–1–

Reminder: $\underline{C(X)} = \{f : X \to \mathbb{R} \,/\, f = \text{continuous}\}$, with algebraic operations

- multiplication by scalars, λf
- sum of functions, $f+g$
- products of functions, $f \cdot g$

$\left.\begin{array}{l}\lambda f, f+g \text{ vector space} \\ f+g, f\cdot g \text{ ring}\end{array}\right\}$ $\boxed{C(X) \text{ is an algebra}}$

Ex: For $A = C(X)$

Ex: For $A = C(X)$

–2–

Def: An ideal of an algebra A is any $J \subseteq A$ satisfying:

- J is a vector subspace of A
- J is a ring ideal, i.e.: $a \cdot x$ and $x \cdot a \in J$ $(\forall) \begin{cases} a \in A \\ x \in J \end{cases}$

Ex: $J = \{0\}$ and $J = A$ are always ideals

Rk:

–3–

Def: A maximal ideal of an algebra A is any ideal $I \neq A$ satisfying:

$\nexists$ ideal J with $I \subsetneqq J \subsetneqq A$

Notation $\underline{M_A}$:= the set of all maximal ideals of A.

Ex:

–4–

Def: A homomorphism between two algebras A and B is any map $F : A \to B$ satisfying

$$\left.\begin{array}{l} F(\lambda a) = \lambda F(a) \ (\forall)\ \lambda \in \mathbb{R}, a \in A \\ F(a+b) = F(a) + F(b) \\ F(ab) = F(a) \cdot F(b) \\ F(1) = 1 \end{array}\right\}$$

Def: A character of an algebra A is any algebra homomorphism

Notation: X_A := the set of all characters of A

$\nwarrow$ also called: the spectrum of A.

–5–

Proposition (8.18 & 8.20) X = cpt & Hausdorff. Then, for the algebra $A = C(X)$:

(i) each $I \in M_{C(X)}$ is of type I_x for some $x \in M$

(ii) each is of type for some $x \in M$

–6–

-1-

Reminder: $C(X)=\{f:X\to\mathbb{R}\,/\,f=\text{continuous}\}$, with algebraic operations:

- multiplication by scalars, λf
- sum of functions, $f+g$
- products of functions, $f\cdot g$

$$\left.\begin{array}{l}\left.\begin{array}{l}\text{multiplication by scalars, } \lambda f\\ \text{sum of functions, } f+g\end{array}\right\}\text{vector space}\\ \left.\begin{array}{l}\text{products of functions, } f\cdot g\end{array}\right\}\text{ring}\end{array}\right\}\ \boxed{C(X)\text{ is an algebra}}$$

Ex: For $A=C(X)$

Ex: For $A=C(X)$

-4-

Def: A homomorphism between two algebras A and B is any map $F: A\to B$ satisfying

$$\left.\begin{array}{l}F(\lambda a)=\lambda F(a)\ (\forall)\ \lambda\in\mathbb{R}, a\in A\\ F(a+b)=F(a)+F(b)\end{array}\right\}$$

$$\left.\begin{array}{l}F(a+b)=F(a)+F(b)\\ F(ab)=F(a)\cdot F(b)\\ F(1)=1\end{array}\right\}$$

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-2-

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Rk:

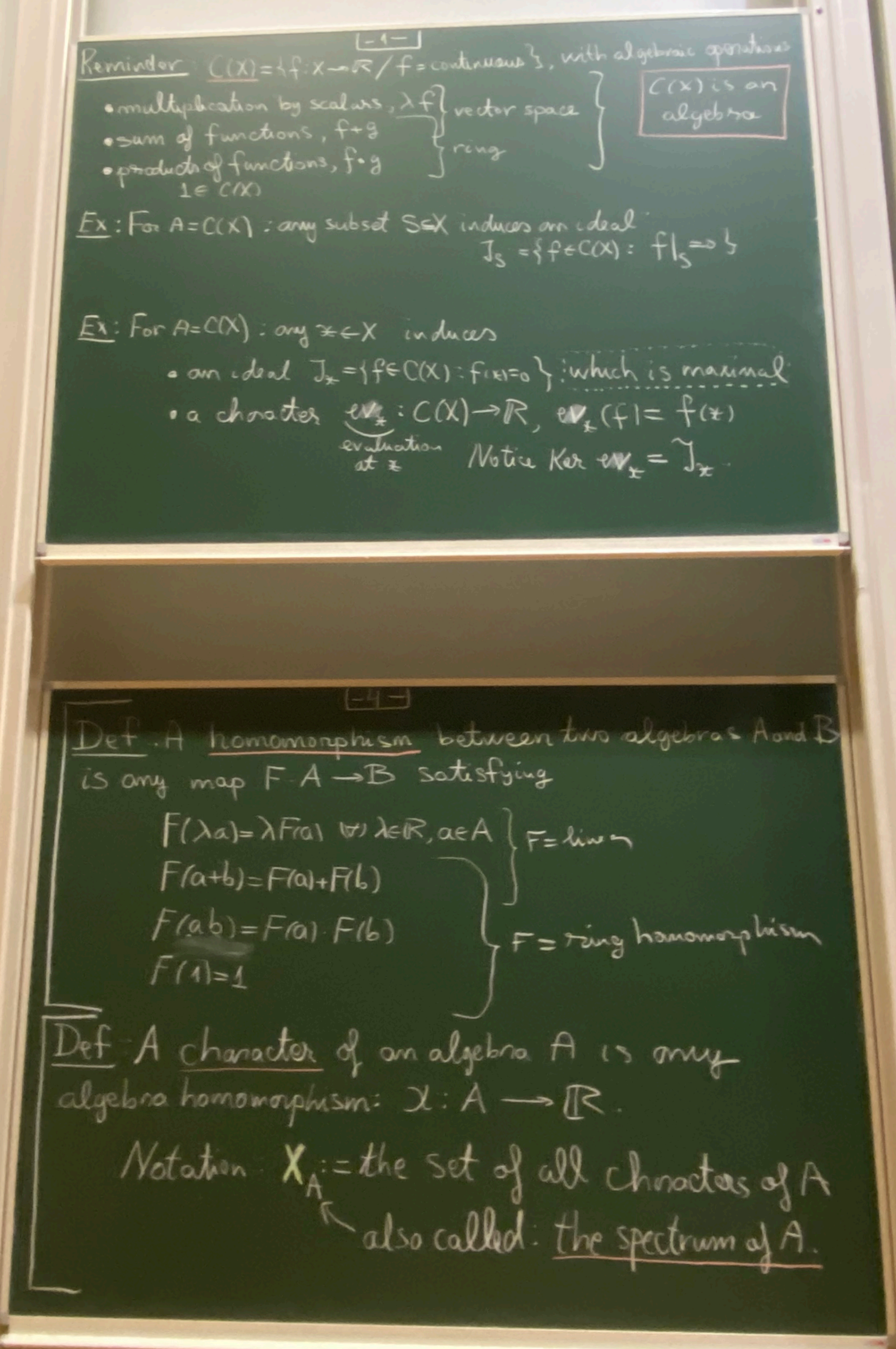
-3-

Def: A maximal ideal of an algebra A is any ideal $J \subseteq A$ satisfying:

$$\nexists \text{ ideal } \mathcal{J} \text{ with } J \subsetneqq \mathcal{J} \subsetneqq A$$

Notation: $M_A :=$ the set of all maximal ideals of A.

Ex:



–1–

Reminder $C(X)=\{f:X\to\mathbb{R}\ /\ f=\text{continuous}\}$, with algebraic operations

- multiplication by scalars, λf
- sum of functions, $f+g$
- products of functions, $f\cdot g$

$1\in C(X)$

$\left.\begin{array}{l}\lambda f\\ f+g\end{array}\right\}$ vector space; $\left.\begin{array}{l}f+g\\ f\cdot g\end{array}\right\}$ ring; $\Big\}$ $\boxed{C(X)\text{ is an algebra}}$

Ex: For $A=C(X)$: any subset $S\subseteq X$ induces an ideal

$$J_S=\{f\in C(X):\ f|_S=0\}$$

Ex: For $A=C(X)$: any $x\in X$ induces

- an ideal $J_x=\{f\in C(X): f(x)=0\}$ which is maximal
- a character $\underbrace{ev_x}_{\text{evaluation at }x}: C(X)\to\mathbb{R}$, $ev_x(f)=f(x)$

Notice $\operatorname{Ker} ev_x = J_x$

–4–

Def A homomorphism between two algebras A and B is any map $F: A\to B$ satisfying

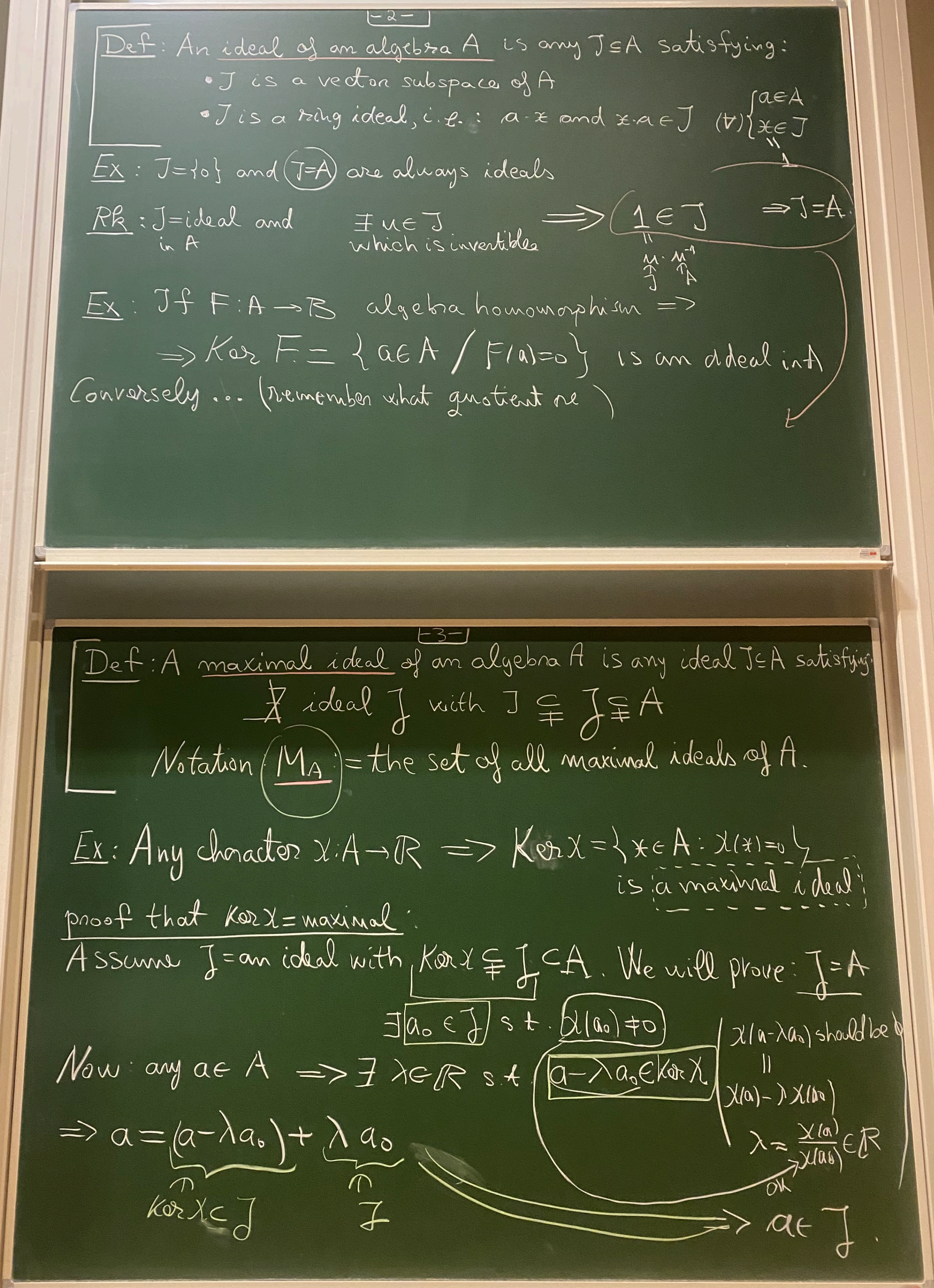
$$\left.\begin{array}{l}F(\lambda a)=\lambda F(a)\ (\forall)\ \lambda\in\mathbb{R}, a\in A\\ F(a+b)=F(a)+F(b)\end{array}\right\}\ F=\text{linear}$$

$$\left.\begin{array}{l}F(a+b)=F(a)+F(b)\\ F(ab)=F(a)\cdot F(b)\\ F(1)=1\end{array}\right\}\ F=\text{ring homomorphism}$$

Def A character of an algebra A is any algebra homomorphism: $\chi: A\longrightarrow\mathbb{R}$.

Notation X_A := the set of all characters of A

$\nwarrow$ also called: the spectrum of A.



-2-

Def: An ideal of an algebra A is any $J \subseteq A$ satisfying:

- J is a vector subspace of A
- J is a ring ideal, i.e.: $a \cdot x$ and $x \cdot a \in J$ $(\forall)\begin{cases} a \in A \\ x \in J \end{cases}$

Ex: $J=\{0\}$ and $\boxed{J=A}$ are always ideals

Rk: J=ideal in A and $\exists\, u \in J$ which is invertible $\Longrightarrow \underset{\substack{u \cdot u^{-1} \\ \uparrow \quad \uparrow \\ J \quad A}}{1} \in J \Rightarrow J=A.$

Ex: If $F: A \to B$ algebra homomorphism $\Rightarrow$

$\Rightarrow \operatorname{Ker} F = \{a \in A \,/\, F(a)=0\}$ is an ideal in A

Conversely ... (remember what quotient are)

-3-

Def: A maximal ideal of an algebra A is any ideal $J \subseteq A$ satisfying:

$\nexists$ ideal $\mathcal{J}$ with $J \subsetneqq \mathcal{J} \subsetneqq A$

Notation $\boxed{M_A}$ = the set of all maximal ideals of A.

Ex: Any character $\chi: A \to \mathbb{R} \Longrightarrow \operatorname{Ker}\chi = \{x \in A : \chi(x)=0\}$ is a maximal ideal

proof that $\operatorname{Ker}\chi$ = maximal:

Assume $\mathcal{J}$ = an ideal with $\operatorname{Ker}\chi \subsetneqq \mathcal{J} \subset A$. We will prove: $\mathcal{J} = A$

$\exists\, \boxed{a_0 \in \mathcal{J}}$ s.t. $\boxed{\chi(a_0) \neq 0}$

Now: any $a \in A \Longrightarrow \exists\, \lambda \in \mathbb{R}$ s.t. $\boxed{a - \lambda a_0 \in \operatorname{Ker}\chi}$

$\chi(a-\lambda a_0)$ should be 0

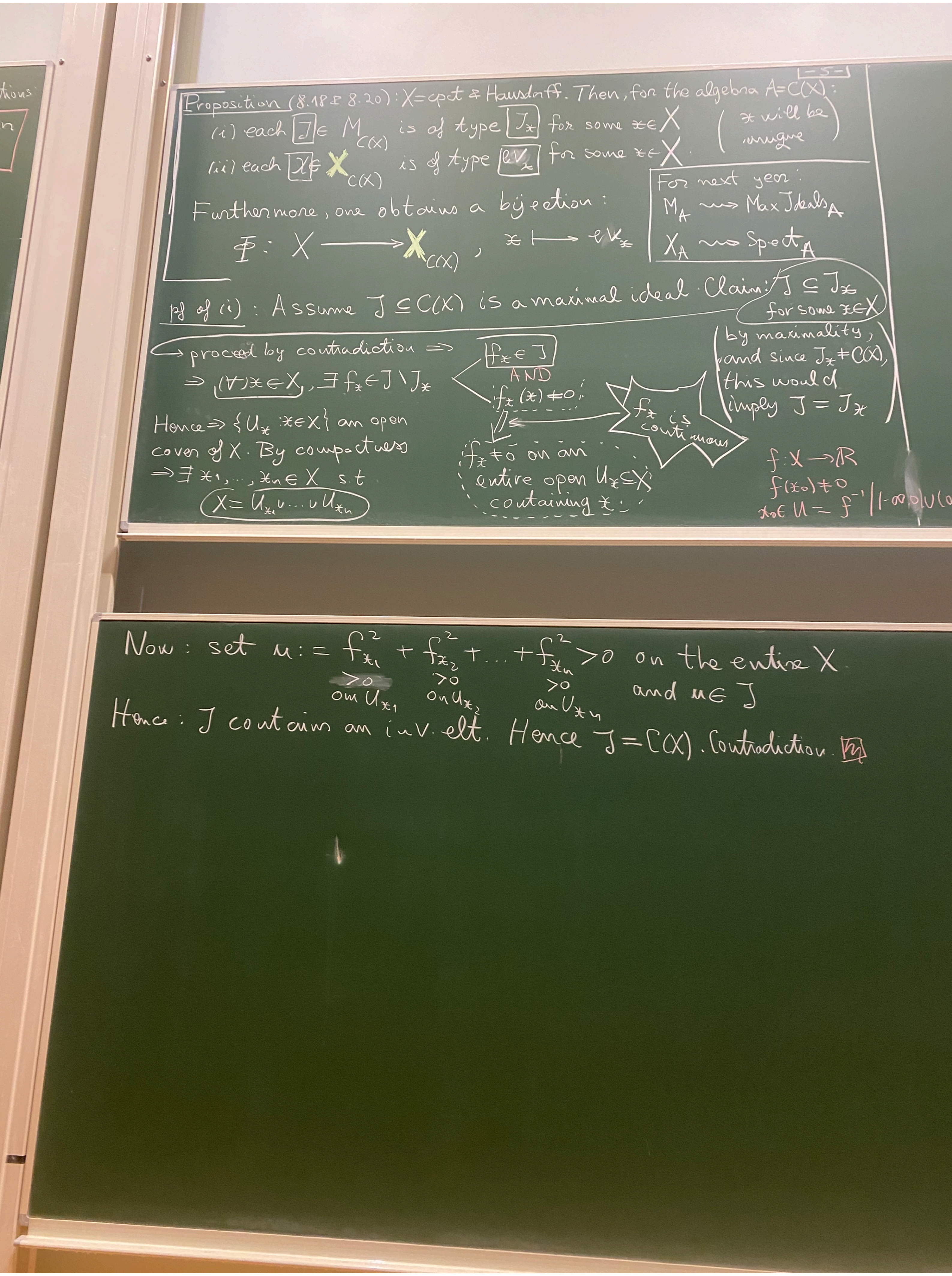
$\|$

$\chi(a) - \lambda\chi(a_0)$

$\lambda = \dfrac{\chi(a)}{\chi(a_0)} \in \mathbb{R}$

ok

$\Rightarrow a = \underbrace{(a-\lambda a_0)}_{\substack{\cap \\ \operatorname{Ker}\chi \subset \mathcal{J}}} + \underbrace{\lambda a_0}_{\substack{\cap \\ \mathcal{J}}} \Longrightarrow a \in \mathcal{J}.$



-5-

Proposition (8.18 & 8.20): X = cpct & Hausdorff. Then, for the algebra $A = C(X)$:

(i) each $\boxed{J} \in M_{C(X)}$ is of type $\boxed{J_x}$ for some $x \in X$ (x will be unique)

(ii) each $\boxed{\chi} \in \mathbb{X}_{C(X)}$ is of type $\boxed{ev_x}$ for some $x \in X$.

Furthermore, one obtains a bijection:

$$\Phi : X \longrightarrow \mathbb{X}_{C(X)}, \quad x \longmapsto ev_x$$

For next year:
$M_A \rightsquigarrow \text{MaxIdeals}_A$
$X_A \rightsquigarrow \text{Spec}_A$

pf of (i): Assume $J \subseteq C(X)$ is a maximal ideal. Claim: $J \subseteq J_x$ for some $x \in X$

(by maximality, and since $J_x \neq C(X)$, this would imply $J = J_x$)

$\hookrightarrow$ proceed by contradiction $\Rightarrow$

$\Rightarrow \underline{(\forall) x \in X}, \exists f_x \in J \setminus J_x$ $<$ $\boxed{f_x \in J}$ AND $f_x(x) \neq 0$

f_x is continuous $\Rightarrow$

$\Downarrow$

$f_x \neq 0$ on an entire open $U_x \subseteq X$ containing x

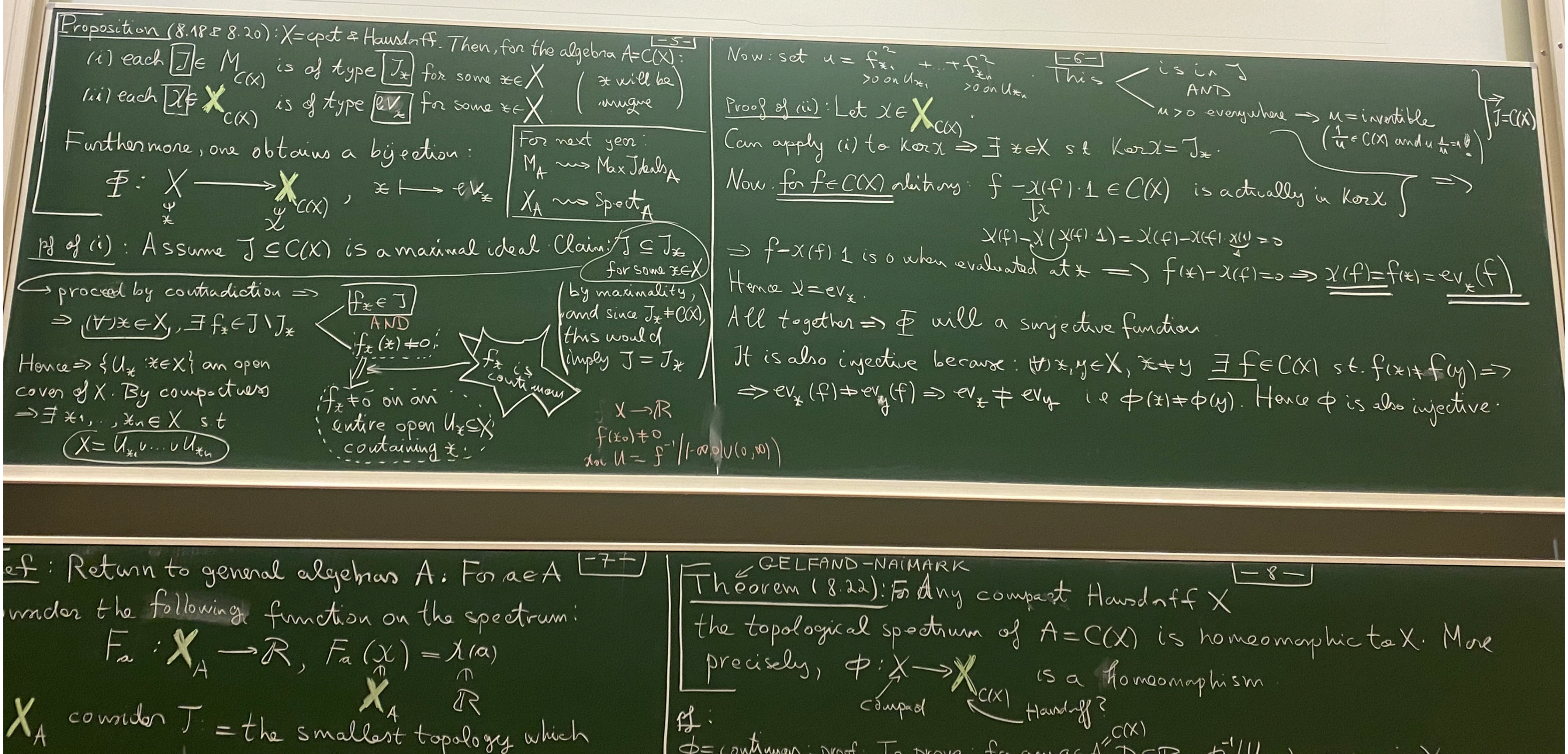
Hence $\Rightarrow \{U_x : x \in X\}$ an open cover of X. By compactness $\Rightarrow \exists x_1, \dots, x_n \in X$ s.t

$X = U_{x_1} \cup \dots \cup U_{x_n}$

$f : X \to \mathbb{R}$
$f(x_0) \neq 0$
$x_0 \in U = f^{-1}((-\infty, 0) \cup ($

Now: set $u := \underset{\substack{>0 \\ \text{on } U_{x_1}}}{f_{x_1}^2} + \underset{\substack{>0 \\ \text{on } U_{x_2}}}{f_{x_2}^2} + \dots + \underset{\substack{>0 \\ \text{on } U_{x_n}}}{f_{x_n}^2} > 0$ on the entire X and $u \in J$

Hence: J contains an inv. elt. Hence $J = C(X)$. Contradiction. $\boxed{\text{M}}$



$\boxed{-5-}$

Proposition (8.18 & 8.20): X = cpct & Hausdorff. Then, for the algebra $A = C(X)$:

(i) each $\boxed{J} \in M_{C(X)}$ is of type $\boxed{J_x}$ for some $x \in X$

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(x will be unique)

Furthermore, one obtains a bijection:

$$\Phi : \underset{\underset{x}{\Psi}}{X} \longrightarrow \underset{\underset{\chi}{\Psi}}{\mathbb{X}_{C(X)}}, \quad x \longmapsto ev_x$$

For next year:
$M_A \rightsquigarrow \text{MaxIdeals}_A$
$X_A \rightsquigarrow \text{Spect}_A$

pf of (i): Assume $J \subseteq C(X)$ is a maximal ideal. Claim: $J \subseteq J_x$ for some $x \in X$

↪ proceed by contradiction ⇒

⇒ $(\forall) x \in X, \exists f_x \in J \setminus J_x$ < $\boxed{f_x \in J}$ AND $f_x(x) \neq 0$

(by maximality, and since $J_x \neq C(X)$, this would imply $J = J_x$)

f_x is continuous ⇒ $f_x \neq 0$ on an entire open $U_x \subseteq X$ containing x.

$X \to \mathbb{R}$
$f(x_0) \neq 0$
for $U = f^{-1}((-\infty, 0) \cup (0, \infty))$

Hence ⇒ $\{U_x : x \in X\}$ an open cover of X. By compactness ⇒ $\exists x_1, \ldots, x_n \in X$ s.t. $X = U_{x_1} \cup \ldots \cup U_{x_n}$

$\boxed{-6-}$

Now: set $u = \underset{>0 \text{ on } U_{x_1}}{f_{x_1}^2} + \ldots + \underset{>0 \text{ on } U_{x_n}}{f_{x_n}^2}$. This < is in J AND $u > 0$ everywhere → u = invertible ($\frac{1}{u} \in C(X)$ and $u \cdot \frac{1}{u} = 1$!) $\Big\} \Rightarrow J = C(X)$

Proof of (ii): Let $\chi \in \mathbb{X}_{C(X)}$.

Can apply (i) to $\text{Ker}\,\chi \Rightarrow \exists x \in X$ s.t. $\text{Ker}\,\chi = J_x$.

Now: for $f \in C(X)$ arbitrary: $f - \chi(f) \cdot 1 \in C(X)$ is actually in $\text{Ker}\,\chi$ ⇒

$$\chi(f) - \chi(\chi(f) \cdot 1) = \chi(f) - \chi(f) \cdot \underset{1}{\chi(1)} = 0$$

⇒ $f - \chi(f) \cdot 1$ is 0 when evaluated at x ⇒ $f(x) - \chi(f) = 0 \Rightarrow \chi(f) = f(x) = ev_x(f)$

Hence $\chi = ev_x$.

All together ⇒ Φ will a surjective function

It is also injective because: $(\forall) x, y \in X, x \neq y$ $\exists f \in C(X)$ s.t. $f(x) \neq f(y) \Rightarrow$
$\Rightarrow ev_x(f) \neq ev_y(f) \Rightarrow ev_x \neq ev_y$ i.e. $\Phi(x) \neq \Phi(y)$. Hence Φ is also injective.

$\boxed{-7-}$

ef: Return to general algebras A: For $a \in A$ nsider the following function on the spectrum:

$$F_a : \mathbb{X}_A \longrightarrow \mathbb{R}, \quad F_a(\underset{\underset{\mathbb{X}_A}{\Pi}}{\chi}) = \underset{\underset{\mathbb{R}}{\Pi}}{\chi(a)}$$

$\mathbb{X}_A$ consider $\mathcal{T}$ = the smallest topology which

$\boxed{-8-}$

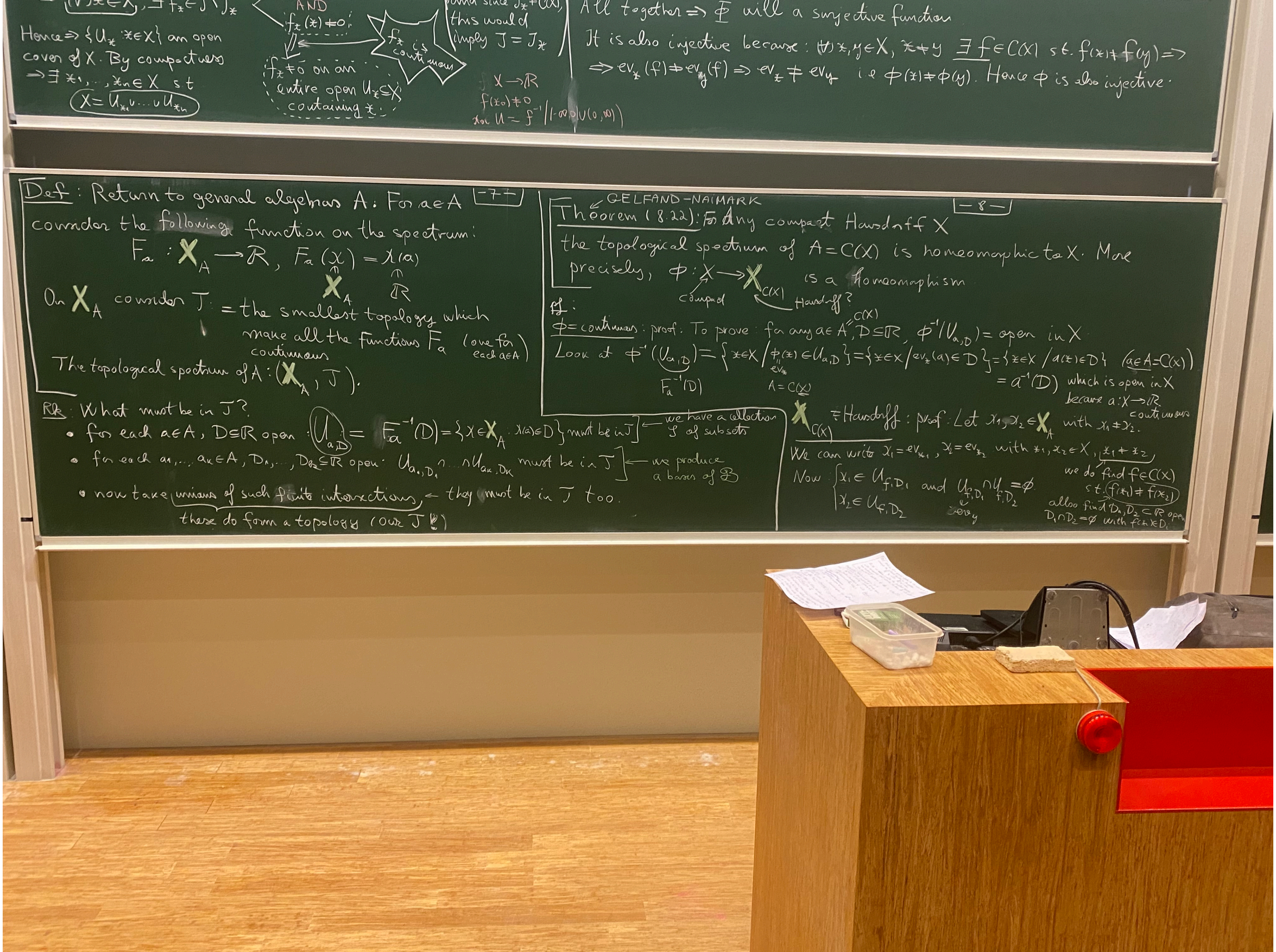
GELFAND-NAIMARK

Theorem (8.22): For Any compact Hausdorff X the topological spectrum of $A = C(X)$ is homeomorphic to X. More precisely, $\Phi : X \longrightarrow \mathbb{X}_{C(X)}$ is a homeomorphism

compact

Hausdorff?

Pf:



$\underbrace{(\forall) x \in X}, \exists f_x \in J \setminus J_x$

Hence $\Rightarrow \{U_x : x \in X\}$ an open cover of X. By compactness $\Rightarrow \exists x_1, \dots, x_n \in X$ s.t $\boxed{X = U_{x_1} \cup \dots \cup U_{x_n}}$

AND $f_x(x) \neq 0$

f_x is continuous

$\Downarrow$ $f_x \neq 0$ on an entire open $U_x \subseteq X$ containing x

and since $J_x \neq C(X)$, this would imply $J = J_x$

$f: X \to \mathbb{R}$
$f(x_0) \neq 0$
for $U = f^{-1}((-\infty, 0) \cup (0, \infty))$

All together $\Rightarrow \Phi$ will a surjective function

It is also injective because: $(\forall) x, y \in X$, $x \neq y$ $\underline{\exists f} \in C(X)$ s.t. $f(x) \neq f(y) \Rightarrow$
$\Rightarrow ev_x(f) \neq ev_y(f) \Rightarrow ev_x \neq ev_y$ i.e $\phi(x) \neq \phi(y)$. Hence ϕ is also injective.

Def: Return to general algebras A: For $a \in A$ consider the following function on the spectrum:

$$F_a : X_A \longrightarrow \mathbb{R}, \quad F_a(\underset{X_A}{\underset{\cap}{\chi}}) = \underset{\mathbb{R}}{\underset{\cap}{\chi(a)}}$$

—7—

On X_A consider J := the smallest topology which make all the functions F_a continuous (one for each $a \in A$)

The topological spectrum of A: (X_A, J).

Rk: What must be in J?

- for each $a \in A$, $D \subseteq \mathbb{R}$ open: $U_{a,D} = F_a^{-1}(D) = \{\chi \in X_A : \chi(a) \in D\}$ must be in J] ← we have a collection $\mathcal{S}$ of subsets
- for each $a_1, \dots, a_k \in A$, $D_1, \dots, D_k \subseteq \mathbb{R}$ open: $U_{a_1, D_1} \cap \dots \cap U_{a_k, D_k}$ must be in J] ← we produce a basis of $\mathcal{B}$
- now take $\underbrace{\text{unions of such finite intersections}}_{\text{these do form a topology (our } J \text{!)}}$ ← they must be in J too.

—8—

GELFAND-NAIMARK
Theorem (8.22): For any compact Hausdorff X the topological spectrum of $A = C(X)$ is homeomorphic to X. More precisely, $\phi : X \longrightarrow X_{C(X)}$ is a homeomorphism

compact

Hausdorff?

Pf:

$\underline{\phi = \text{continuous}}$: proof: To prove: for any $a \in \overset{C(X)}{\overset{\parallel}{A}}$, $D \subseteq \mathbb{R}$, $\phi^{-1}(U_{a,D})$ = open in X

Look at $\phi^{-1}(\underset{F_a^{-1}(D)}{\underbrace{U_{a,D}}}) = \{x \in X / \underset{ev_x}{\underset{\parallel}{\phi(x)}} \in U_{a,D}\} = \{x \in X / ev_x(a) \in D\} = \{x \in X / a(x) \in D\}$ $(\underline{a \in A} = C(X))$

$A = C(\underline{X})$

$= a^{-1}(D)$ which is open in X because $a : X \to \mathbb{R}$ continuous

$\underline{X_{C(X)} = \text{Hausdorff}}$: proof: Let $\chi_1, \chi_2 \in X_A$ with $\chi_1 \neq \chi_2$.

We can write $\chi_1 = ev_{x_1}$, $\chi_2 = ev_{x_2}$ with $x_1, x_2 \in X$, $\underline{x_1 \neq x_2}$

we do find $f \in C(X)$ s.t. $\boxed{f(x_1) \neq f(x_2)}$

also find $D_1, D_2 \subset \mathbb{R}$ open $D_1 \cap D_2 = \emptyset$ with $f(x_i) \in D_i$

Now $\begin{cases} \chi_1 \in U_{f,D_1} \\ \chi_2 \in U_{f,D_2} \end{cases}$ and $\underset{\ni ev_y}{U_{f,D_1}} \cap U_{f,D_2} = \emptyset$.