Homework 5 Inleiding Topologie*

Let $f\colon X \to Y$ be a continuous map between topological spaces.

- We say f is *open* if for every open subset $U \subseteq X$, the subset $f(U) \subseteq Y$ is open.
- We say f is *closed* if for every closed subset $A \subseteq X$, the subset $f(A) \subseteq Y$ is closed.
- We say that it is a *local homeomorphism* if for each $x \in X$ there exists an open neighborhood U of x in X, and V of $\pi(x)$ in Y such that the restriction of π to U becomes a homeomorphism $\pi|_U : U \to V$.
- Given a subset $A \subset X$, *the inclusion of A into X* is the map $\iota : A \to X$ given by $\iota(a) = a$ for $a \in A$ (remember that a function is not just a formula, it consists of three pieces: a domain (which is a set), a codomain (also a set) and a rule (usually a formula) that associates to an element in the domain an element in the codomain). If X is a space then, by default, we endow A with the topology induced from X (opens in A being those of type $A \cap U$ with U open in X).

Exercise:

(a) For which intervals $I \subset \mathbb{R}$ is the inclusion $\iota : I \to \mathbb{R}$ an open map? And for which ones is it a closed map?

(b) Show that if $I \subset \mathbb{R}$ is neither empty nor the entire $\mathbb{R}$, then the inclusion $\iota : I \to \mathbb{R}$ cannot be both an open map, as well as a closed map, at the same time.

(c) Show that if $f\colon X \to Y$ is a continuous bijective map then

$$f \text{ is a homeomorphism} \iff f \text{ is an open map} \iff f \text{ is a closed map.}$$

(d) Remembering our discussion that S^1 can be obtained as a quotient of $[0, 1]$ obtained by gluing the end points, provide an explicit example of a topological quotient map that is not open.

(e) On the other hand, for quotient maps $\pi : X \to X/\Gamma$ corresponding to actions of groups Γ on a space X, prove that π is automatically open.

(f) For which spaces X one can still make a statement as in (b), about all inclusions $\iota : A \to X$ of subsets A of X that are neither empty nor X itself?

(g) Show that, if X is a compact space and Y is a Hausdorff space, then any continuous map $f : X \to Y$ is automatically a closed map. What do you obtain if you combine this with item (c)?

(h) Bonus question: in item (e), if Γ is assumed to be finite, show that the quotient map $\pi : X \to X/\Gamma$ is a local homeomorphism.

*Deadline January 9th, 9AM, via blackboard.