Reminder: X = compact & Hausdorff

$\boxed{C(X) := \text{the set of all continuous functions } f : X \to \mathbb{R}}$

Algebraic operations on $C(X)$: for $f, g \in C(X)$

- multiplications λf by scalars $\lambda \in \mathbb{R}$
- sum of functions $f+g \in C(X)$
- products —"— $f \cdot g \in C(X)$

$1 \in C(X)$

vector space structure

ring structure

algebra structure on $C(X)$

Topological structure:

- metric d_{sup}: $d_{sup}(f,g) = \sup\{|f(x)-g(x)| : x \in X\} \Rightarrow C(X)$ metric space $\Rightarrow$ the uniform topology on $C(X)$
- norm $\|\cdot\|_{sup}$

$d_{sup}(f,g) = \|f-g\|_{sup}$

$\Rightarrow C(X) =$ Banach space

... all together: $\boxed{C(X) = \text{Banach algebra}}$

Stone–Weierstrass thm: Assume $\mathcal{P} \subseteq C(X)$ is such that:

$|f| = \sqrt{f^2}$

(C1) $\mathcal{P}$ is a subalgebra of $C(X)$, i.e. – it is a vector subspace of $C(X)$
– it is a subring

(C2) $\mathcal{P}$ is point separating, i.e., $(\forall)\, x, y \in X, x \neq y\ \exists f \in \mathcal{P}$ with $f(x) \neq f(y)$.

Then $\boxed{\mathcal{P} \text{ is dense in } C(X)}$ w.r.t. the uniform topology on $C(X)$, or,

equivalently: $\boxed{(\forall)\, h \in C(X), (\forall)\, \varepsilon > 0\ \ \exists f \in \mathcal{P} \text{ s.t. } d_{sup}(f, h) < \varepsilon}$

Pf: Claim 1: $(\forall)\, x, y \in X, x \neq y, (\forall)\, a, b \in \mathbb{R}, \exists f \in \mathcal{P}$ s.t. $f(x) = a, f(y) = b$.

pf of Claim 1: Fix $\xrightarrow{(C2)}$ find $g \in \mathcal{P}$ s.t. $g(x) \neq g(y)$

$\Rightarrow \underbrace{\alpha + \beta g}_{f} = \alpha \cdot \underset{\mathcal{P}}{1} + \beta \cdot \underset{\mathcal{P}}{g} \in \mathcal{P}$ $\quad (\forall)\, \alpha, \beta \in \mathbb{R}$

Notice $\exists\, \alpha, \beta$ s.t. $\begin{cases} \alpha + \beta g(x) = a \\ \alpha + \beta g(y) = b \end{cases}$. $\left(\beta = \frac{a-b}{g(x) - g(y)},\ \alpha = \ldots \right)$

Start the proof: Let $h \in C(X)$. Let $\varepsilon > 0$. Look for $f \in \mathcal{P}$. $\underbrace{\overbrace{-\varepsilon < f(x) - h(x)}^{\text{Step 1}} < \varepsilon}_{\text{Step 2}}$ $(\forall)\, x$

Step 1:

Claim 2: $f \in \mathcal{P} \Rightarrow |f| \in \overline{\mathcal{P}}$ (closure of $\mathcal{P}$ in $C(X)$ endowed with d_{sup}) (= the set of functions in $C(X)$ that are limits of sequences $(f_n)_{n\geq 1}$ with $f_n \in \mathcal{P}$).

pf: $\left.\begin{array}{l} f \in C(X) \\ X = \text{cpct} \end{array}\right\} \Rightarrow f$ is bounded

$\Rightarrow \exists M > 0$ s.t. $|f| \leq M$. After rescaling $f \Rightarrow$ may assume $|f| \leq 1$.

($|f| : X \to [0,1]$).

Recall: $\exists\, p_n : [0,1] \to \mathbb{R}$ st $p_n \rightrightarrows \sqrt{t}$

Take now $f_n = \underbrace{p_n(f^2)}_{\text{a polynomial expression in } f} \xrightarrow[\mathcal{P} = \text{subalgebra}]{\text{since}} \boxed{f_n \in \mathcal{P}}$

Moreover: $f_n \rightrightarrows \sqrt{f^2} = |f|$. $\boxed{f_n \rightrightarrows |f|}$

$\Rightarrow |f| \in \overline{\mathcal{P}}$.

Claim 3: $f, g \in \mathcal{P} \Rightarrow \sup\{f,g\}, \inf\{f,g\} \in \overline{\mathcal{P}}$

(where $\sup\{f,g\} : X \to \mathbb{R}$, $\sup\{f,g\}(x) = \max\{f(x), g(x)\}$)

pf: Recall/notice: $\forall a, b \in \mathbb{R}$:

$$\max\{a,b\} = \frac{a+b+|a-b|}{2}, \quad \min\{a,b\} = \frac{a+b-|a-b|}{2}$$

$$\Rightarrow \sup\{f,g\} = \frac{f+g+|f-g|}{2} \in \overline{\mathcal{P}}.$$

Examples:

$\boxed{-0-}$

$U \cup V = \boxed{U \cup (X^\infty \setminus K) = X^\infty \setminus K' \text{ where } K' = K \cap (X \setminus U).}$

$X = [0,1)$

$\tilde{X} = [0,1]$

$\iota = \text{inclusion} : [0,1) \longrightarrow [0,1]$

$\boxed{i(x) = x}$

$$p_n = a_0 + a_1 X + a_2 X^2 + \dots + a_n X^n \in \mathbb{R}[X]$$

As a function : $[0,1] \longrightarrow \mathbb{R}$

$t \longmapsto p_n(t) = a_0 + a_1 t + a_2 t^2 + \dots + a_n t^n \in \mathbb{R}$

For $g : X \to [0,1]$ we have

$\longrightarrow p(g) = a_0 + a_1 \cdot g + a_2 \cdot g^2 + \dots + a_n \cdot g^n \in C(X)$

$(p(g)(t) = p(g(t)))$.

$g \in \mathcal{P} \Rightarrow g^2 = g \cdot g \in \mathcal{P} \Rightarrow g^3 = g \cdot g^2 \in \mathcal{P} \Rightarrow \dots \Rightarrow g^k \in \mathcal{P}$

$a_k g^k \in \mathcal{P}$

$a_1 g \in \mathcal{P}$ $\quad a_2 g^2 \in \mathcal{P}$

$p(g) \in \mathcal{P}$

Stone-Weierstrass thm: Assume $\mathcal{P} \subseteq C(X)$ is such that: $\left(|f| = \sqrt{f^2} \right)$

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Then $\boxed{\mathcal{P} \text{ is dense in } C(X)}$ w.r.t. the uniform topology on $C(X)$, or, equivalently: $\boxed{(\forall)\, h \in C(X),\ (\forall)\, \varepsilon > 0\ \exists f \in \mathcal{P} \text{ s.t. } d_{sup}(f, h) < \varepsilon}$

Pf: **Claim 1**: $(\forall)\, x, y \in X,\ x \neq y,\ (\forall)\, a, b \in \mathbb{R},\ \exists f \in \mathcal{P}$ s.t. $f(x) = a,\ f(y) = b$

pf of Claim 1: Fix $\xrightarrow{(C2)}$ find $g \in \mathcal{P}$ s.t. $g(x) \neq g(y)$

$\Rightarrow \underbrace{\alpha + \beta g}_{f} = \alpha \cdot \underset{\mathcal{P}}{1} + \beta \underset{\mathcal{P}}{g} \in \mathcal{P}$ $\quad (\forall)\, \alpha, \beta \in \mathbb{R}$

Notice $\exists\, \alpha, \beta$ s.t. $\begin{cases} \alpha + \beta g(x) = a \\ \alpha + \beta g(y) = b \end{cases}$ $\left(\beta = \frac{a-b}{g(x) - g(y)},\ \alpha = \dots \right)$

Claim 2: $f \in \mathcal{P} \Rightarrow |f| \in \overline{\mathcal{P}}$ (closure of $\mathcal{P}$ in $C(X)$ endowed with d_{sup}) (= the set of functions in $C(X)$ that are limits of sequences $(f_n)_{n \geq 1}$ with $f_n \in \mathcal{P}$)

Pf: $\left. \begin{array}{l} f \in C(X) \\ X = \text{cpct} \end{array} \right\} \Rightarrow f$ is bounded

$\Rightarrow \exists M > 0$ s.t. $|f| \leq M$. After rescaling $f \Rightarrow$ may assume $|f| \leq 1$. $(|f| : X \to [0,1])$

Recall $\exists\, p_n : [0,1] \to \mathbb{R}$ st $p_n \rightrightarrows \sqrt{t}$

Take now $f_n = \underbrace{p_n(f^2)}_{\text{a polynomial expression in } f} \xrightarrow[\mathcal{P} = \text{subalgebra}]{\text{since}} \boxed{f_n \in \mathcal{P}}$ $\Big\}\ |f| \in \overline{\mathcal{P}}$

Moreover: $f_n \rightrightarrows \sqrt{f^2} = |f|$ $\quad \boxed{f_n \rightrightarrows |f|}$

Claim 3: $f, g \in \mathcal{P} \Rightarrow \sup\{f, g\},\ \inf\{f, g\} \in \overline{\mathcal{P}}$ $\quad$ (where $\sup\{f,g\} : X \to \mathbb{R}$, $\sup\{f,g\}(x) = \max\{f(x), g(x)\}$)

Pf: Recall/notice $\forall a, b \in \mathbb{R}$:

$\max\{a, b\} = \frac{a + b + |a-b|}{2}$, $\min\{a, b\} = \frac{a + b - |a-b|}{2}$ $\quad \Rightarrow \sup\{f, g\} = \frac{f + g + |f-g|}{2} \in \overline{\mathcal{P}}$

... t the proof: Let $h \in C(X)$. Let $\varepsilon > 0$. Look for $f \in \mathcal{P}$. $\underbrace{\overbrace{-\varepsilon < f(x) - h(x)}^{\text{Step 1}} < \varepsilon}_{\text{Step 2}}$ $(\forall)\, x$

Proof Claim 1: Fix (C) $\Longrightarrow$ find $g \in \mathcal{P}$ s.t. $g(x) \neq g(y)$

$\Rightarrow \alpha + \beta g = \alpha \cdot \underset{\in \mathcal{P}}{1} + \beta \cdot \underset{\in \mathcal{P}}{g} \in \mathcal{P}$ $\quad (\forall)\ \alpha, \beta \in \mathbb{R}$

f //

Notice $\exists\ \alpha, \beta$ s.t. $\begin{cases} \alpha + \beta g(x) = a \\ \alpha + \beta g(y) = b \end{cases}$. $\left(\begin{array}{l} \beta = \frac{a-b}{g(x)-g(y)} \\ \alpha = \dots \end{array}\right)$

Claim 3:

Pf: Recall/

$\max\{a,b\} =$

Start the proof: Let $\boxed{h \in C(X)}$ Let $\varepsilon > 0$. Look for $f \in \mathcal{P}$. $\underbrace{\overbrace{-\varepsilon < f(x) - h(x)}^{[-3]} \overbrace{< \varepsilon}^{\text{Step 1}} \ (\forall)\, x}_{\text{Step 2}}$

Step 1: Fix $x_0 \in X$ (temporarily). Aim: to build $f_{x_0} \in \overline{\mathcal{P}}$ s.t.

$$\boxed{f_{x_0}(x_0) = h(x_0), \quad f_{x_0}(y) - h(y) < \varepsilon \quad (\forall)\ y \in X}$$

Draw a picture

Pf: For each $y \in X$, use claim 1 $\Longrightarrow$ consider $f_{x_0,y} \in \mathcal{P}$ s.t.

$$f_{x_0,y}(x_0) = \underbrace{h(x_0)}_{a}, \quad \underbrace{f_{x_0,y}(y) = \underbrace{h(y)}_{b}}$$

we have $f_{x_0,y} - h =$ continuous, vanishes at $y \Longrightarrow$

$\Rightarrow f_{x_0,y} - h < \varepsilon$ in a neighbd. V_y of y.

$\left(\begin{array}{l} \exists\ V_y \subseteq X \text{ open s.t.} \\ f_{x_0,y}(y') - h(y') < \varepsilon \quad (\forall)\ y' \in V_y \end{array}\right)$

Now: open cover $\{V_y : y \in X\}$ of X, $X =$ cpct $\Longrightarrow$

$\Longrightarrow \exists$ finite subcover: $X = V_{y_1} \cup \dots \cup V_{y_k}$ for some $y_1, \dots, y_k \in X$

Set. $f_{x_0} = \inf\{f_{x_0,y_1}, \dots, f_{x_0,y_k}\}$. On each V_{y_i}:

$f_{x_0} \leq f_{x_0,y_i} \leq h + \varepsilon$ $\Big\} \Longrightarrow f_{x_0} \leq h + \varepsilon$ on X.

Step 2:

s.t.

Now:

Now: {

$\Rightarrow X =$

Set:

f

$f > h$

Recall/notice: $\forall a,b\in\mathbb{R}$:

$\max\{a,b\} = \frac{a+b+|a-b|}{2}$, $\min\{a,b\} = \frac{a+b-|a-b|}{2}$ $\Rightarrow \sup\{f,g\} = \frac{f+g+|f-g|}{2} \in \overline{\mathcal{P}}$.

$\sup\{f,g\}: X \to \mathbb{R}$

$\sup\{f,g\}(x) = \max\{f(x), g(x)\}$

-4-

Step 2: Take care of the other inequality. Apply Step 1 $\boxed{\text{for each } x \in X}$ $\Rightarrow f_x \in \overline{\mathcal{P}}$ s.t.

s.t. $f_x(x) = h(x)$, $f_x - h < \varepsilon$ on X

Now: $f_x - h = 0$ at $x \in X$, $\Rightarrow \exists\, W_x \subseteq X$ open s.t. $f_x - h > -\varepsilon$ on W_x

$W_x \ni x$

Now: $\{W_x : x \in X\}$ - open cover of X $\Rightarrow$ can write

$\Rightarrow X = W_{x_1} \cup \ldots \cup W_{x_p}$ for some $x_1, \ldots, x_p \in X$

Set: $f = \sup\{f_{x_1}, \ldots, f_{x_p}\}$ ... this is again in $\overline{\mathcal{P}}$.

$f \in \overline{\mathcal{P}}$

On each W_{x_i}: $f \geq f_{x_i} > h - \varepsilon$.

$f > h - \varepsilon$ on X. The inequalities $f_x < h + \varepsilon$ $\forall x$ $\Rightarrow$ $f < h + \varepsilon$ on X

Conclusion:
(*) holds for $\overline{\mathcal{P}}$
i.e. $\overline{\mathcal{P}}$ = dense in $C(X)$
$\Updownarrow$
$\mathcal{P}$ = dense in $C(X)$

-1-

Stone-Weierstrass thm

: Assume $\mathcal{P} \subseteq C(X)$ is such that:

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Then $\boxed{\mathcal{P} \text{ is dense in } C(X)}$ w.r.t. the uniform topology on $C(X)$, or, equivalently: $\boxed{(\forall)\, h \in C(X),\ (\forall)\, \varepsilon > 0\ \exists f \in \mathcal{P} \text{ s.t. } d_{sup}(f, h) < \varepsilon}$ $(*)$

$|f| = \sqrt{f^2}$

Pf: **Claim 1**: $(\forall)\, x, y \in X,\ x \neq y,\ (\forall)\, a, b \in \mathbb{R},\ \exists f \in \mathcal{P}$ s.t. $f(x) = a,\ f(y) = b$.

pf of Claim 1: Fix $\xrightarrow{(C2)}$ find $g \in \mathcal{P}$ s.t. $g(x) \neq g(y)$

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f //

Notice $\exists \alpha, \beta$ s.t. $\begin{cases} \alpha + \beta g(x) = a \\ \alpha + \beta g(y) = b \end{cases}$ $\left(\beta = \frac{a-b}{g(x)-g(y)},\ \alpha = \dots\right)$

-2-

Claim 2: $f \in \mathcal{P} \Rightarrow |f| \in \overline{\mathcal{P}}$ (closure of $\mathcal{P}$ in $C(X)$ endowed with d_{sup}) (= the set of functions in $C(X)$ that are limits of sequences $(f_n)_{n \geq 1}$ with $f_n \in \mathcal{P}$)

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Moreover: $f_n \rightrightarrows \sqrt{f^2} = |f|$. $\boxed{f_n \rightrightarrows |f|}$

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(where $\sup\{f,g\} : X \to \mathbb{R}$, $\sup\{f,g\}(x) = \max\{f(x), g(x)\}$)

Pf: Recall/notice: $\forall a, b \in \mathbb{R}$:

$\max\{a,b\} = \frac{a+b+|a-b|}{2}$, $\min\{a,b\} = \frac{a+b-|a-b|}{2}$ $\Rightarrow \sup\{f,g\} = \frac{f+g+|f-g|}{2} \in \overline{\mathcal{P}}$.

-3-

Start the proof: Let $\boxed{h \in C(X)}$. Let $\varepsilon > 0$. Look for $f \in \mathcal{P}$.

$\overbrace{-\varepsilon < f(x)}^{\text{Step 2}} \overbrace{- h(x) < \varepsilon}^{\text{Step 1}}$ $(\forall)\, x$

Step 1: Fix $x_0 \in X$ (temporarily). Aim: to build $f_{x_0} \in \overline{\mathcal{P}}$ s.t.

$\boxed{f_{x_0}(x_0) = h(x_0),\quad f_{x_0}(y) - h(y) < \varepsilon\ (\forall)\, y \in X}$

Draw a picture

Pf: For each $y \in X$, use claim 1 $\Rightarrow$ consider $f_{x_0,y} \in \mathcal{P}$ s.t.

$f_{x_0,y}(x_0) = \underset{a}{h(x_0)}$, $f_{x_0,y}(y) = \underset{b}{h(y)}$

we have $f_{x_0,y} - h$ = continuous, vanishes at $y \Rightarrow$

$\Rightarrow f_{x_0,y} - h < \varepsilon$ in a neighbd. V_y of y.

$\left(\exists V_y \subseteq X \text{ open s.t. } f_{x_0,y}(y') - h(y') < \varepsilon\ (\forall)\, y' \in V_y\right)$

Now: open cover $\{V_y : y \in X\}$ of X, X = cpct $\Rightarrow$

$\Rightarrow \exists$ finite subcover: $X = V_{y_1} \cup \dots \cup V_{y_k}$ for some $y_1, \dots, y_k \in X$

Set $f_{x_0} = \inf\{f_{x_0,y_1}, \dots, f_{x_0,y_k}\}$. On each V_{y_i}: $f_{x_0} \leq f_{x_0,y_i} \leq h + \varepsilon$ $\Rightarrow f_{x_0} \leq h + \varepsilon$ on X.

-4-

Step 2: Take care of the other inequality. Apply Step 1 $\boxed{\text{for each } x \in X}$ $\Rightarrow f_x \in \overline{\mathcal{P}}$ s.t. $f_x(x) = h(x)$, $f_x - h < \varepsilon$ on X

Now: $f_x - h = 0$ at $x \in X$ $\Rightarrow \exists\ W_x \subseteq X$ open, $W_x \ni x$, s.t. $f_x - h > -\varepsilon$ on W_x

Now: $\{W_x : x \in X\}$ - open cover of $X \Rightarrow$ can write

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$f > h - \varepsilon$ on X. The inequalities $f_x < h + \varepsilon\ (\forall)\, x \Rightarrow f < h + \varepsilon$ on X

Conclusion: $(*)$ holds for $\overline{\mathcal{P}}$, i.e. $\overline{\mathcal{P}}$ = dense in $C(X)$ $\Updownarrow$ $\mathcal{P}$ = dense in $C(X)$

Remin
$C(X$
Algebra
Topolo

-5-

Def: An ideal of an algebra A is any $J \subseteq A$ s.t.

- J = a vector subspace of A
- J ~~= a ring ideal~~ satisfies: $(\forall)\, a \in A, x \in J$: $a \cdot x \in J$, $x\, a \in J$

(ideal is more than subalgebra!)

The ideal J is called maximal if:

$\nexists$ other ideal $\mathcal{J}$ s.t. $J \subsetneq \mathcal{J} \subsetneq A$

Denote by $M_A = \{ J \subseteq A \mid J = \text{maximal ideal} \}$

Ex 1:

... is a subalgebra of $C(X)$, i.e. it is a vector subspace of $C(X)$; it is a subring

(C2) $\mathcal{P}$ is point separating, i.e., $(\forall)\, x, y \in X, x \neq y\ \exists f \in \mathcal{P}$ with $f(x) \neq f(y)$.

$|f| = \sqrt{f^2}$

Claim 2: $f \in \mathcal{P} \Rightarrow |f| \in \bar{\mathcal{P}}$

Pf: $\left.\begin{array}{l} f \in C(X) \\ X = \text{cpt} \end{array}\right\} \Rightarrow f$ is bounded

$\Rightarrow \exists M > 0$ s.t. $|f| \leq M$. After

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Ex 2: When $A = C(X)$. Any subset $S \subseteq X$ gives rise to:

$$J_S := \{ f \in C(X) / \underbrace{f|_S = 0}_{f(y) = 0\ (\forall)\, y \in S} \} \quad \text{ideal}$$

When $S = \{x\}$ with $x \in X \Rightarrow$

$$\Rightarrow J_x = \{ f \in C(X) : f(x) = 0 \}$$

For arbitrary S: $J_S \subseteq J_x$ $(\forall)\, x \in S$.

Prop 8.18. (X = cpt, Haus.) Each J_x are maximal ideals (one for each $x \in X$) and, moreover, each maximal ideal is of this type.

In conclusion:

$$X \longrightarrow M_{C(X)}, \quad x \longmapsto J_x \text{ is a bijection.}$$