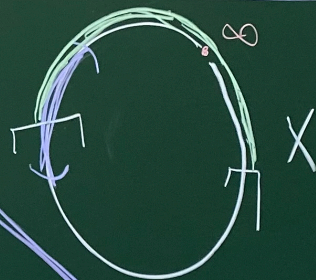
$X=[0,1)$ $\qquad$ $\tilde{X}=[0,1]$ $\qquad$ i = inclusion : $[0,1)\longrightarrow[0,1]$, $\boxed{i(x)=x}$

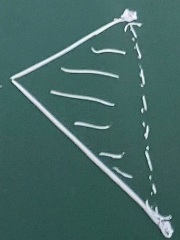
$X=(0,1)$ $\qquad$ $\tilde{X}=S^1$ $\qquad$ $i:(0,1)\longrightarrow S^1$, $i(t)=e^{2\pi i t} = (\cos 2\pi t, \sin 2\pi t)$

$X=(0,\frac{1}{2})\cup(\frac{1}{2},1)$ $\qquad$ $\tilde{X}$:  $\qquad$ i ... (on the picture)

$\underset{X}{\underset{\cap}{U}}\cap(\underset{\infty}{\underset{\not\ni}{X^\infty}}\setminus K) = U\cap(X\setminus K)$

$X=$  $\qquad$ $\tilde{X}=S^2$ 



$X=$  $\qquad$ Trouble!

-2-

<u>Def</u>: X = a space. A <u>1-point compactification</u> of X is <u>a Hausdorff compact space</u> $\boxed{\tilde{X}}$ together with <u>an embedding</u> $\boxed{i : X \longrightarrow \tilde{X}}$ s.t. $\tilde{X} \setminus i(X)$ has only one point.

<u>Q</u>: does $\tilde{X}$ exist? How many?

<u>Theorem</u>: If X is a $\boxed{\text{locally compact}}$ Hausdorff space then:

1. $\exists$ a 1-point compactification
2. any two —"— s are homeomorphic

<u>proof</u>: $X^\infty = X \cup \{\infty\}$ where ∞ is an element not in X.

Topology $\mathcal{T}^\infty$ on X^∞:

call $U \subseteq X^\infty$ open in X^∞ if
- $U \subseteq X$ and is open in X — $\mathcal{T}$

  OR
- $U = X^\infty \setminus K$ with $K \subseteq X$ compact. $\mathcal{T}(\infty)$

$\mathcal{T}^\infty = \mathcal{T} \cup \mathcal{T}$

(T2): $U, V \in \mathcal{T}^\infty$ =

Four cases: 1. $U \in \mathcal{T}$

2. $U \in \mathcal{T}$

3. ...

4. $U \in$

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(T3) $U_i \in \mathcal{T}^\infty$ $(\forall)$

This holds if

Left with:

$U \in \mathcal{T}$, $V \in \mathcal{T}($

$X^\infty \setminus K$

$K \subseteq X$ co

-5-

<u>Claim 3</u>: $(X^\infty, \mathcal{T}^\infty)$ is compact.

<u>Claim 4</u>: the

-2-

Def: X = a space. A 1-point compactification of X is a Hausdorff compact space $\boxed{\tilde{X}}$ together with an embedding $\boxed{i : X \longrightarrow \tilde{X}}$ s.t. $\tilde{X} \setminus i(X)$ has only one point.

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OR

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$\mathcal{T}^\infty = \mathcal{T} \cup \mathcal{T}(\infty)$

-3-

Claim 1: $\mathcal{T}^\infty$ is a topology on X^∞

(T2): $U, V \in \mathcal{T}^\infty \Rightarrow U \cap V \in \mathcal{T}^\infty$?

Four cases:

1. $U \in \mathcal{T}, V \in \mathcal{T} \Rightarrow U \cap V \in \mathcal{T}$ ✓
2. $U \in \mathcal{T}$, $\underset{X^\infty \setminus K}{V \in \mathcal{T}(\infty)} \Rightarrow U \cap V = U \cap (X \setminus K) \in \mathcal{T}$ ✓ (cpt, X = Haud. $\Downarrow$ K = closed in X) is open in X
3. ...
4. $\underset{X^\infty \setminus K}{U \in \mathcal{T}(\infty)}, \underset{X^\infty \setminus L}{V \in \mathcal{T}(\infty)} \Rightarrow U \cap V = X^\infty \setminus (K \cup L) \in \mathcal{T}(\infty)$

$K, L \subseteq X$ cpcts $\Rightarrow K \cup L$ is compact.

(T3) $U_i \in \mathcal{T}^\infty$ $(\forall) i \in I \Rightarrow \bigcup_{i \in I} U_i \in \mathcal{T}^\infty$??

This holds if all $U_i \in \mathcal{T}$ or all $U_i \in \mathcal{T}(\infty)$ ($\cap$ of compacts is compact)

Left with: $U \in \mathcal{T}$, $\underset{\substack{X^\infty \setminus K \\ K \subseteq X \text{ cpt}}}{V \in \mathcal{T}(\infty)} \Rightarrow \underset{X^\infty \setminus K'}{U \cup V} \in \mathcal{T}^\infty$??

where $K' = K \cap \underbrace{(X \setminus U)}_{\text{closed in } X}$ ✓ closed in K $\Rightarrow$ cpt.

-4-

Claim 2: X^∞ is Hausdorff: $\begin{cases} x, y \in X^\infty, \\ x \neq y \end{cases}$ To prove: they can be separated.

If $x, y \in X$: ok since X = Hausdorff.

Left: $x \in X$, $y = \infty$.

Look for: $\begin{cases} U = \text{open neighbhd. of } x \text{ in } X \\ V = X^\infty \setminus K, \text{ with } K \subseteq X \text{ compact} \end{cases}$ s.t. $U \cap (X^\infty \setminus K) = \phi$

$x \in \underset{\text{open}}{U} \subseteq \underset{\text{cpt}}{K}$. In other words. i.e.: we are looking for a neighbhd. of x which is compact

Def: Call a space X locally compact if any point $x \in X$ admits a compact neighborhoods.

$((\forall) x \in X \ \exists \underbrace{K \subseteq X \text{ neighbd. of } x}_{\exists U \subseteq X \text{ open s.t. } x \in U \subseteq K}$ s.t. K = compact)

$(x, y) \in X \times Y$

$K \times L$

-5-

Claim 3: $(X^\infty, \mathcal{T}^\infty)$ is compact.

-6-

Claim 4: the embedding $i : X \longrightarrow X^\infty$

-7-

Claim 5: Any other 1-pt. comp. $(\tilde{X}, \tilde{i})$ is homeomorphic to X

-2-

-point compactification

compact space $\boxed{\tilde{X}}$

g $\boxed{i : X \longrightarrow \tilde{X}}$ s.t. $\tilde{X} \setminus i(X)$ has only one point.

y ?

Hausdorff space then:

fication

$______$s are homeomorphic

∞ is an element not in X.

and is open in X $\quad \mathcal{T}$

OR

with $K \subseteq X$ compact $\quad \mathcal{T}(\infty)$

-3-

$\mathcal{T}^{\infty} = \mathcal{T} \cup \mathcal{T}(\infty)$ Claim 1: $\mathcal{T}^{\infty}$ is a topology on X^{∞}

(T2): $U, V \in \mathcal{T}^{\infty} \Rightarrow U \cap V \in \mathcal{T}^{\infty}$?

Four cases: 1. $U \in \mathcal{T}, V \in \mathcal{T} \Rightarrow U \cap V \in \mathcal{T}$

2. $U \in \mathcal{T}$, $\underset{X^{\infty} \setminus K}{V \in \mathcal{T}(\infty)} \Rightarrow U \cap V = U \cap (X \setminus K) \in \mathcal{T}$

cpt, X = Haud. ⇒ K = closed in X ⇒ $X \setminus K$ is open in X

3. ...

4. $U \in \mathcal{T}(\infty)$, $V \in \mathcal{T}(\infty)$; $U = X^{\infty} \setminus K$, $V = X^{\infty} \setminus L$ $\Rightarrow U \cap V = X^{\infty} \setminus (K \cup L) \in \mathcal{T}(\infty)$

$K, L \subseteq X$ cpts ⇒ $K \cup L$ is compact.

(T3) $U_i \in \mathcal{T}^{\infty}$ (∀) $i \in I \Rightarrow \bigcup_{i \in I} U_i \in \mathcal{T}^{\infty}$??

This holds if all $U_i \in \mathcal{T}$ or all $U_i \in \mathcal{T}(\infty)$

Left with: (∩ of compacts is compact)

$U \in \mathcal{T}$, $V \in \mathcal{T}(\infty) \Rightarrow U \cup V \in \mathcal{T}^{\infty}$??

$V = X^{\infty} \setminus K$, $K \subseteq X$ cpt

$U \cup V = X^{\infty} \setminus K'$ where $K' = K \cap (X \setminus U)$, $X \setminus U$ closed in X, closed in K ⇒ cpt.

Claim 2: X^{∞} is Hausdorff: ∄x, ...

If $x, y \in X$: ok since X = Hausdorff.

Left: $x \in X$, $y = \infty$.

Look for: U = open neighbhd ... ; $V = X^{\infty} \setminus K$, with ...

$x \in \underset{\text{open}}{U} \subseteq \underset{\text{cpt}}{K}$ In other ... we a...

Def: Call a space X locall... admits a compact neighbo...

(∀) $x \in X$ ∃ $K \subseteq X$ neighb... ; ∃ $U \subseteq X$ open s...

-6-

Claim 4: the embedding $i : X \longrightarrow X^{\infty}$

Claim 5: Any other 1-pt. co...

$\mathcal{T}^\infty$ is a topology on X^∞ -3-

$\in \mathcal{T}$

$= U \cap (X \setminus K) \in \mathcal{T}$

cpt, X = Haud. $\Rightarrow$ K = closed in X

is open in X

$\cap V = X^\infty \setminus (K \cup L) \in \mathcal{T}(\infty)$

cpts $\Rightarrow K \cup L$ is compact.

??

all $U_i \in \mathcal{T}(\infty)$

of compacts is compact

; ??

$K' = K \cap \underbrace{(X \setminus U)}_{\text{closed in } X}$ → cpt.

closed in K

-4-

<u>Claim 2</u>: X^∞ is Hausdorff: $\begin{cases} x, y \in X^\infty, \\ x \neq y \end{cases}$ To prove: they can be separated.

If $x, y \in X$: ok since X = Hausdorff.

Left: $x \in X$, $y = \infty$.

Look for: $\begin{cases} U = \text{open neighbd. of } x \text{ in } X \\ V = X^\infty \setminus K, \text{ with } K \subset X \text{ compact} \end{cases}$ s.t. $U \cap (X^\infty \setminus K) = \emptyset$

$x \in \underset{\text{open}}{U} \subseteq \underset{\text{cpt}}{K}$ In other words i.e.: we are looking for a neighbd. of x which is compact

<u>Def</u>: Call a space X locally compact if any point $x \in X$ admits a compact neighborhoods.

$((\forall)\, x \in X\ \exists\ \underbrace{K \subseteq X \text{ neighbd. of } x}_{\exists U \subseteq X \text{ open s.t. } x \in U \subseteq K}$ s.t. K = compact$)$

$(x, y) \in X \times Y$

$K \times L$

-7-

<u>Claim 5</u>: Any other 1-pt. comp. $(\tilde{X}, \tilde{i})$ is homeomorphic to X^∞.

-6-

<u>Examples</u>:

$X = [0, 1)$

$X = (0, 1)$

$X = (0, \tfrac{1}{2}) \cup$

$X =$

Def X = a space. A 1-point comp[...] of X is a Hausdorff compact space $\boxed{\tilde{X}}$ together with an embedding $\boxed{i : X \to \tilde{X}}$ s.t. $\tilde{X} \setminus i(X)$ has only one point.

Q: does $\tilde{X}$ exist? How many?

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1. $\exists$ a 1-point compactification
2. any two — " —s are homeomorphic

proof: $X^\infty = X \cup \{\infty\}$ where ∞ is an element not in X.

Four cases: 1. $U \in J$, V[...]

2. $U \in J$, $V \in J(\infty) \Rightarrow$ $\overset{\shortparallel}{X^\infty \setminus K}$

3. ...

4. $\underset{X^\infty \setminus K}{U \in J(\infty)}$, $\underset{X^\infty \setminus L}{V \in J(\infty)}$ $\Big\} \Rightarrow U \cap V = X^\infty \setminus$[...]

$K, L \subseteq X$ cpcts $\Rightarrow K \cup L$ comp[...]

(K = clos[...] is ope[...])

(T3) $U_i \in J^\infty$ $(\forall) i \in I \Rightarrow \bigcup_{i \in I} U_i \in J^\infty$??

This holds if all $U_i \in J$ or all $U_i \in$[...]

Left with: [...]

-5-

Claim 3: (X^∞, J^∞) is compact

We check the def.: let $\{U_i : i \in I\}$ be an arbitrary open cover of X^∞.

There exists $i_0 \in I$ s.t. $\infty \in U_{i_0}$, hence $U_{i_0} = X^\infty \setminus K$ with $\boxed{K \subseteq X \text{ cpct.}}$

Remark that: $\{U_i : i \neq i_0\}$ – covers K

Since K = cpct $\Rightarrow$

$\Rightarrow \exists\, i_1, \ldots, i_p \in I$ s.t. $K \subseteq U_{i_1} \cup \ldots \cup U_{i_p}$

Hence $\{U_{i_0}, U_{i_1}, \ldots, U_{i_p}\}$ = cover X^∞ ☑

-6-

Claim 4: the embedding $\begin{cases} i : X \to X^\infty \\ i(x) = x \end{cases}$

To prove: $J^\infty|_X$ we get back J.

$J^\infty|_X$ consists of $U \cap X$ with $U \in J^\infty$ i.e.

- $\underbrace{U \cap X}_{=U}$ with $U \in J$ $\Rightarrow$

  or

- $U \cap X$ with $U = X^\infty \setminus K$: $K \subseteq X$ cpct.

A 1-point compactification ... $\tilde X$ ... a Hausdorff compact space $\boxed{\tilde X}$ with an embedding $\boxed{i : X \to \tilde X}$ s.t. $\tilde X \setminus i(X)$ has only one point.

... $\tilde X$ exist? How many?

If X is a locally compact Hausdorff space then
$\exists$ a 1-point compactification
any two — " — are homeomorphic

$X^\infty = X \cup \{\infty\}$ where ∞ is an element not in X.

$\mathcal{J}^\infty = \mathcal{J} \cup \mathcal{J}(\infty)$ ... on X

(T2) $U, V \in \mathcal{J}^\infty \Rightarrow U \cap V \in \mathcal{J}^\infty$?

$U, V \in \mathcal{J} \Rightarrow U \cap V \in \mathcal{J}$ ✓

Four cases 1. $U \in \mathcal{J}, V \in \mathcal{J} \Rightarrow U \cap V \in \mathcal{J}$

2. $U \in \mathcal{J}$, $V \in \mathcal{J}(\infty) \Rightarrow U \cap V = U \cap (X \setminus K) \in \mathcal{J}$ ✓ ($V = X^\infty \setminus K$; cpt, X=Haus $\Rightarrow$ K=closed in X, is open in X)

3. ...

4. $U \in \mathcal{J}(\infty)$, $V \in \mathcal{J}(\infty)$, $U = X^\infty \setminus K$, $V = X^\infty \setminus L$ $\Rightarrow U \cap V = X^\infty \setminus (K \cup L) \in \mathcal{J}(\infty)$

$K, L \subseteq X$ cpts $\Rightarrow K \cup L$ is compact.

(T3) $U_i \in \mathcal{J}^\infty$ $\forall i \in I \Rightarrow \bigcup_{i \in I} U_i \in \mathcal{J}^\infty$??

This holds if all $U_i \in \mathcal{J}$ or all $U_i \in \mathcal{J}(\infty)$

Left with: ...

Left: $x \in X$, $y = \infty$.

Look for: $\begin{cases} U = \text{open neighbhd. of } x \text{ in } X \\ V = X^\infty \setminus K, \text{ with } K \subseteq X \text{ compact} \end{cases}$ s.t. $U \cap (X^\infty \setminus K) = \phi$

i.e.: $x \in \underset{\text{open}}{U} \subseteq \underset{\text{cpt}}{K}$

In other words: we are looking for a neighbhd. of x which is compact

Def: Call a space X locally compact if any point $x \in X$ admits a compact neighborhoods.

(5)

$(X^\infty, \mathcal{J}^\infty)$ is compact

... def. let $\{U_i : i \in I\}$ be any open cover of X^∞

... $i_0 \in I$ s.t. $\infty \in U_{i_0}$, hence $U_{i_0} = X^\infty \setminus K$ with $\boxed{K \subseteq X \text{ cpt.}}$

... that $\{U_i : i \neq i_0\}$ — covers K

... cpt $\Rightarrow$

$\exists i_1, \ldots, i_p \in I$ s.t. $K \subseteq U_{i_1} \cup \ldots \cup U_{i_p}$

$\{U_{i_0}, U_{i_1}, \ldots, U_{i_p}\}$ — cover X^∞ ☒

(6)

Claim 4: the embedding $\begin{cases} i : X \to X^\infty \\ i(x) = x \end{cases}$

To prove: $\mathcal{J}^\infty|_X$ we get back $\mathcal{J}$.

consists of $U \cap X$ with $U \in \mathcal{J}^\infty$ i.e.

$\underbrace{U \cap X}_{=U}$ with $U \in \mathcal{J}$ $\Rightarrow$ get all opens of X.

or

$U \cap X$ with $U = X^\infty \setminus K$: $U \cap X = (X^\infty \setminus K) \cap X = X \setminus K$ ← open in X.

$K \subseteq X$ cpt. (K cpt. in Haus $\Rightarrow$ K = closed in X)

all together we get $\mathcal{J}$.

Claim 5: Any other 1-pt. comp. $(\tilde X, \tilde i)$ is homeomorphic to X^∞.

Def: X = a space. A 1-point compactification [-2-] of X is a Hausdorff compact space $\boxed{\tilde X}$ together with an embedding $\boxed{i : X \to \tilde X}$ s.t. $\tilde X \setminus i(X)$ has only one point.

Q: does $\tilde X$ exist? How many?

Theorem: If X is a $\boxed{\text{locally compact}}$ Hausdorff space then:

1. $\exists$ a 1-point compactification
2. any two ——"——————s are homeomorphic

proof: $X^\infty = X \cup \{\infty\}$ where ∞ is an element not in X.

Topology $\mathcal{T}^\infty$ on X^∞:

call $U \subseteq X^\infty$ open in X^∞ if
$$\begin{cases} U \subseteq X \text{ and is open in } X & \mathcal{T} \\ \text{OR} & \\ U = X^\infty \setminus K \text{ with } K \subseteq X \text{ compact} & \mathcal{T}(\infty) \end{cases}$$

$\mathcal{T}^\infty = \mathcal{T} \cup \mathcal{T}(\infty)$

Claim 1: $\mathcal{T}^\infty$ is a topology on X^∞ [-3-]

(T2): $U, V \in \mathcal{T}^\infty \Rightarrow U \cap V \in \mathcal{T}^\infty$?

Four cases
1. $U \in \mathcal{T}, V \in \mathcal{T} \Rightarrow U \cap V \in \mathcal{T}$ ✓
2. $U \in \mathcal{T}$, $\underbrace{V}_{X^\infty \setminus K} \in \mathcal{T}(\infty) \Rightarrow U \cap V = U \cap (X \setminus K) \in \mathcal{T}$ ✓ (cpt in X = Haus $\Rightarrow$ K = closed in X)
3. ...
4. $\underbrace{U}_{X^\infty \setminus K} \in \mathcal{T}(\infty)$, $\underbrace{V}_{X^\infty \setminus L} \in \mathcal{T}(\infty)$ $\Rightarrow U \cap V = X^\infty \setminus (K \cup L) \in \mathcal{T}(\infty)$ is open in X; $K, L \subseteq X$ cpts $\Rightarrow K \cup L$ is compact

(T3) $U_i \in \mathcal{T}^\infty$ $(\forall) i \in I \Rightarrow \bigcup_{i \in I} U_i \in \mathcal{T}^\infty$??

This holds if all $U_i \in \mathcal{T}$ or all $U_i \in \mathcal{T}(\infty)$ ($\cap$ of compacts is compact)

Left with: $U \in \mathcal{T}$, $\underbrace{V}_{X^\infty \setminus K,\ K \subseteq X \text{ cpt}} \in \mathcal{T}(\infty) \Rightarrow \underbrace{U \cup V}_{X^\infty \setminus K'} \in \mathcal{T}^\infty$??

where $K' = K \cap (X \setminus U)$ closed in X, closed in K $\Rightarrow$ cpt ✓

Claim2: X^∞ is Hausdorff: $(\forall)\, x, y \in X^\infty$, $x \neq y$. To prove: they can be separated. [-4-]

If $x, y \in X$: ok since X = Hausdorff.

Left: $x \in X$, $y = \infty$.

Look for: $\begin{cases} U = \text{open neighbhd of } x \text{ in } X \\ V = X^\infty \setminus K, \text{ with } K \subseteq X \text{ compact} \end{cases}$ s.t. $U \cap (X^\infty \setminus K) = \emptyset$

$x \in \underset{\text{open}}{U} \subseteq \underset{\text{cpt}}{K}$. In other words, i.e.: we are looking for a neighbhd of x which is compact.

Def: Call a space X locally compact if any point $x \in X$ admits a compact neighborhoods.

$(\forall)\, x \in X\ \exists\ \underbrace{K \subseteq X \text{ neighbhd of } x}_{\exists U \subseteq X \text{ open s.t. } x \in U \subseteq K}$ s.t. K = compact

$(x, y) \in X \times Y$, $K \times L$ ☒

Claim3: $(X^\infty, \mathcal{T}^\infty)$ is compact. [-5-]

We check the def.: let $\{U_i : i \in I\}$ be an arbitrary open cover of X^∞.

There exists $i_0 \in I$ s.t. $\infty \in U_{i_0}$, hence $U_{i_0} = X^\infty \setminus K$ with $\boxed{K \subseteq X \text{ cpt}}$.

Remark that: $\{U_i : i \neq i_0\}$ – covers K

Since K = cpt $\Rightarrow$

$\Rightarrow \exists\, i_1, \dots, i_p \in I$ s.t. $K \subseteq U_{i_1} \cup \dots \cup U_{i_p}$

Hence $\{U_{i_0}, U_{i_1}, \dots, U_{i_p}\}$ – cover X^∞ ☒

Claim4: the embedding $i : X \to X^\infty$, $i(x) = x$. [-6-]

To prove: $\mathcal{T}^\infty|_X$ we get back $\mathcal{T}$.

consists of $U \cap X$ with $U \in \mathcal{T}^\infty$ i.e.
$$\begin{cases} \underbrace{U \cap X}_{=U} \text{ with } U \in \mathcal{T} \Rightarrow \text{get all opens of } X. \\ \text{or} \\ U \cap X \text{ with } U = X^\infty \setminus K,\ K \subseteq X \text{ cpt}: U \cap X = (X^\infty \setminus K) \cap X = X \setminus K \leftarrow \text{open in } X. \end{cases}$$

(cpt in Haus $\Rightarrow$ K = closed in X)

all together we get $\mathcal{T}$.

Claim5: Any other 1-pt. comp. $(\tilde X, \tilde i)$ is homeomorphic to X^∞.

$\in X^\infty$. To prove they can be separated.

x in X s.t. ...

K compact $U \cap (X^\infty \setminus K) = \phi$

... i.e.:

... looking for a neighbhd. of x which is compact

... compact if any point $x \in X$ ...

... homeomorphic to ...

... all tog... we get ... $\mathcal{T}$ ...

... in X.

— 7 —

Claim 5: Let $(\tilde X, \tilde i)$ be another one as in the def.

$$\begin{tikzcd} & X^\infty \\ X \arrow[ur, "i"] \arrow[dr, "\tilde i"'] & \\ & \tilde X \arrow[uu, dashed, "f"'] \end{tikzcd}$$

$\tilde X \setminus \tilde i(X) = \{p_\infty\}$

Define $f : \tilde X \to X^\infty$
$y \in \tilde X$

$$f(y) = \begin{cases} x & \text{if } y = \tilde i(x) \\ \infty & \text{if } y = p_\infty \end{cases}$$

compact   ALSO BIJECTIVE   Hausdorff

Hence: suffices to check: f = continuous: $U \in \mathcal{T}^\infty \Rightarrow f^{-1}(U)$ = open in $\tilde X$

two cases . . . .

$\boxed{\ }$

2. any two — " — s are homeomorphic

proof: $X^\infty = X \cup \{\infty\}$ where ∞ is an element not in X.

Topology $\mathcal{T}^\infty$ on X^∞:

call $U \subseteq X^\infty$ open in X^∞ if $\begin{cases} U \subseteq X \text{ and is open in } X & \mathcal{T} \\ \text{OR} & \\ U = X^\infty \setminus K \text{ with } K \subseteq X \text{ compact} & \mathcal{T}(\infty) \end{cases}$

This holds if all (

Left with:

$U \in \mathcal{T},\ V \in \mathcal{T}(\infty) \Rightarrow$

$X^\infty \setminus K$, $K \subseteq X$ cpt

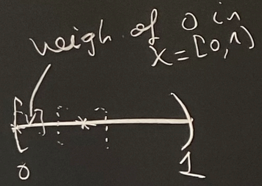
X

On local compactness

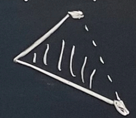
$\underbrace{\text{local compactness}}_{\text{l.c.}}$

① X = compact $\Rightarrow$ X = l.c.

② any interval $\subseteq \mathbb{R}$ is l.c.



③ any open in $\mathbb{R}^n$ & any closed in $\mathbb{R}^n$ are l.c.

But  is not l.c. (Exercise).

④ $X = \mathbb{Q}$ with the topology induced from $\mathbb{R}$. It is not l.c.!

Look around $0 \in X$. If N = neighbd. of 0 in $\mathbb{Q}$ $\Rightarrow$

$\Rightarrow \exists\, \varepsilon > 0$ s.t. $\underbrace{(-\varepsilon,\varepsilon)\cap\mathbb{Q}} \subseteq N \subseteq \mathbb{Q}$. Take closures inside $\mathbb{R}$ $\Rightarrow$

$\Rightarrow \overline{(-\varepsilon,\varepsilon)\cap\mathbb{Q}} \subseteq \overline{N} \Rightarrow [-\varepsilon,\varepsilon] \subseteq \overline{N}$. Now: if N was compact (inside the Hausdor

⑤ X = l.

$a \in A$

⑥ X = l.c.

Rk: this impl

$U \in J$, $V \in J(\infty) \Rightarrow U \cup V \in J^{\infty}$?? ($\cap$ of compacts is compact)

$X^{\infty} \setminus K$, $K \subseteq X$ cpct

$X^{\infty} \setminus K'$ where $K' = K \cap (X \setminus U)$ closed in X, closed in K $\Rightarrow$ cpct.

admits a compact neighborhoods. ... any point $x \in X$

$(\forall)\, x \in X\ \exists\ \underbrace{K \subseteq X \text{ neighbd. of } x}_{\exists U \subseteq X \text{ open s.t. } x \in U \subseteq K}$ s.t. K = compact

$(x,y) \in X \times Y$

$K \times L$

⑤ X = l.c., A - closed in $X \Rightarrow$ also A is l.c. $\boxed{-9-}$

a — neigh. of a in A which is compact?

$a \in A$ } $\Rightarrow a \in K \subseteq X \Rightarrow a \in (A \cap K) \subseteq A$

K: neighbd. of a in X, compact

$A \cap K$: a neigh. of a in A; compact?

A = closed in $X \Rightarrow A \cap K$ is closed in K (cpct) $\Rightarrow A \cap K$ = cpct $\boxed{}$

⑥ X = l.c. & Hausdorff, U - open in $X \Rightarrow$ also U is l.c.

Rk. this implies that, if X admits a 1-pt. compactification, it MUST BE L.C.!

$\boxed{}$

... is not l.c.!

$\mathbb{R} \Rightarrow$

... as compact (inside the Hausdorff $\mathbb{R}$) $\Rightarrow N$ is closed in $\mathbb{R} \Rightarrow \bar{N} = N \Rightarrow [-\varepsilon, \varepsilon] \subseteq N$ which is impossible since $N \subseteq X = \mathbb{Q}$.

Def: X = a space. A 1-point compactification [2] of X is a Hausdorff compact space $\boxed{\tilde{X}}$ together with an embedding $\boxed{i : X \to \tilde{X}}$ s.t. $\tilde{X} \setminus i(X)$ has only one point.

Q: does $\tilde{X}$ exist? How many?

Theorem: If X is a $\boxed{\text{locally compact}}$ Hausdorff space then:

1. $\exists$ a 1-point compactification
2. any two — " — 's are homeomorphic

proof: $X^\infty = X \cup \{\infty\}$ where ∞ is an element not in X.

Topology $\mathcal{T}^\infty$ on X^∞:

call $U \subseteq X^\infty$ open in X^∞ if
- $U \subseteq X$ and is open in X ($\mathcal{T}$)
- OR
- $U = X^\infty \setminus K$ with $K \subseteq X$ compact ($\mathcal{T}(\infty)$)

$\mathcal{T}^\infty = \mathcal{T} \cup \mathcal{T}(\infty)$

Claim 1: $\mathcal{T}^\infty$ is a topology on X^∞ [3]

(T2): $U, V \in \mathcal{T}^\infty \Rightarrow U \cap V \in \mathcal{T}^\infty$?

Four cases
1. $U \in \mathcal{T}, V \in \mathcal{T} \Rightarrow U \cap V \in \mathcal{T}$ ✓
2. $U \in \mathcal{T}$, $V \in \mathcal{T}(\infty) \Rightarrow U \cap V = U \cap (X \setminus K) \in \mathcal{T}$ ✓ ($V = X^\infty \setminus K$; K cpt, X = Haus $\Rightarrow$ K = closed in X; $X \setminus K$ is open in X)
3. . . .
4. $U \in \mathcal{T}(\infty)$, $V \in \mathcal{T}(\infty)$ ($U = X^\infty \setminus K$, $V = X^\infty \setminus L$) $\Rightarrow U \cap V = X^\infty \setminus (K \cup L) \in \boxed{\mathcal{T}(\infty)}$; $K, L \subseteq X$ cpcts $\Rightarrow K \cup L$ is compact

(T3) $U_i \in \mathcal{T}^\infty$ $(\forall)\, i \in I \Rightarrow \bigcup_{i \in I} U_i \in \mathcal{T}^\infty$??

This holds if all $U_i \in \mathcal{T}$ or all $U_i \in \mathcal{T}(\infty)$ ($\cap$ of compacts is compact)

Left with:

$U \in \mathcal{T}$, $V \in \mathcal{T}(\infty) \Rightarrow U \cup V \in \mathcal{T}^\infty$??

$V = X^\infty \setminus K$, $K \subseteq X$ cpt; $U \cup V = X^\infty \setminus K'$ where $K' = K \cap (X \setminus U)$ ✓ ($X \setminus U$ closed in X; K' closed in K $\Rightarrow$ cpt)

Claim 2: X^∞ is Hausdorff: $\begin{cases} x, y \in X^\infty \\ x \neq y \end{cases}$ To prove they can be separated. [4]

If $x, y \in X$: ok since X = Hausdorff

Left: $x \in X$, $y = \infty$.

Look for: $\begin{cases} U = \text{open neighbhd of } x \text{ in } X \\ V = X^\infty \setminus K, \text{ with } K \subseteq X \text{ compact} \end{cases}$ s.t. $U \cap (X^\infty \setminus K) = \phi$

$x \in U \subseteq K$ (U open, K cpt). In other words i.e. we are looking for a neighbhd of x which is compact

Def: Call a space X locally compact if any point $x \in X$ admits a compact neighborhoods.

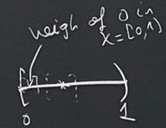
$\left((\forall)\, x \in X\ \exists \underbrace{K \subseteq X \text{ neighbd of } x}_{\exists U \subseteq X \text{ open s.t. } x \in U \subseteq K} \text{ s.t. } K = \text{compact} \right)$

$(x, y) \in X \times Y$, $K \times L$

On $\underbrace{\text{local compactness}}_{\text{l.c.}}$ [8]

① X = compact $\Rightarrow$ X = l.c.

② any interval $\subseteq \mathbb{R}$ is l.c.



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$\Rightarrow \overline{(-\varepsilon, \varepsilon) \cap \mathbb{Q}} \subseteq \overline{N}$

Now: if N was compact (inside the Hausdorff $\mathbb{R}$) $\Rightarrow$ N is closed in $\mathbb{R}$ $\Rightarrow \overline{N} = N \Rightarrow [-\varepsilon, \varepsilon] \subseteq N$ which is impossible since $N \subseteq X = \mathbb{Q}$.

⑤ X = l.c., A - closed in X $\Rightarrow$ also A is l.c. [9]

a: neigh. of a in A which is compact?

pf: $a \in A$ $\Rightarrow a \in K \subseteq X$ (K: neighbd of a in X, compact) $\Rightarrow a \in A \cap K \subseteq A$ ($A \cap K$: a neigh. of a in A; compact?)

A = closed in $X \Rightarrow A \cap K$ is closed in K (cpt) $\Rightarrow A \cap K$ = cpt ∎

⑥ X = l.c. & Hausdorff, U - open in X $\Rightarrow$ also U is l.c.

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