Inleiding Topologie, hand in exercise 1

16 November 2023

Hand in exercise 1: topology on the integers

For $n, k \in \mathbb{Z}$, $k \neq 0$, we denote by $A(n,k)$ the set of integers that are congruent to n modulo k:

$$A(n,k) = \{\ \dots \quad , \quad -3k+n, \quad -2k+n, \quad -k+n, \quad n, \quad k+n, \quad 2k+n, \quad 3k+n, \quad \dots\ \} \subset \mathbb{Z}.$$

We denote by $\mathcal{T}$ the collection of subsets $U \subset \mathbb{Z}$ with the property that: for every $n \in U$, there exists a nonzero integer k such that $A(n,k) \subset U$. Please do the following:

1. prove that $\mathcal{T}$ is a topology on $\mathbb{Z}$.
2. all the subsets $A(n,k)$, with $n, k \in \mathbb{Z}$, $k \neq 0$, are both open as well as closed in $(\mathbb{Z}, \mathcal{T})$.

Bonus question - for extra points:

3. prove that

$$\mathbb{Z} \backslash \{1, -1\} = \bigcup_{p \text{ a prime number}} A(0,p)$$

   and, using this, prove an important property of the prime numbers.

Hand in instructions

- The due date for this assignment is 23/11/2023 at midnight.
- Please submit the assignment via blackboard. The solution can be handwritten or in LaTeX, English or Dutch.
- Please submit the assignment in a single pdf file (do not submit multiple scans!).
- The homework will be corrected taking into account also the mathematical quality of the writing, which will count for 10 percent of the mark. Hence please use clear sentences, make sure your solution is readable and structured, explain what you do and give all details/prove all your claims.