HOMEWORK 4, GIVEN ON DECEMBER 7, 2023

Recall that in the lectures, the projective space (defined as the collection of all lines through the origin) was interpreted as

$$\mathbb{P}^n = X/R,$$

the quotient of $X := \mathbb{R}^{n+1} \setminus \{0\}$ (endowed with the Euclidean topology) modulo the relation R defined by:

$$x \sim_R x' \iff x' = \lambda x \quad \text{for some } \lambda \in \mathbb{R}^*.$$

Our first goal is to show that $\mathbb{P}^n$ can also be interpreted as "the space obtained from the n-sphere

$$S^n = \{x \in \mathbb{R}^{n+1} : ||x|| = 1\} \quad \left(||x|| = \sqrt{(x_0)^2 + (x_1)^2 + \ldots + (x_n)^2} \right)$$

after a certain gluing. To that end define the following relation on S^n:

$$S := \{(x, y) \in S^n \times S^n : y = x \text{ or } y = -x\}.$$

1). Describe the gluing that corresponds to S.

2). Show that the inclusion $i : S^n \to X$ induces an map $j : S^n/S \to X/R$

$$\begin{CD} S^n @>{i}>> X \\ @V{\pi_S}VV @VV{\pi_R}V \\ S^n/S @>>{j}> X/R \end{CD}$$

Furthermore, show that the map j is continuous (hopefully making use of the commutative diagram above).

3). Show that j is a bijection, describe its inverse j^{-1} explicitly, fit j^{-1} in a commutative diagram similar to the one above, then prove that also j^{-1} is continuous.

So far we have shown that $\mathbb{P}^n$ is homeomorphic to S^n/S. Next, we want to prove that the projective space could also be obtained from the n-ball

$$D^n = \{x \in \mathbb{R}^n : ||x|| \leq 1\}$$

by gluing any two antipodal points that are on the boundary $S^{n-1} \subset D^n$ (we leave anything in the interior of the ball unchanged!).

4). Define the equivalence relation R' on D^n that encodes the gluing that we have just described.

5). Define a bijection $f : S^n/S \to D^n/R'$ (don't forget to check it is well defined) and prove that it is continuous.

6). Describe the inverse of f and prove that it continuous as well.

Hand-in instructions: Hand in your solution via Blackboard. If you scan your solutions, combine your scans into a single file and make sure everything is readable. The deadline is December 14, at 15:15.