Vector space $(V, +, \cdot)$ over $\mathbb{K}$ $\boxed{-1-}$

$\longrightarrow$ operation $+ : V \times V \longrightarrow V$, $(v_1, v_2) \longmapsto v_1 + v_2$
making $(V, +)$ into an abelian group ($0 \in V$)

$\longrightarrow$ multiplication by scalars $\cdot : \mathbb{K} \times V \longrightarrow V$, $(\lambda, v) \longmapsto \lambda \cdot v$
satisfying distributivity

$$(\lambda_1 + \lambda_2) \cdot v = \lambda_1 \cdot v + \lambda_2 \cdot v \quad (\forall)\ \lambda_1, \lambda_2 \in \mathbb{K},\ v \in V$$

$$\lambda \cdot (v_1 + v_2) = \lambda \cdot v_1 + \lambda \cdot v_2$$

and

$$\lambda_1 \cdot (\lambda_2 \cdot v) = (\lambda_1 \lambda_2) \cdot v$$

Ex: $C(X)$ = a vector space

$\boxed{-2-}$

Ring $(R, +, \cdot)$

$\rightarrow$ operation $+ : R \times R \longrightarrow R$ making $(R, +)$ abelian group
(neutral element denoted: 0)

$\rightarrow$ multiplication on R, $\cdot : R \times R \longrightarrow R$ which is

- associative: $(r_1 \cdot r_2) \cdot r_3 = r_1 \cdot (r_2 \cdot r_3) \quad (\forall)\ r_1, r_2, r_3 \in R$
- distributive w.r.t. $+$: $r_1 \cdot (r_2 + r_3) = r_1 \cdot r_2 + r_1 \cdot r_3$
  $(r_2 + r_3) \cdot r_1 = r_2 \cdot r_1 + r_3 \cdot r_1$
- unital: $\exists\ 1 \in R$ s.t. $1 \cdot r = r \cdot 1 = r \quad (\forall)\ r \in R$

-3- The algebra $C(X)$ of observables

X = Hausdorff compact space.

$\boxed{C(X) := \{ f : X \to \mathbb{R} \,/\, f = \text{continuous} \}}$ (of observables / measurements on X)

generically $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and then use notation $C(X, \mathbb{K})$.

$C(X)$: algebraic structure:

① addition: $f+g$ where $(f+g)(x) = f(x) + g(x)$: $f+g \in C(X)$ whenever $f, g \in C(X)$.

② multiplication by scalars $\lambda \in \mathbb{R}$. $\lambda \cdot f$: $(\lambda \cdot f)(x) := \lambda f(x)$

③ multiplication $(f \cdot g)(x) := f(x) \cdot g(x)$

Def: An algebra over $\mathbb{K}$ is a ring $(A, +, \cdot)$ and a vector space $(A, +, \cdot)$ (same A, same $+$!) such that:

$$\lambda \cdot (a \cdot b) = (\lambda \cdot a) \cdot b = a \cdot (\lambda \cdot b) \quad (\forall) \lambda \in \mathbb{K}, a, b \in A.$$

-4-

Conclusion: $\boxed{C(X) = \text{an algebra over } \mathbb{R}}$

($C(X, \mathbb{K})$ = an algebra over $\mathbb{K}$).

Other examples of algebras: $\mathbb{R}$, $\mathbb{R}[X_1, \dots, X_n]$, $\mathcal{M}_{n \times n}(\mathbb{R})$.

Thm (1st version of Gelfand-Naimark): the top. space X can be recovered from the algebra $C(X)$. (see Thm 8.22 to be discussed later)

Separability (TITEL): given X = top. space, assume we have

$$\boxed{A, B \subseteq X \text{ closed}, \; A \cap B = \emptyset} \quad (*)$$

Def (2.51): ① say A and B can be separated topologically if $\exists$ $U_A, U_B \subseteq X$ opens s.t. $\begin{cases} A \subseteq U_A \\ B \subseteq U_B \end{cases}$ and $U_A \cap U_B = \emptyset$

② say A and B can be separated by continuous fcts. if $\exists f : X \to [0,1]$ continuous s.t. $f|_A = 0$, $f|_B = 1$.

Rk: ② ⇒ ① : if $\exists f$: can take $U_A := f^{-1}(-\infty, \frac{1}{2})$, $U_B = f^{-1}(\frac{1}{2}, \infty)$

open because f = continuous.

... n ($\mathbb{R}$).

... can be

... 8.22 to be discussed later

... (*)

...logically:

and $U_A \cap U_B = \emptyset$

...tinuous fcts.

...o, $f|_B = 1$.

... $f^{-1}(\frac{1}{2}, \infty)$.

...us.

[5-]

Rk: An implication ⇒ ② would give quite a lot: would give functions! Producing functions is, in general hard!

& This may have consequences on the metrizability problem: when is a space X metrizable?

Def 2.56: A space X is called **normal** if ∀ A, B as in (*) can be separated topologically.

Ex: Compact Hausdorff spaces are normal

Thm (Urysohn Lemma) (5.21): If X = normal & Hausdorff then any A and B as in (*) can be separated by continuous functions.

Using this ... one can then prove:

Thm (Urysohn metrizability thm): If X = normal, Hausdorff and 2^{nd} countable, then X = metrizable (i.e. ∃ metric d on X inducing the topology of X).

[6-7]

...rvables

Metric space (X,d):

$$d : X \times X \longrightarrow \mathbb{R}$$

& axioms (see 1^{st} lecture)

Called complete if $\boxed{\forall \text{ Cauchy sequence } (x_n)_{n\geq 1} \text{ in } (X,d)}$ is convergent.

i.e. $(\forall)\, \varepsilon > 0\ \exists\ n_\varepsilon \in \mathbb{N}$ s.t.

$d(x_n, x_m) < \varepsilon\ (\forall)\, n,m \geq n_\varepsilon$.

Normed space $(V, \|\cdot\|)$: V is vector space and

$$\|\ \| : V \longrightarrow \mathbb{R}_{\geq 0} \quad \text{(norm)}$$

satisfying:

$$\begin{cases} \|v+w\| \leq \|v\| + \|w\| \\ \|\lambda v\| = |\lambda| \cdot \|v\| \quad (\forall)\, \lambda \in \mathbb{K},\ v \in V \\ \|v\| = 0 \text{ only for } v = 0. \end{cases}$$

From $\|\cdot\| \Longrightarrow$ a metric $d_{\|\cdot\|}$ where $d_{\|\cdot\|}(v,w) := \|v-w\|$.

Banach space: $(V, \|\cdot\|)$ which is complete (w.r.t. $d_{\|\cdot\|}$)

Rk: An implication ⇒ ② would give quite a lot: would give functions! Producing functions is, in general hard!

5-1

& This may have consequences on the metrizability problem: when is a space X metrizable?

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6-7

Some topological structure present on $C(X)$: metric d_{sup} on $C(X)$:

$$d_{sup}(f,g) = \sup\{ \underbrace{d_{\mathbb{R}}(f(x), g(x))}_{|f(x)-g(x)|} : x \in X \}$$

(exercise: show that $d_{sup}(f,g) < \infty$ $\forall f, g$)

Hint: X = cpct

Actually: a norm $\|\cdot\|_{sup}$ on $C(X)$:

$$\|f\|_{sup} = \sup\{ |f(x)| : x \in X \}$$ and $d_{sup} = d_{\|\cdot\|_{sup}}$.

Conclusion: $C(X)$ = Banach space.

Def: A Banach algebra over $\mathbb{K}$ is an algebra A over $\mathbb{R}$ which also has a norm $\|\cdot\|$ s.t.

$$\|a \cdot b\| \le \|a\| \cdot \|b\| \quad \forall a, b \in A$$

& require completeness.

Final conclusion: $\boxed{C(X) = \text{a Banach algebra}}$

Stone-Weierstrass

Classical Weirstrass: Any continuous function $f:[0,1]\to\mathbb{R}$ can be approximated uniformly by polynomial f-ts / $\exists\, p_n:[0,1]\to\mathbb{R}$ pol. such that $p_n \rightrightarrows f$.

or: $C_{pol}([0,1]) \underset{\text{densly}}{\subseteq} (C([0,1]), d_{sup})$

$C_{pol}([0,1])$ is dense in $C([0,1])$

A is dense in X ← top. space

$\cap$

X means: $\bar{A}=X$.

What if $[0,1]$ is replaced by a compact Hausdorff X?

$$\mathcal{P} \subseteq C(X)$$

← with d_{sup}

give it as input in theorem.

Q: what conditions to impose on $\mathcal{P}$ s.t.: they imply that $\mathcal{P}$ is dense in $C(X)$ (and obtein W-thm as immediate consequence)

Condition 1: $\mathcal{P}$ is a subalgebra of $C(X)$, i.e.: a vector subspace and: $f\cdot g\in\mathcal{P}$ $(\forall f,g\in\mathcal{P})$, and $1\in\mathcal{P}$.

Condition 2: $\mathcal{P}$ is point separating: i.e.: $\forall\, x,y\in X$, $x\neq y$ $\exists f\in\mathcal{P}$ s.t. $f(x)\neq f(y)$.

λf

$f+\lambda$

Theorem (S-W): If $\mathcal{P}$ satisfies C1, C2 $\Rightarrow$ $\mathcal{P}$ is dense in $C(X)$.

For $X=[0,1]$: produce a seq. of polynomials

$p_n : [0,1] \to \mathbb{R}$ s.t. $p_n \rightrightarrows$ the squareroot fct. $\sqrt{t} \in C([0,1])$.

Define p_n:

$$\begin{cases} p_1 = 0, & p_{n+1}(t) = p_n(t) + \frac{1}{2}\left(t - p_n(t)^2\right) \quad (**) \\ p_2(t) = \frac{1}{2}t \\ p_3(t) = t - \frac{1}{8}t^3 \\ \ldots \end{cases}$$

Claim: $p_n \rightrightarrows \sqrt{t}$

Notice: when fixing $t \in [0,1]$ if the sequence $(p_n(t))_{n \geq 1}$ is convergent, passing to limit in $(**)$ $\Rightarrow$

$\lim = \lim + \frac{1}{2}(t - \lim^2) \Rightarrow \lim$ must be $\sqrt{t}$

EXERCISE