

Reminder: $C(X) = \{f: X \rightarrow \mathbb{C} / f \text{ continuous}\}$, with algebraic operations

- multiplication by scalars, λf
- sum of functions, $f+g$
- product of functions, $f \cdot g$

$\left\{ \begin{array}{l} \text{vector space} \\ \text{ring} \end{array} \right\} \quad C(X) \text{ is an algebra}$

Ex: For $A = C(X)$

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Def: A homomorphism between two algebras A and B is any map $F: A \rightarrow B$ satisfying

$$\left. \begin{aligned} F(\lambda a) &= \lambda F(a) \quad \forall \lambda \in \mathbb{C}, a \in A \\ F(a+b) &= F(a) + F(b) \\ F(ab) &= F(a) \cdot F(b) \\ F(1) &= 1 \end{aligned} \right\}$$

Def: A character of an algebra A is any algebra homomorphism

Notation: $X_A =$ the set of all characters of A
 ↪ also called: the spectrum of A .

Proposition (8.18 & 8.20): $X = \text{cpt} \neq \text{Head}(A)$. Then, for the algebra $A = C(X)$

- each $J \in M_{C(X)}$ is of type J_x for some $x \in M$
- each J_x is of type J for some $x \in M$

Def: An ideal of an algebra A is any $J \subseteq A$ satisfying

- J is a vector subspace of A
- J is a ring ideal, i.e.: $a \in J$ and $x \in J \Rightarrow ax \in J$

Ex: $J = \{0\}$ and $T(A)$ are always ideals

Re:

Def: A maximal ideal of an algebra A is any ideal $J \subseteq A$ such that if $J \subsetneq J'$ then $J' = A$.
 Notation: M_A = the set of all maximal ideals of A .

Ex:

Pangea

Reminder: $C(X) = \{f: X \rightarrow \mathbb{R} / f \text{ continuous}\}$, with algebraic operations:

- multiplication by scalars, λf
- sum of functions, $f+g$
- products of functions, $f \cdot g$

$\left. \begin{matrix} \text{vector space} \\ \text{ring} \end{matrix} \right\} \quad \boxed{C(X) \text{ is an algebra}}$

Ex: For $A = C(X)$

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Def: A homomorphism between two algebras A and B [4] is any map $F: A \rightarrow B$ satisfying

$$\left. \begin{matrix} F(\lambda a) = \lambda F(a) \quad \forall \lambda \in \mathbb{R}, a \in A \\ F(a+b) = F(a) + F(b) \\ F(ab) = F(a) \cdot F(b) \\ F(1) = 1 \end{matrix} \right\}$$

Def: A character of an algebra A is any algebra homomorphism

Notation: $X_A :=$ the set of all characters of A
 ↙ also called: the spectrum of A .

[2-]

Def: An ideal of an algebra A is any $J \subseteq A$ satisfying:

- J is a vector subspace of A
- J is a ring ideal, i.e.: $a \cdot x$ and $x \cdot a \in J \quad (\forall \begin{cases} a \in A \\ x \in J \end{cases})$

Ex: $J = \{0\}$ and $J = A$ are always ideals

Rk:

[3-]

Def: A maximal ideal of an algebra A is any ideal $J \subseteq A$ satisfying
 \nexists ideal J' with $J \subsetneq J' \subseteq A$

Notation: M_A := the set of all maximal ideals of A .

Ex:

Definition $(A = \{f : X \rightarrow \mathbb{R} / f \text{ continuous}\})$, with algebraic operations
 - multiplication by scalars $\lambda \in \mathbb{R}$
 - sum of functions $f + g$
 - product of functions $f \cdot g$
 - inverse of function f^{-1}
 $\Rightarrow A$ is an algebra

For $A = C(X)$, we have two subspaces:
 $\{f \in C(X) / f(0) = 0\}$ and $\{f \in C(X) / f(1) = 0\}$

For $A = C(X)$, we have two ideals:
 $I_0 = \{f \in C(X) / f(0) = 0\}$ and $I_1 = \{f \in C(X) / f(1) = 0\}$

For $A = C(X)$, we have two maximal ideals:
 $J_0 = \{f \in C(X) / f(0) = 0\}$ and $J_1 = \{f \in C(X) / f(1) = 0\}$

Proposition (3.10.2.3) X is a Hausdorff space. Then for the algebra $A = C(X)$
 (i) each $\{f \in C(X) / f(x) = 0\}$ is of type I_0 for some $x \in X$
 (ii) each $\{f \in C(X) / f(x) = 0\}$ is of type I_1 for some $x \in X$

Furthermore, one obtains a bijection
 $F : X \rightarrow J_{C(X)}$, $x \mapsto f_x$

For each $x \in X$,
 $f_x : C(X) \rightarrow \mathbb{R}$, $f_x(f) = f(x)$
 f_x is called the evaluation at x .

N. 4(i) Assume $J \subseteq C(X)$ is a maximal ideal. Then $J = J_0$ or $J = J_1$

proved by contradiction: $J \neq J_0 \cup J_1$
 $\Rightarrow \{x \in X / f(x) \neq 0\}_0 \cap \{x \in X / f(x) \neq 0\}_1 \neq \emptyset$
 Hence $\{U_x / x \in X\}$ an open cover of X . By compactness
 $\Rightarrow \exists x_0 \in X$ st $\{U_{x_0} / x_0 \in X\}$ is an open cover of X .
 $\{U_{x_0} / x_0 \in X\} \cap \{U_{x_1} / x_1 \in X\} \neq \emptyset$

Now set $w = f_{x_0}^2 + f_{x_1}^2 + \dots + f_{x_n}^2 > 0$ on the circle X
 $\Rightarrow f_{x_0}(w) > 0$ and $f_{x_1}(w) > 0$ and \dots
 Hence J contains w and all $f_{x_i}(w)$ (induction)

Definition $F : A \rightarrow \text{continuous functions from } X \text{ to } \mathbb{R}$
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Definition X_A is the set of all characters of A
 also called the spectrum of A

Definition A is a real algebra.
 (i) A is a vector space over \mathbb{R}
 (ii) A has a multiplication which is commutative and associative
 (iii) A has a unit element 1_A
 (iv) A has additive inverses
 (v) A has a zero element

For $A = C(X)$, $C(X)$ is a real algebra.
 $\Rightarrow C(X) / I_0$ is a real algebra.
 $\Rightarrow C(X) / J_0$ is a real algebra.

Definition A is a real algebra.
 (i) A is a vector space over \mathbb{R}
 (ii) A has a multiplication which is commutative and associative
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 (v) A has a zero element

Reminder: $C(X) = \{f: X \rightarrow \mathbb{R} / f \text{ continuous}\}$, with algebraic operations

- multiplication by scalars, λf } vector space
- sum of functions, $f+g$ } ring
- product of functions, $f \cdot g$

$$1 \in C(X)$$

$C(X)$ is an algebra

Ex: For $A = C(X)$: any subset $S \subseteq X$ induces an ideal

$$J_S = \{f \in C(X) : f|_S = 0\}$$

Ex: For $A = C(X)$: any $x \in X$ induces

- an ideal $J_x = \{f \in C(X) : f(x) = 0\}$: which is maximal

• a character $\text{ev}_x : C(X) \rightarrow \mathbb{R}$, $\text{ev}_x(f) = f(x)$

evaluation at x Notice $\text{Ker ev}_x = J_x$.

Def: A homomorphism between two algebras A and B is any map $F: A \rightarrow B$ satisfying

$$F(\lambda a) = \lambda F(a) \quad \forall \lambda \in \mathbb{R}, a \in A \quad \left. \right\} F = \text{linear}$$

$$F(a+b) = F(a) + F(b)$$

$$F(ab) = F(a) \cdot F(b)$$

$$F(1) = 1$$

$F = \text{ring homomorphism}$

Def: A character of an algebra A is any algebra homomorphism: $\chi: A \rightarrow \mathbb{R}$.

Notation: $X_A :=$ the set of all characters of A

also called: the spectrum of A .

[-2 -]

Def: An ideal of an algebra A is any $J \subseteq A$ satisfying:

• J is a vector subspace of A

• J is a ring ideal, i.e.: $a \cdot x$ and $x \cdot a \in J \quad \forall \begin{cases} a \in A \\ x \in J \end{cases}$

Ex: $J = \{0\}$ and $(J = A)$ are always ideals

Rk: J is ideal and $\exists u \in J$ which is invertible $\Rightarrow (1 \in J) \Rightarrow J = A$

Ex: If $F: A \rightarrow B$ algebra homomorphism \Rightarrow

$\Rightarrow \text{Ker } F = \{a \in A \mid F(a) = 0\}$ is an ideal in A
Conversely ... (remember what quotient we)

b

[-3 -]

Def: A maximal ideal of an algebra A is any ideal $J \subseteq A$ satisfying:
 \nexists ideal I with $J \subsetneq I \subsetneq A$

Notation (M_A) = the set of all maximal ideals of A .

Ex: Any character $\chi: A \rightarrow \mathbb{R} \Rightarrow \text{Ker } \chi = \{x \in A : \chi(x) = 0\}$
is a maximal ideal

proof that $\text{Ker } \chi = \text{maximal}$:

Assume J is an ideal with $\text{Ker } \chi \subsetneq J \subsetneq A$. We will prove: $J = A$

$\exists a_0 \in J$ s.t. $\chi(a_0) \neq 0$

$\chi(n \cdot a_0)$ should be 0
 $\chi(a) - \chi(a_0)$
 $\lambda = \frac{\chi(a)}{\chi(a_0)} \in \mathbb{R}$
OK

Now: any $a \in A \Rightarrow \exists \lambda \in \mathbb{R}$ s.t. $a = (a - \lambda a_0) + \lambda a_0$
 $\text{Ker } \chi \subset J$ J

$\Rightarrow a \in J$.

Proposition (8.18 & 8.20): $X = \text{cpt} \& \text{Hausdorff}$. Then, for the algebra $A = C(X)$:

- (i) each $J \in M_{C(X)}$ is of type $\boxed{J_x}$ for some $x \in X$ (x will be unique)
- (ii) each $J \in X_{C(X)}$ is of type $\boxed{eV_x}$ for some $x \in X$.

Furthermore, one obtains a bijection:

$$\Phi: X \longrightarrow X_{C(X)}, \quad x \mapsto eV_x$$

For next year:
 $M_A \rightsquigarrow \text{Max Ideals}_A$
 $X_A \rightsquigarrow \text{Spect}_A$

pf of (i): Assume $J \subseteq C(X)$ is a maximal ideal. Claim: $J \subseteq J_x$ for some $x \in X$

→ proceed by contradiction \Rightarrow

$$\Rightarrow (\forall x \in X, \exists f_x \in J \setminus J_x)$$

Hence $\{U_x : x \in X\}$ is an open cover of X . By compactness

$\Rightarrow \exists x_1, \dots, x_n \in X$ s.t.

$$(X = U_{x_1} \cup \dots \cup U_{x_n})$$

$f_x \in J$
AND
 $f_x(x) \neq 0$
 \downarrow
 $f_x \neq 0$ on an entire open $U_x \subseteq X$ containing x

by maximality,
and since $J_x \neq C(X)$,
this would imply $J = J_x$

$$f: X \rightarrow \mathbb{R}$$

$$f(x_0) \neq 0$$

$$x_0 \in U = f^{-1}(1-\alpha, 1+\alpha) \cap (U_{x_1} \cup \dots \cup U_{x_n})$$

Now: set $u := f_{x_1}^2 + f_{x_2}^2 + \dots + f_{x_n}^2 > 0$ on the entire X .

$\overset{>0}{\text{on } U_{x_1}}$ $\overset{>0}{\text{on } U_{x_2}}$ $\overset{>0}{\text{on } U_{x_n}}$ and $u \in J$

Hence: J contains an inv. elt. Hence $J = C(X)$. Contradiction. \square

Proposition (8.18 & 8.20): $X = \text{cpt} \neq \text{Hausdorff}$. Then, for the algebra $A = C(X)$:

- each $\boxed{J} \in M_{C(X)}$ is of type $\boxed{J_*}$ for some $x \in X$ ($*$ will be)
- each $\boxed{J} \in X_{C(X)}$ is of type $\boxed{\text{ev}_x}$ for some $x \in X$ (unique)

Furthermore, one obtains a bijection:

$$\Phi: X \xrightarrow{\psi} X_{C(X)}, \quad x \mapsto \text{ev}_x \quad \begin{cases} \text{For next year:} \\ M_A \rightsquigarrow \text{Max Ideals}_A \\ X_A \rightsquigarrow \text{Spectrum} \end{cases}$$

Pf of (i): Assume $J \subseteq C(X)$ is a maximal ideal. Claim: $J \subseteq J_*$ for some $x \in X$

→ proceed by contradiction ⇒

$$\Rightarrow \forall x \in X, \exists f_x \in J \setminus J_*$$

- AND $f_x(x) \neq 0$.
- by maximality, and since $J_* \neq C(X)$, this would imply $J = J_*$
- if $f_x \neq 0$ on an entire open $U_x \subseteq X$ containing x ,
- then $f_x|_{U_x}$ is continuous.

Hence $\{U_x : x \in X\}$ is an open cover of X . By compactness

$$\Rightarrow \exists x_1, \dots, x_n \in X \text{ s.t. } (X = U_{x_1} \cup \dots \cup U_{x_n})$$

Now: set $u = f_{x_1}^2 + \dots + f_{x_n}^2$. This is in J AND $u > 0$ everywhere ⇒ $u = \text{invertible}$ $\Rightarrow J = C(X)$

Proof of (ii): Let $x \in X_{C(X)}$.

Can apply (i) to $\text{Ker } x \Rightarrow \exists z \in X$ s.t. $\text{Ker } x = J_z$. $(\frac{1}{u} \in C(X) \text{ and } u \frac{1}{u} = 1)$

Now, for $f \in C(X)$ arbitrary: $f - x(f) \cdot 1 \in C(X)$ is actually in $\text{Ker } x \Rightarrow$

$$x(f) - x(f)(x(f) \downarrow) = x(f) - x(f) \xrightarrow{x(f)} = 0$$

⇒ $f - x(f) \cdot 1$ is 0 when evaluated at $*$ ⇒ $f(*) - x(f) = 0 \Rightarrow x(f) = f(*) = \underline{\text{ev}_x(f)}$

Hence $x = \underline{\text{ev}_x}$.

All together ⇒ Φ will a surjective function

It is also injective because: $\forall x, y \in X, x \neq y \exists f \in C(X) \text{ s.t. } f(x) \neq f(y) \Rightarrow$

$$\Rightarrow \text{ev}_x(f) \neq \text{ev}_y(f) \Rightarrow \text{ev}_x \neq \text{ev}_y \text{ i.e. } \phi(x) \neq \phi(y)$$
. Hence ϕ is also injective.

ef: Return to general algebras A : For $a \in A$

under the following function on the spectrum:

$$F_a: X_A \rightarrow \mathbb{R}, \quad F_a(x) = \lambda(a)$$

X_A consider J = the smallest topology which

Theorem (8.22): For any compact Hausdorff X

the topological spectrum of $A = C(X)$ is homeomorphic to X . More precisely, $\phi: X \xrightarrow{\text{compact}} X_{C(X)} \xrightarrow{\text{Hausdorff?}} C(X)$ is a homeomorphism.

Pf:

$\phi = \text{continuous: proof: To prove: } f_a \text{ s.t. } a = \int f_a d\mu$

$\forall x \in X, \exists f \in J \setminus J_x$
 Hence $\{U_x : x \in X\}$ are open
 cover of X . By compactness
 $\Rightarrow \exists x_1, \dots, x_n \in X$ s.t.
 $(X = U_{x_1} \cup \dots \cup U_{x_n})$

AND
 if $f_x(x) \neq 0$,
 \downarrow
 $f_x \neq 0$ on an entire open $U_x \subseteq X$ containing x .
 \Downarrow
 $f(x) \neq 0$ for all $x \in U = f^{-1}([-1, 1] \setminus \{0\})$

point since $J_x = C(X)$
 this would imply $J = J_x$

All together $\Rightarrow \phi$ will be a surjective function.
 It is also injective because: $\forall x, y \in X, x \neq y \exists f \in C(X)$ s.t. $f(x) \neq f(y) \Rightarrow$
 $\Rightarrow ev_x(f) \neq ev_y(f) \Rightarrow ev_x \neq ev_y$ i.e. $\phi(x) \neq \phi(y)$. Hence ϕ is also injective.

Def: Return to general algebras A : For $a \in A$ [-7+]
 consider the following function on the spectrum:
 $F_a : X_A \rightarrow \mathbb{R}, F_a(x) = x(a)$

On X_A consider \mathcal{T} = the smallest topology which
 make all the functions F_a (one for each $a \in A$) continuous.
 The topological spectrum of A : (X_A, \mathcal{T}) .

RQ: What must be in \mathcal{T} ?

- for each $a \in A$, $D \subseteq \mathbb{R}$ open: $\bigcup_{a \in D} = F_a^{-1}(D) = \{x \in X_A : x(a) \in D\}$ must be in \mathcal{T} [we have a collection of subsets]
- for each $a_1, \dots, a_n \in A$, $D_1, \dots, D_n \subseteq \mathbb{R}$ open: $U_{a_1, D_1} \cap \dots \cap U_{a_n, D_n}$ must be in \mathcal{T} [we produce a basis of \mathcal{B}]
- now take unions of such finite intersections, they must be in \mathcal{T} too.
 these do form a topology (our \mathcal{T} !)

GELFAND-NAIMARK [-8-]
Theorem (8.22): For any compact Hausdorff X
 the topological spectrum of $A = C(X)$ is homeomorphic to X . More precisely, $\phi : X \rightarrow X^{C(X)}$ is a homeomorphism.
 pf:
 $\phi = \text{continuous}$: proof. To prove: for any $a \in A = C(X)$, $D \subseteq \mathbb{R}$, $\phi^{-1}(U_{a, D})$ = open in X .
 Look at $\phi^{-1}(U_{a, D}) = \{x \in X : \phi(x) \in U_{a, D}\} = \{x \in X : ev_x(a) \in D\} = \{x \in X : a(x) \in D\}$ ($a \in A = C(X)$)
 $= a^{-1}(D)$ which is open in X because $a : X \rightarrow \mathbb{R}$ continuous.
 $X^{C(X)}$ = Hausdorff: proof. Let $x_1, x_2 \in X_A$ with $x_1 \neq x_2$.
 We can write $x_1 = ev_{x_1}, x_2 = ev_{x_2}$ with $x_1, x_2 \in X$, $x_1 \neq x_2$.
 Now: $\begin{cases} x_1 \in U_{f, D_1} \text{ and } U_{f, D_1} \cap U_{f, D_2} = \emptyset \\ x_2 \in U_{f, D_2} \end{cases}$ we do find $f \in C(X)$ s.t. $f(x_1) \neq f(x_2)$.
 also find $D_1, D_2 \subseteq \mathbb{R}$ open, $D_1 \cap D_2 = \emptyset$ with $f^{-1}(D_1) \cap f^{-1}(D_2) = \emptyset$.

Exercise $(X) = \{f : X \rightarrow \mathbb{R} / f \text{ continuous}\}$, with algebra operations

- multiplication by scalars, $f \cdot g$
- sum of functions, $f + g$
- product of functions, $f \cdot g$
- $\mathbb{R} \in C(X)$

Ex: For $A = C(X)$, any subset $S \subseteq X$ induces an ideal
 $I_S = \{f \in C(X) : f|_S = 0\}$

For $A = C(X)$, any σ -ideal induces

- an ideal $I_\sigma = \{f \in C(X) : f(\sigma) = 0\}$ which is maximal
- a character $\phi : C(X) \rightarrow \mathbb{R}$, $\phi(f) = f(\sigma)$
- $\text{Ker } \phi = I_\sigma$

Proposition (8.1.1 & 8.2.2) X is a compact Hausdorff. Then, for the algebra $A = C(X)$

- each $\overline{\{f\}} \subset M_{C(X)}$ is of type $\overline{J_f}$ for some $\sigma \in X$ (σ will be $\text{Ker } \phi$)
- each $\overline{\{f\}} \subset M_{C(X)}$ is of type $\overline{J_\sigma}$ for some $\sigma \in X$ (unique)

Furthermore, one obtains a big ideal

$$I_f : X \longrightarrow X, \quad x \longmapsto \psi_x, \quad \psi_x \text{ non-zero}$$

Id (ii): Assume $J \subseteq C(X)$ is a maximal ideal. Claim $J \subseteq J_f$

→ proved by contradiction with $J_f \subsetneq J$

$\Rightarrow (\forall x \in X, \exists f \in J) \setminus J_f$

for $x \in J_f$: $f(x) = 0$ (by definition of J_f)
 $\Rightarrow f(x) = 0$ and $x \in J_f$ (by definition of J_f)
 $\Rightarrow x \in J_f$ (contradiction)

However $\{U_\alpha \subset X\}$ an open cover of X by compact sets
 $\Rightarrow \exists x_\alpha \in X$ s.t.
 $\forall \alpha \in \text{Index } U_\alpha, \exists f_\alpha \in J$ s.t.
 $f_\alpha(x_\alpha) = 0$

Now set $\omega = \sum \frac{1}{n} f_n$ (with $n \in \mathbb{N}$)

Then $\omega \in J$ but $\omega(x_\alpha) = 0$ for all α

Point of (ii): Let $x \in X$

→ $\exists \omega \in J$ s.t. $\text{Ker } \omega = J_\omega$

Can apply (i) $\Rightarrow \exists \omega \in J$ s.t. $\text{Ker } \omega = J_\omega$

Now for $f \in C(X)$ relations $f - \frac{f(x)}{x} \in J_\omega$ is actually in $\text{Ker } \omega$ \Rightarrow

$\Rightarrow f - \frac{f(x)}{x} \in J_\omega \Rightarrow f(x) = f(x) - \frac{f(x)}{x} = 0$

Hence $\omega(x) = 0$

All together we get well a surjective function

If ϕ also injective because $\forall x, \forall y \in X, \forall f \in C(X) \text{ s.t. } f(x) = f(y) \Rightarrow$

$\Rightarrow \psi_x(f) = \psi_y(f) \Rightarrow \forall f \in C(X), \phi(f) = \phi(f)$ Hence ϕ is also injective

Def An ideal of an algebra A is a non-empty subring

- I is a vector subspace of A
- I is a ring under \oplus, \cdot (i.e. a subring of (A, \oplus, \cdot))

Ex $I = \{0\}$ and A are always ideals

Ex If $A = \mathbb{R}$ algebra homomorphism $\phi : A \rightarrow A$
 $\Rightarrow \text{Ker } \phi = \{a \in A / \phi(a) = 0\}$ is an ideal

Consequently ... (reminds old question on)

Def Return to general algebras A . For $a \in A$

consider the following function on the spectrum

$F_a : X_A \rightarrow \mathbb{R}, F_a(x) = x(a)$

On X_A consider \mathcal{T} = the smallest topology which makes all the functions F_a (and the constant functions) continuous

The topological position of A (X_A, \mathcal{T})

What must be in \mathcal{T} ?

- for each $a \in A$, $D(a) = \{x \in X_A : F_a(x) \neq 0\}$ must be \mathcal{T} -open
- for each $a, b \in A$, $D(a+b) = \{x \in X_A : F_a(x) + F_b(x) \neq 0\}$ must be \mathcal{T} -open
- now take union of such sets (disjoint) → they need to be \mathcal{T} -open
- these do form a topology (check \mathcal{T})

LEMMA (8.2.1) \mathcal{T} being compact Hausdorff X

the topological position of $A = C(X)$ is homeomorphic to X . More precisely, $\phi : X \rightarrow X_A$ is a homeomorphism

→ $\phi : X \rightarrow X_A$ is a homeomorphism

THEOREM (8.2.2) To prove \mathcal{T} being compact Hausdorff, $\phi^{-1}(U_\alpha)$ is open in X

Since $\phi^{-1}(U_\alpha) = \{x \in X : F_{U_\alpha}(x) \neq 0\} = \{x \in X : \text{Ker } \phi \cap U_\alpha \neq \emptyset\} = \phi^{-1}(\text{Ker } \phi \cap U_\alpha)$

$\text{Ker } \phi \cap U_\alpha$ is open in X because X is compact Hausdorff. And $\text{Ker } \phi \cap U_\alpha$ with \mathcal{T} is closed in X

We can write $\text{Ker } \phi = \bigcup_{\sigma \in X} \text{Ker } \phi \cap U_\sigma$ with $\text{Ker } \phi \cap U_\sigma$ closed in X (why?)

Now $\text{Ker } \phi \cap U_\alpha = \bigcup_{\sigma \in U_\alpha} \text{Ker } \phi \cap U_\sigma$ with $\text{Ker } \phi \cap U_\sigma$ closed in X (why?)

$\Rightarrow \text{Ker } \phi \cap U_\alpha = \bigcup_{\sigma \in U_\alpha} \text{Ker } \phi \cap U_\sigma$ with $\text{Ker } \phi \cap U_\sigma$ closed in X (why?)

Def A maximal ideal of an algebra A is a non-empty subring

- $I \neq \{0\}$ and $I \neq A$

Notation M_A = the set of all maximal ideals of A

Ex Any ideal $I \neq A = \mathbb{R}$ $\Rightarrow \text{Ker } \phi = \{0\}$ is a maximal ideal

point that $\text{Ker } \phi$ is maximal

A maximal ideal with $\text{Ker } \phi = \{0\}$ has no prime ideal

$\frac{1}{x} \in \mathbb{R}[x]$ is irreducible

Now we have $A = \mathbb{R}[x] / \langle \text{Ker } \phi \rangle$ with $\text{Ker } \phi = \{0\}$

$\Rightarrow A = \mathbb{R}[x] / \langle \text{Ker } \phi \rangle$ with $\text{Ker } \phi = \{0\}$