

Reminder: $C(X) = \{f: X \rightarrow \mathbb{R} / f \text{ continuous}\}$, with algebraic operations

- multiplication by scalars, λf
- sum of functions, $f+g$
- product of functions, $f \cdot g$

vector space } $C(X)$ is an algebra
ring }

Ex: For $A = C(X)$

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Def: A homomorphism between two algebras A and B is any map $F: A \rightarrow B$ satisfying

- $F(\lambda a) = \lambda F(a) \quad \forall \lambda \in \mathbb{R}, a \in A$
- $F(a+b) = F(a) + F(b)$
- $F(ab) = F(a)F(b)$
- $F(1) = 1$

Def: A character of an algebra A is any algebra homomorphism

Notation: $X_A :=$ the set of all characters of A
also called: the spectrum of A .

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Proposition (8.48 & 8.49) $X = \text{pt} = \text{Hausdorff}$. Then for the algebra $A = C(X)$

- (i) each $J \in M_{C(X)}$ is of type J_a for some $a \in M$
- (ii) each J_a is of type J_a for some $a \in M$

Def: An ideal of an algebra A is any $J \subseteq A$ satisfying

- J is a vector subspace of A
- J is a ring ideal, i.e. $a \cdot x$ and $x \cdot a \in J \quad \forall a \in A, x \in J$

Ex: $J = \{0\}$ and $J = A$ are always ideals

Pr:

Def: A maximal ideal of an algebra A is any ideal $J \subseteq A$ which is maximal w.r.t. $J \subseteq I \subseteq A$

Notation: $M_A =$ the set of all maximal ideals of A

Ex:



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$$\left. \begin{aligned} F(\lambda a) &= \lambda F(a) \quad \forall \lambda \in \mathbb{R}, a \in A \\ F(a+b) &= F(a) + F(b) \\ F(ab) &= F(a) \cdot F(b) \\ F(1) &= 1 \end{aligned} \right\}$$

Def: A character of an algebra A is any algebra homomorphism.

Notation: $X_A :=$ the set of all characters of A
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Def: An ideal of an algebra A is any $J \subseteq A$ satisfying:

- J is a vector subspace of A
- J is a ring ideal, i.e.: $a \cdot x$ and $x \cdot a \in J \quad (\forall) \begin{cases} a \in A \\ x \in J \end{cases}$

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Rk:

Def: A maximal ideal of an algebra A is any ideal $J \subseteq A$ satisfying:

~~J~~ ideal J with $J \subsetneq J \subsetneq A$

Notation: M_A := the set of all maximal ideals of A .

Ex:

Reminder: $C(X) = \{f: X \rightarrow \mathbb{R} / f = \text{continuous}\}$, with algebraic operations

- multiplication by scalars, λf
 - sum of functions, $f+g$
 - product of functions, $f \cdot g$
- $1 \in C(X)$

$C(X)$ is an algebra

Ex: For $A = C(X)$: any subset $S \subseteq X$ induces an ideal
 $J_S = \{f \in C(X) : f|_S = 0\}$

Ex: For $A = C(X)$: any $x \in X$ induces

- an ideal $J_x = \{f \in C(X) : f(x) = 0\}$: which is maximal
 - a character $\text{ev}_x : C(X) \rightarrow \mathbb{R}$, $\text{ev}_x(f) = f(x)$
- evaluation at x Notice $\text{Ker } \text{ev}_x = J_x$

Def: A homomorphism between two algebras A and B is any map $F: A \rightarrow B$ satisfying

$$\left. \begin{aligned} F(\lambda a) &= \lambda F(a) \quad \forall \lambda \in \mathbb{R}, a \in A \\ F(a+b) &= F(a) + F(b) \\ F(ab) &= F(a) \cdot F(b) \\ F(1) &= 1 \end{aligned} \right\} \begin{array}{l} F = \text{linear} \\ F = \text{ring homomorphism} \end{array}$$

Def: A character of an algebra A is any algebra homomorphism: $\chi: A \rightarrow \mathbb{R}$.

Notation: $X_A :=$ the set of all characters of A
also called: the spectrum of A .

Def: An ideal of an algebra A is any $J \subseteq A$ satisfying:

- J is a vector subspace of A
- J is a ring ideal, i.e.: $a \cdot x$ and $x \cdot a \in J$ $(\forall) \begin{cases} a \in A \\ x \in J \end{cases}$

Ex: $J = \{0\}$ and $(J = A)$ are always ideals

Rk: $J = \text{ideal in } A$ and $\exists u \in J$ which is invertible $\Rightarrow 1 \in J \Rightarrow J = A$

Ex: $J \neq \emptyset$ $F: A \rightarrow B$ algebra homomorphism \Rightarrow
 $\Rightarrow \text{Ker } F = \{a \in A \mid F(a) = 0\}$ is an ideal in A
 Conversely ... (remember what quotient re)

Def: A maximal ideal of an algebra A is any ideal $J \subseteq A$ satisfying:

\nexists ideal J' with $J \subsetneq J' \subsetneq A$

Notation (M_A) = the set of all maximal ideals of A .

Ex: Any character $\chi: A \rightarrow \mathbb{R} \Rightarrow \text{Ker } \chi = \{x \in A \mid \chi(x) = 0\}$ is a maximal ideal

proof that Ker χ = maximal:

Assume $J = \text{an ideal with } \text{Ker } \chi \subsetneq J \subsetneq A$. We will prove: $J = A$

$\exists a_0 \in J$ s.t. $\chi(a_0) \neq 0$

Now: any $a \in A \Rightarrow \exists \lambda \in \mathbb{R}$ s.t. $a - \lambda a_0 \in \text{Ker } \chi$
 $\Rightarrow a = \underbrace{(a - \lambda a_0)}_{\text{Ker } \chi \subset J} + \underbrace{\lambda a_0}_J \Rightarrow a \in J$

$\chi(a - \lambda a_0)$ should be 0
 $\chi(a) - \lambda \chi(a_0)$
 $\lambda = \frac{\chi(a)}{\chi(a_0)} \in \mathbb{R}$
 ok

Proposition (8.18 & 8.20): $X = \text{cpt} \approx \text{Hausdorff}$. Then, for the algebra $A = C(X)$:

- (i) each $\mathfrak{J} \in M_{C(X)}$ is of type \mathfrak{J}_x for some $x \in X$ (x will be unique)
- (ii) each $\mathfrak{J} \in X_{C(X)}$ is of type \mathfrak{J}_x for some $x \in X$.

Furthermore, one obtains a bijection:

$$\Phi: X \longrightarrow X_{C(X)}, \quad x \longmapsto \mathfrak{J}_x$$

For next year:

$$M_A \rightsquigarrow \text{Max Ideals}_A$$

$$X_A \rightsquigarrow \text{Spec } A$$

pf of (i): Assume $\mathfrak{J} \subseteq C(X)$ is a maximal ideal. Claim: $\mathfrak{J} \subseteq \mathfrak{J}_x$ for some $x \in X$

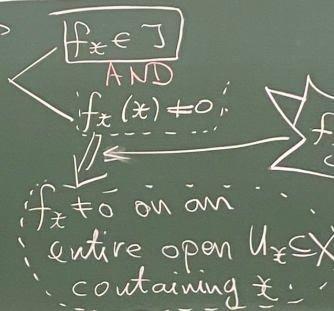
→ proceed by contradiction ⇒

$$\Rightarrow (\forall) x \in X, \exists f_x \in \mathfrak{J} \setminus \mathfrak{J}_x$$

Hence ⇒ $\{U_x : x \in X\}$ an open cover of X . By compactness

⇒ $\exists x_1, \dots, x_n \in X$ s.t.

$$X = U_{x_1} \cup \dots \cup U_{x_n}$$



(by maximality, and since $\mathfrak{J}_x \neq C(X)$, this would imply $\mathfrak{J} = \mathfrak{J}_x$)

$$f: X \rightarrow \mathbb{R}$$

$$f(x_0) \neq 0$$

$$x_0 \in U = f^{-1}(|f|^{-1} \cup \dots)$$

Now: set $u := f_{x_1}^2 + f_{x_2}^2 + \dots + f_{x_n}^2 > 0$ on the entire X and $u \in \mathfrak{J}$

Hence: \mathfrak{J} contains an inv. elt. Hence $\mathfrak{J} = C(X)$. Contradiction. \square

Proposition (8.18 & 8.20): $X = \text{cpt} \neq \text{Hausdorff}$. Then, for the algebra $A = C(X)$:

- (i) each $\mathcal{J} \in M_{C(X)}$ is of type \mathcal{J}_x for some $x \in X$ (x will be unique)
- (ii) each $\chi \in X_{C(X)}$ is of type ev_x for some $x \in X$ (unique)

Furthermore, one obtains a bijection:

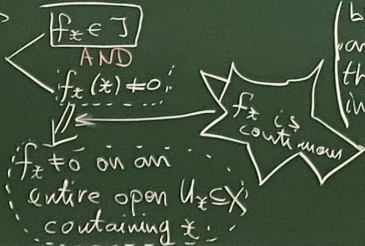
$$\Phi: X \xrightarrow{\psi} X_{C(X)}, \quad x \mapsto ev_x$$

For next year:
 $M_A \rightsquigarrow \text{Max Ideals}_A$
 $X_A \rightsquigarrow \text{Spect}_A$

pf of (i): Assume $\mathcal{J} \subseteq C(X)$ is a maximal ideal. Claim: $\mathcal{J} \subseteq \mathcal{J}_x$ for some $x \in X$

→ proceed by contradiction ⇒
 $\Rightarrow (\forall x \in X, \exists f_x \in \mathcal{J} \setminus \mathcal{J}_x)$

Hence $\Rightarrow \{U_x : x \in X\}$ an open cover of X . By compactness $\Rightarrow \exists x_1, \dots, x_n \in X$ st $X = U_{x_1} \cup \dots \cup U_{x_n}$



by maximality, and since $\mathcal{J}_x \neq C(X)$, this would imply $\mathcal{J} = \mathcal{J}_x$

$X \rightarrow \mathbb{R}$
 $f(x_0) \neq 0$
 $\text{for } U = f^{-1}([1-\epsilon, 1+\epsilon])$

Now: set $u = f_{x_1}^2 + \dots + f_{x_n}^2$. This is $\begin{cases} \text{is in } \mathcal{J} \\ \text{AND} \\ u > 0 \text{ everywhere} \end{cases} \Rightarrow u \text{ invertible}$

Proof of (ii): Let $\chi \in X_{C(X)}$

Can apply (i) to $\ker \chi \Rightarrow \exists x \in X$ st $\ker \chi = \mathcal{J}_x$

Now, for $f \in C(X)$ arbitrary: $f - \chi(f) \cdot 1 \in C(X)$ is actually in $\ker \chi$

$$\Rightarrow f - \chi(f) \cdot 1 \text{ is 0 when evaluated at } x \Rightarrow f(x) - \chi(f) = 0 \Rightarrow \chi(f) = f(x) = ev_x(f)$$

Hence $\chi = ev_x$. All together $\Rightarrow \Phi$ will be a surjective function

It is also injective because: $(\forall x, y \in X, x \neq y) \exists f \in C(X)$ st $f(x) \neq f(y) \Rightarrow ev_x(f) \neq ev_y(f) \Rightarrow ev_x \neq ev_y$ i.e. $\Phi(x) \neq \Phi(y)$. Hence Φ is also injective.

ef: Return to general algebras A . For $a \in A$ under the following function on the spectrum:

$$F_a: X_A \rightarrow \mathbb{R}, \quad F_a(x) = \chi(x)(a)$$

X_A consider \mathcal{T} = the smallest topology which

GELFAND-NAIMARK

Theorem (8.22): For any compact Hausdorff X the topological spectrum of $A = C(X)$ is homeomorphic to X . More precisely, $\phi: X \rightarrow X_{C(X)}$ is a homeomorphism.

pf: $\phi = \text{continuous}$; proof. To prove: for any $a \in A = C(X)$ $\phi^{-1}(U) = \dots$

$(\forall x \in X, \exists f_x \in J_x) \wedge$
 Hence $\Rightarrow \{U_x : x \in X\}$ an open cover of X . By compactness $\Rightarrow \exists x_1, \dots, x_n \in X$ s.t. $X = U_{x_1} \cup \dots \cup U_{x_n}$
 AND $f_x(x) \neq 0$ on an entire open $U_x \subset X$ containing x .
 since $J_x \neq \{0\}$, this would imply $J = J_x$
 $X \rightarrow \mathbb{R}$
 $f(x) \neq 0$
 for $U = f^{-1}([1, \infty) \cup (0, 1])$
 All together $\Rightarrow \Phi$ will be a surjective function
 It is also injective because: $(\forall) x, y \in X, x \neq y \exists f \in C(X)$ s.t. $f(x) \neq f(y) \Rightarrow \Rightarrow ev_x(f) \neq ev_y(f) \Rightarrow ev_x \neq ev_y$ i.e. $\phi(x) \neq \phi(y)$. Hence ϕ is also injective.

Def: Return to general algebras A : For $a \in A$ consider the following function on the spectrum:
 $F_a : X_A \rightarrow \mathbb{R}, F_a(x) = \chi(a)$
 On X_A consider J = the smallest topology which make all the functions F_a (one for each $a \in A$) continuous.
 The topological spectrum of A : (X_A, J) .

GELFAND-NAIMARK Theorem (8.22): For any compact Hausdorff X the topological spectrum of $A = C(X)$ is homeomorphic to X . More precisely, $\phi : X \rightarrow X_{C(X)}$ is a homeomorphism.
pf: $\phi = \text{continuous}$: proof. To prove: for any $a \in A, D \subseteq \mathbb{R}$ open, $\phi^{-1}(U_{a,D}) = \text{open in } X$.
 Look at $\phi^{-1}(U_{a,D}) = \{x \in X \mid \phi(x) \in U_{a,D}\} = \{x \in X \mid ev_x(a) \in D\} = \{x \in X \mid a(x) \in D\} = a^{-1}(D)$ which is open in X because $a : X \rightarrow \mathbb{R}$ continuous.

Rk: What must be in J ?
 • for each $a \in A, D \subseteq \mathbb{R}$ open: $U_{a,D} = F_a^{-1}(D) = \{x \in X_A \mid \chi(a) \in D\}$ must be in J ← we have a collection \mathcal{B} of subsets
 • for each $a_1, \dots, a_k \in A, D_1, \dots, D_k \subseteq \mathbb{R}$ open: $U_{a_1, D_1} \cap \dots \cap U_{a_k, D_k}$ must be in J ← we produce a basis of \mathcal{B}
 • now take unions of such finite intersections ← they must be in J too. these do form a topology (our J)

$X_{C(X)}$ is Hausdorff: proof. Let $x_1, x_2 \in X_A$ with $x_1 \neq x_2$. We can write $x_1 = ev_{x_1}, x_2 = ev_{x_2}$ with $x_1, x_2 \in X, x_1 \neq x_2$. Now $\{x_1 \in U_{f, D_1} \text{ and } U_{f, D_1} \cap U_{f, D_2} = \emptyset\}$ we do find $f \in C(X)$ s.t. $f(x_1) \neq f(x_2)$ also find $D_1, D_2 \subseteq \mathbb{R}$ open, $D_1 \cap D_2 = \emptyset$ with $f(x_1) \in D_1, f(x_2) \in D_2$.

Proposition 1.1.1 Let X be a compact Hausdorff space. Then, for the algebra $C(X, \mathbb{C})$ of complex-valued continuous functions on X , the following hold:

- (1) each $f \in C(X, \mathbb{C})$ is of type I for some $x \in X$ (i.e. $f(x) \neq 0$)
- (2) each $f \in C(X, \mathbb{C})$ is of type II for some $x \in X$ (i.e. $f(x) = 0$)

Furthermore, one obtains a bijection:

$$\Phi: X \rightarrow \{ \text{maximal ideals of } C(X, \mathbb{C}) \}$$

Proof: (1) Assume $J \subseteq C(X, \mathbb{C})$ is a maximal ideal. Claim: $J \subseteq J_x$ for some $x \in X$.
 proceed by contradiction: assume $J \not\subseteq J_x$ for all $x \in X$.
 Then $\bigcup_{x \in X} J_x \supseteq J$. By compactness, there is a finite subcover $J \subseteq \bigcup_{i=1}^n J_{x_i}$.
 Hence $\exists f_i \in J_{x_i} \setminus J$. Then $\sum_{i=1}^n f_i^2 \notin J$ because $\sum_{i=1}^n f_i^2(x) > 0$ for all $x \in X$.
 This contradicts the maximality of J .
 Hence $J \subseteq J_x$ for some $x \in X$.
 Now, if $J \subseteq J_x$ and $J \subseteq J_y$ for $x \neq y$, then $J \subseteq J_x \cap J_y = J_{\{x, y\}}$.
 This contradicts the maximality of J .
 Hence $J = J_x$ for some $x \in X$.
 Note: $\text{Ker } \phi_x = J_x$.

Proposition 1.1.2 Let X be a compact Hausdorff space. Then, for the algebra $C(X, \mathbb{C})$ of complex-valued continuous functions on X , the following hold:

- (1) each $f \in C(X, \mathbb{C})$ is of type I for some $x \in X$ (i.e. $f(x) \neq 0$)
- (2) each $f \in C(X, \mathbb{C})$ is of type II for some $x \in X$ (i.e. $f(x) = 0$)

Proof of (1): Let $x \in X$.
 Can apply (1) to $\text{ker } \phi_x \Rightarrow \exists f \in C(X)$ s.t. $f(x) \neq 0$.
 Now for $f \in C(X)$ s.t. $f(x) \neq 0$, f is actually in $\text{ker } J$.
 $\Rightarrow f - x(f) = f - f(x) \in J$
 $\Rightarrow f(x) = f(x) \neq 0$
 Hence $x \in \text{ker } \phi_x$.
 All together $\Rightarrow \Phi$ well a surjective function.
 It is also injective because $\phi_x \neq \phi_y \Rightarrow \exists f \in C(X)$ s.t. $f(x) \neq f(y)$.
 $\Rightarrow \phi_x(f) \neq \phi_y(f) \Rightarrow \phi_x \neq \phi_y$ (i.e. ϕ is injective).
 Hence Φ is also injective.

Def: An ideal of an algebra A is called a **maximal ideal** if it is a proper ideal and is not contained in any other proper ideal.

Ex: Ideals and $\text{ker } \phi$ are always ideals.

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Def: A character of an algebra A is any algebra homomorphism $\chi: A \rightarrow \mathbb{C}$.

Notation: X_A is the set of all characters of A , also called the **spectrum** of A .

Ex: For $A = C(X, \mathbb{C})$, any $x \in X$ induces an ideal $J_x = \{ f \in C(X) : f(x) = 0 \}$ which is maximal and a character $\phi_x: C(X) \rightarrow \mathbb{C}, \phi_x(f) = f(x)$.

Def: Return to general algebra A . For $a \in A$ consider the following function on the spectrum X_A :

$$F_a: X_A \rightarrow \mathbb{C}, F_a(\chi) = \chi(a)$$

Ex: X_A consider J to be the smallest topology which makes all the functions F_a (with $a \in A$) continuous.

The topological spectrum of A is (X_A, J) .

Ex: What must be in J ?

- for $a \in A$, $\text{ker } \phi_a$ open $(\text{ker } \phi_a = F_a^{-1}(0))$ and hence $0 \in J$
- for $a \in A$, $\text{ker } \phi_a$ open $(\text{ker } \phi_a = F_a^{-1}(0))$ and hence $0 \in J$
- now have closure of $\text{ker } \phi_a$ (characters) - they equal to J for $a \in A$.

Theorem 1.1.3 For any compact Hausdorff X , the topological spectrum of $A = C(X, \mathbb{C})$ is homeomorphic to X . Moreover, ϕ_x is a homeomorphism.

Proof: To prove for any A , $\text{ker } \phi_x$ is open in X .
 Let $U = \{ x \in X : \exists f \in C(X) \text{ s.t. } f(x) \neq 0 \}$.
 Then $U = \bigcup_{f \in C(X)} \{ x \in X : f(x) \neq 0 \}$.
 Each $\{ x \in X : f(x) \neq 0 \}$ is open in X .
 Hence U is open in X .
 Now, $\text{ker } \phi_x = \{ f \in C(X) : f(x) = 0 \} = U^c$.
 Hence $\text{ker } \phi_x$ is closed in $C(X, \mathbb{C})$.
 Hence ϕ_x is a homeomorphism.

Def: A maximal ideal of an algebra A is called a **maximal ideal** if it is a proper ideal and is not contained in any other proper ideal.

Notation: M_A is the set of all maximal ideals of A .

Ex: Any character $\chi: A \rightarrow \mathbb{C} \Rightarrow \text{ker } \chi = J_\chi$ is a maximal ideal.

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