Homework 3 Topology

- The deadline is october 7th before the werkcollege
- Your solution may be handwritten (please write clearly) or latexed. Handwritten solutions should be handed in physically, latexed solutions via blackbloard

This exercise will be about the following plausible-sounding conjecture:
Let A, B and X be topological spaces. If A can not be embedded into B then $A \times X$ can not be embedded into $B \times X$.
This conjecture turns out to be false however. We can take $A = S^1$ and $B = X = \mathbb{R}$ as a counterexample.

(a) Proof that this is indeed a counterexample, i.e. show that S^1 can not be embedded into $\mathbb{R}$, but $S^1 \times \mathbb{R}$ can be embedded into $\mathbb{R}^2$.

What makes this conjecture sound reasonable, is an analogy with positive real numbers. Being embeddable is about if the ambient space is big enough to fit to space we want to embed. For positive real numbers, it indeed is true that if $a > b$, then also $ax > bx$. This analogy with numbers already reveals another simpler counterexample. If we don't look at positive numbers, but allow x to be 0, then the second inequality actually becomes an equality. The analogue of 0 for topological spaces is the empty topological space, the one with the empty set as underlying set and where the empty set is the only open. If $X = \emptyset$, then also $A \times X = B \times X = \emptyset$, set $A \times X$ can be embedded into $B \times X$.
If we allow ourselves to be creative, then $x = \infty$ would also be a counterexample. Then both ax and bx would be infinite, so we also get an equality. This off course is not rigorous, but it raises the question if we can make it rigorous for topological spaces.
We only defined the product of two topological spaces. With this, we can recursively define the product of a finite amount of spcaes, but we have not given a definition to an infinite product of topological spaces. In the second part of this homework, we will try to make sense of $\mathbb{R}^\infty$, so that we also produce the counterexample $A = \mathbb{R}^2$, $B = \mathbb{R}$ and $X = \mathbb{R}^\infty$. Because if we succeed to define $\mathbb{R}^\infty$ sensibly, it should be an absorbant space such that $\mathbb{R}^2 \times \mathbb{R}^\infty \cong \mathbb{R} \times \mathbb{R}^\infty \cong \mathbb{R}^\infty$.
We have seen several ways to make topological spaces, so we will try out different ways to define $\mathbb{R}^\infty$. The first we need to define is the underlying set. The space $\mathbb{R}^\infty$ should consist of all the infinite sequence $(x_1, x_2, \dots)$ of real numbers.

(b) For the first topology, we explicitly say which sets are open. For $j \in \mathbb{N}$, we define

$$p_j : \mathbb{R}^\infty \to \mathbb{R}, (x_i)_{i \in \mathbb{N}} \mapsto x_j.$$

We now say $U \subset \mathbb{R}^\infty$ is open if $p_j(U)$ is open in $\mathbb{R}$ for every $j \in \mathbb{N}$. Prove that we indeed have $\mathbb{R} \times \mathbb{R}^\infty \cong \mathbb{R}^\infty$ if we endow $\mathbb{R}^\infty$ with this topology. You don't have to prove this is a topology.

(c) Now show that we also have $\mathbb{R}^n \times \mathbb{R}^\infty \cong \mathbb{R}^\infty$.
Hint, you may want to use induction, and associativity of the product of topological spaces, $(X \times Y) \times Z \cong X \times (Y \times Z)$, which you don't need to prove.

(d) Another way to construct topological spaces is via metrics. We can define a metric d on $\mathbb{R}^\infty$ by setting

$$d((x_i)_{i\in\mathbb{N}}, (y_i)_{i\in\mathbb{N}}) = \min(1, \max_{i\in\mathbb{N}} |x_i - y_i|)$$

Show that if we endow $\mathbb{R}^\infty$ with the topology induced from this metric (you again don't have to prove this is a metric), we also have $\mathbb{R}^n \times \mathbb{R}^\infty \cong \mathbb{R}^\infty$.

(e) **Bonus exercise**. These are some additional questions to think about. If you show you taught about them, you can earn upto three bonus points.

(I) A priori these two topologies could be the same, are they?

(II) How is this related to function spaces?

(III How is this related to Hilbert's hotel?