Reminder: Topology $\xrightarrow{\boxed{-1-}}$ Algebra, | space $X \longmapsto$ the algebra $C(X)$

Gelfand-Naimark: about going backwards: | space $X_A \longleftarrow\!\shortmid$ algebra A

Explicitly, the topological spectrum X_A of A is given by:

→ as a set: $X_A = \{ \chi : A \to \mathbb{R} \,/\, \underbrace{\chi = \text{linear},\ \chi(ab) = \chi(a)\chi(b)\ (\forall)\, a, b,\ \chi(\overset{1=1_A\in A}{1}) = \overset{\in\mathbb{R}}{1}}_{\chi \text{ is an algebra homomorphism } \chi : A \to \mathbb{R} \text{ (character of } A)} \}$

→ as a space: endowed with the smallest topology on X_A such that, for all $a \in A$, $F_a : X_A \to \mathbb{R}$, $F_a(\chi) = \chi(a)$ is continuous

(Note: in this way A is related to $C(X_A)$ by an algebra homomorphism $A \to C(X_A)$, $a \mapsto F_a$)

$f(x_1, \dots, x_n) = (x_1, \dots, x_n)$

$\boxed{\text{G-N}: X = \text{cpct, Hausdorff} \Rightarrow X_{C(X)} \text{ is homeomorphic to } X.}$

Remark 1. In any algebra A

- we have two distinguished elements $0 = 0_A$, $1 = 1_A \in A$
- for any $a \in A \Longrightarrow$ lots of new elements of A: $\lambda_0 \cdot 1 + \lambda_1 \cdot a + \dots + \lambda_n a^n \in A$ $(\forall)\ n \in \mathbb{N}$, $\lambda_0, \lambda_1, \dots, \lambda_n \in \mathbb{R}$.

Examples of algebras A [-2-]	Special elements in A? [-3-]	Relations? [-4-]
$C(X)$ in general	no	—
$C(\mathbb{R})$	$id \in C(\mathbb{R})$	no
$C(\mathbb{R}^n)$	$pr_1, \dots, pr_n \in C(\mathbb{R}^n)$	no
$C(S^1)$	$f, g \in C(S^1)$, $\begin{cases} f(x,y) = x \\ g(x,y) = y \end{cases}$	~~$f^2+g^2 = f\cdot f + g\cdot g : S^1 \to \mathbb{R}$~~ $f^2+g^2=1$
$\mathbb{R}[T]$	$T \in \mathbb{R}[T]$ (generates the entire $\mathbb{R}[T]$)	
$A = \mathbb{R}[T]/(T^3)$ (polynomials modulo T^3)	$u = \hat{T}$	$u^3 = 0$
$\mathbb{R}[T,S]$	T, S (generate the entire $\mathbb{R}[T,S]$)	
$C_{pol}(S^1)$ = functions $S^1 \to \mathbb{R}$ induced by polynomials $P \in \mathbb{R}[T,S]$; $(x,y) \mapsto P(x,y)$	$f, g \in C_{pol}(S^1)$ from above $f(x,y)=x$, $g(x,y)=y$	$f^2+g^2=1$
$\mathbb{R}[T,S]/(T^2+S^2-1)$	$\hat{T}, \hat{S}$	$\hat{T}^2+\hat{S}^2=1$.
$\mathcal{C}^\infty(\mathbb{R}^n) = \{f : \mathbb{R}^n \to \mathbb{R} / f = \text{smooth}\}$		

Remark: F…

- $\chi(0_A)$…
- if …

$\Rightarrow$ …

- kee…

aft…

Example: …

$\Rightarrow$ arbitra…

Are the…

Because…

Is this … one …

All t…

now…

$\alpha : A \to \mathbb{R}$

$$\frac{\begin{array}{c}\chi(f) = u \in \mathbb{R} \\ \chi(g) = v \in \mathbb{R}\end{array}}{\chi(\underbrace{f^2+g^2}_{1}) = u^2+v^2}$$

$$\overset{\shortparallel}{1}$$

$f : V \to W$

$f(\lambda v) = \lambda f(v)$

$f(v+w) = f(v) + f(w)$

$\mathbb{R}[T,S]$ $\quad T^2+S^2=1$

$u, v \in \mathbb{R}$ $\quad u^2+v^2 = 1$.

$\exists^{?}\ \chi : A \to \mathbb{R}$ s.t. $\chi(f) = u$, $\chi(g) = v$?

$\chi(a_0 + a_1 f + a_2 f^2 + \dots + a_n f^n + b_1 g + b_2 fg + \dots + b_n f^n g)$

$\chi(f^2) = u^2$, $\chi(g^2) = v^2$

$$\Downarrow$$

$1 = u^2+v^2$

$\shortparallel \qquad \shortparallel$

$\chi(1) = \chi(f^2+g^2)$

$a + bf + \dots +$

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Remark: For any $\chi \in X_A$:

- $\chi(0_A) = 0$, $\chi(1_A) = 1$
- if $a \in A$ and we know $\chi(a) =: r \in \mathbb{R} \Rightarrow$
  $\Rightarrow$ we know what χ does on all elements of type $\lambda_0 1 + \lambda_1 a + \dots + \lambda_n a^n$:
  $$\chi(\lambda_0 1 + \lambda_1 a + \dots + \lambda_n a^n) = \lambda_0 + \lambda_1 r + \dots + \lambda_n r^n$$
- keep in mind: also relations are preserved after applying χ!

Example: $A = C_{pol}(S^1)$. We have: f, g generating the entire A

$\Rightarrow$ arbitrary $\chi \in X_A$ are uniquely determined by two numbers:

$\chi(f), \chi(g) \in \mathbb{R}$ $\qquad$ $\chi(f) = 1$, $\chi(g) = 2$

Are these numbers arbitrary? No!

Because $f^2 + g^2 = 1 \Longrightarrow \chi(f)$ and $\chi(g)$ must satisfy
$$\chi(f)^2 + \chi(g)^2 = 1. \qquad \boxed{(u, v) \in S^1}$$

Is this all? i.e. if we have $u, v \in \mathbb{R}$ s.t. $u^2 + v^2 = 1$, can one find $\chi: A \to \mathbb{R}$ s.t. $\begin{cases} \chi(f) = u \\ \chi(g) = v \end{cases}$? Define $\boxed{\chi(h) := h(\underbrace{u, v}_{\in S^1})}$, $h \in A$ arbitrary.

All together: $X_A \longrightarrow S^1$, $\chi \mapsto (\chi(f), \chi(g))$ $\qquad$ $f(x)$

now: prove this is a homeomorphism.

Principle 1: Complicated spaces can often be broken into simpler spaces.

For instance: topological d-dimensional manifolds ... i.e. spaces X s.t.:

① X = locally euclidean: $\exists$ open cover $\{U_i : i \in I\}$ of X such that each $U_i \cong \mathbb{R}^d$ (homeomorphic to)

② ... X = Hausdorff and 2^{nd} countable.

Q: How to go from the U_i's to the entire X?

Principle 2: Use $C(X)$.

partitions of 1 come in as a tool

PARTITIONS OF UNITY

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Problem: If $U \subseteq X$ open, $g: U \to \mathbb{R}$ continuous

$$\Rightarrow \quad \tilde{g}: X \longrightarrow \mathbb{R}, \quad \tilde{g}(x) = \begin{cases} g(x) & x \in U \\ 0 & \text{otherwise} \end{cases}$$

may fail to be continuous (Ex: $X = \mathbb{R}$, $U = (0,1)$, $g(t) = t$).

Rk: If $\exists$ $\boxed{A \subseteq U \text{ with } A \text{ closed in } X}$ (*)

such that $g = 0$ on $U \setminus A \Rightarrow \tilde{g}$ is continuous.

(Exercise)

Def (simplified versio…

η is supported i…

Def: X = space, U…

A partition of u…

is a collectio…

consisting of co…

i). $\eta_1 + \dots$

ii) eac…

Expl 1: Why 1 and not arbitrary $f \in C(X)$? -9-

Answer: not necessary: for arbitrary f multiply the eq in i) by $f \Rightarrow \underbrace{\eta_1 f}_{f_1} + \dots + \underbrace{\eta_n f}_{f_n} = f$

$\Rightarrow f = \underbrace{f_1}_{\text{Supp}} + \dots + \underbrace{f_n}_{\text{Supp}}$

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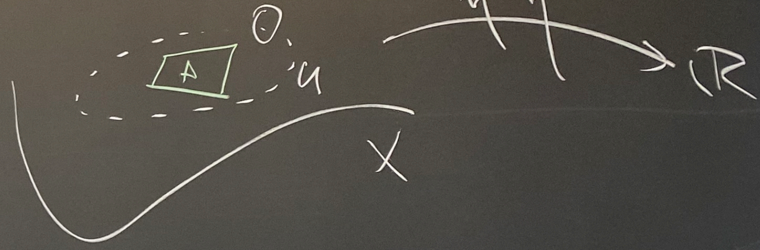
Def (simplified version of 5.4): Given $U \subseteq X$ open, $\eta \in C(X)$, we say that η is supported in U if $\exists\, A$ as in $(*)$ s.t. $\eta = 0$ on $X \setminus A$.



Def: X = space, $\mathcal{U} = \{U_1, \dots, U_n\}$ open cover of X.

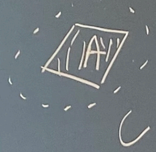
A partition of unity subordinated to $\mathcal{U}$ is a collection $\{\eta_1, \dots, \eta_n\}$ consisting of continuous functions $\eta_1, \dots, \eta_n : X \longrightarrow [0,1]$ s.t.:

i). $\eta_1 + \dots + \eta_n = 1$ (the constant function)

ii) each η_i is supported in U_i.

Such that $g=0$ on ...

(Exercise)



Expl 1: Why 1 and not arbitrary $f \in C(X)$? $\boxed{-g-}$

Answer: not necessary: for arbitrary f multiply the eq in i) by $f \Rightarrow \underbrace{\eta_1 f}_{f_1} + \dots + \underbrace{\eta_n f}_{f_n} = f$

$\Rightarrow f = \underbrace{f_1}_{\substack{\text{supp} \\ \text{in } U_1}} + \dots + \underbrace{f_n}_{\substack{\text{supp} \\ \text{in } U_n}}$.

Theorem: If $X =$ Hausdorff and normal then $(\forall)$ finite open cover $\mathcal{U}$ $\exists$ a partition of unity subordinated to $\mathcal{U}$!