Recall: given a metric space (X,d)

$\longrightarrow$ one defines $\mathcal{T}_d$ as the collection of all subsets $U \subseteq X$ with the property that

$$(\forall)\, x \in U \ (\exists)\, \varepsilon > 0 : B_d(x,\varepsilon) \subseteq U$$

called the topology induced by d

$\longrightarrow$ given a sequence $(x_n)_{n \geq 1}$ in X and $x \in X$, we say that $x_n \longrightarrow x$ in (X,d) if $\lim_{n\to\infty} d(x_n, x) = 0$.

CLAIM: If d, d' are two metrics on X, then the following are equivalent:

(A) (X,d) and (X,d') have the same convergence: $x_n \to x$ in $(X,d) \Longleftrightarrow x_n \to x$ in (X,d')

(B) $\mathcal{T}_d = \mathcal{T}_{d'}$.

Proof of $A \Rightarrow B$: $\boxed{\text{Assume } A.}$

To prove $\mathcal{T}_d = \mathcal{T}_{d'}$ we prove $\mathcal{T}_d \subseteq \mathcal{T}_{d'}$ and $\mathcal{T}_{d'} \subseteq \mathcal{T}_d$.

The two are analogous hence it suffices

TO PROVE: $\mathcal{T}_d \subseteq \mathcal{T}_{d'}$

Let $\boxed{U \in \mathcal{T}_d}$. To prove: $U \in \mathcal{T}_{d'}$

? check def of $\mathcal{T}_{d'}$: Let $\boxed{x \in U}$ ...

To prove: $\exists\, \varepsilon > 0$ s.t. $B_{d'}(x,\varepsilon) \subseteq U$

How? By CONTRADICTION: assume $\nexists$ as above.

$\Rightarrow$ each $\varepsilon = \frac{1}{n}$ with $n \in \mathbb{N}$ fails to satisfy $B_{d'}(x, \frac{1}{n}) \subseteq U \Rightarrow$

$\Rightarrow (\forall)\, n \in \mathbb{N}$ there exists an element, call it x_n, which is in $B_{d'}(x, \frac{1}{n})$ but not in U

i.e. x_n such that: $\boxed{d'(x_n, x) < \frac{1}{n}}$, $\boxed{x_n \notin U}$ $(\lightning)$

But $d'(x_n, x) < \frac{1}{n}$ implies that $x_n \longrightarrow x$ in $(X,d') \xRightarrow[(A)]{\text{by}} x_n \longrightarrow x$ in $(X,d) \Rightarrow$

$\Rightarrow \boxed{d(x_n, x) \longrightarrow 0}$

Also, $\left.\begin{matrix} U \in \mathcal{T}_d \\ x \in U \end{matrix}\right\} \Rightarrow \exists\, r > 0$ s.t. $\boxed{B_d(x,r) \subseteq U}$

by

$\Rightarrow$ for large n one has $x_n \in U$

CONTRADICTION WITH $(\lightning)$

Proof of $B \Rightarrow A$: Assume B. We prove the "$\Leftrightarrow$" appearing in A. Given the symmetry:

to prove: given $\boxed{x_n \longrightarrow x \text{ in } (X,d)}$ we want to prove $x_n \longrightarrow x$ in (X,d')

hence $\boxed{d(x_n, x) \longrightarrow 0}$

to prove: $d'(x_n, x) \longrightarrow 0$

i.e. $\boxed{(\forall)\, \varepsilon > 0}$ $\underbrace{\exists\, n_\varepsilon \text{ s.t. } d'(x_n, x) < \varepsilon \ (\forall)\, n \geq n_\varepsilon}_{\text{To prove}}$

fix ε

i.e.

Use B: since $B_{d'}(x,\varepsilon) \in \mathcal{T}_{d'} \Rightarrow$ it is also in $\mathcal{T}_d \Rightarrow \exists\, \varepsilon_1 > 0$ s.t. $B_d(x, \varepsilon_1) \subseteq B_{d'}(x,\varepsilon)$

But $d(x_n, x) \longrightarrow 0 \Rightarrow \exists\, n_\varepsilon$ s.t. $d(x_n, x) < \varepsilon_1$ $(\forall)\, n \geq n_\varepsilon \Rightarrow x_n \in B_d(x, \varepsilon_1)$ hence in $B_{d'}(x,\varepsilon)$ $(\forall)\, n \geq n_\varepsilon$.

Recall: given a metric space (X, d)

$\longrightarrow$ one defines $\mathcal{T}_d$ as the collection of all subsets $U \subseteq X$ with the property that

$(\forall)\, x \in U\ (\exists)\, \varepsilon > 0 : B_d(x, \varepsilon) \subseteq U$

called the topology induced by d.

$\longrightarrow$ given a sequence $(x_n)_{n \geq 1}$ in X and $x \in X$, we say that $x_n \longrightarrow x$ in (X, d) if $\lim_{n \to \infty} d(x_n, x) = 0$.

CLAIM: If d, d' are two metrics on X, then the following are equivalent:

(A) (X, d) and (X, d') have the same convergence: $x_n \to x$ in $(X, d) \Longleftrightarrow x_n \to x$ in (X, d')

(B) $\mathcal{T}_d = \mathcal{T}_{d'}$.

Proof of $A \Rightarrow B$: $\boxed{\text{Assume } A.}$

To prove $\mathcal{T}_d = \mathcal{T}_{d'}$ we prove $\mathcal{T}_d \subseteq \mathcal{T}_{d'}$ and $\mathcal{T}_{d'} \subseteq \mathcal{T}_d$.

The two are analogous hence it suffices

TO PROVE: $\mathcal{T}_d \subseteq \mathcal{T}_{d'}$

Let $\boxed{U \in \mathcal{T}_d}$. To prove: $U \in \mathcal{T}_{d'}$

? check def of $\mathcal{T}_{d'}$: Let $\boxed{x \in U}$ ...

To prove: $\exists\, \varepsilon > 0$ s.t. $B_{d'}(x, \varepsilon) \subseteq U$

How? By CONTRADICTION: assume $\nexists\, \varepsilon$ as above.

$\Rightarrow$ each $\varepsilon = \frac{1}{n}$ with $n \in \mathbb{N}$ fails to satisfy $B_{d'}(x, \frac{1}{n}) \subseteq U \Rightarrow$

$\Rightarrow (\forall)\, n \in \mathbb{N}$ there exists an element, call it x_n, which is in $B_{d'}(x, \frac{1}{n})$ but not in U

i.e.: x_n such that: $\boxed{d'(x_n, x) < \frac{1}{n}}$ $\boxed{x_n \notin U}$ ($\lightning$)

But $d'(x_n, x$

Also, $U \in \mathcal{T}_d$

by $x \in U$

Proof of $B \Rightarrow$

to prove:

(B) $\mathcal{T}_d = \mathcal{T}_{d'}$

But $d'(x_n, x) < \frac{1}{n}$ implies that $x_n \to x$ in (X, d') $\xrightarrow[(A)]{\text{by}}$ $x_n \to x$ in (X, d) $\Longrightarrow$

$\Longrightarrow$ $\boxed{d(x_n, x) \to 0}$

Also, $U \in \mathcal{T}_d$, by $x \in U$ $\Longrightarrow$ $\exists\, r > 0$ s.t. $\boxed{B_d(x, r) \subseteq U}$ $\Longrightarrow$ for large n one has $x_n \in U$.

CONTRADICTION WITH ($\lightning$).

$(x, \frac{1}{n})$ but not in U

<u>Proof of B ⟹ A</u>: Assume B. We prove the "⟺" appearing in A. Given the symmetry: to prove: given $\boxed{x_n \to x \text{ in } (X, d)}$ we want to prove: $x_n \to x$ in (X, d'),

hence $\boxed{d(x_n, x) \to 0}$

to prove: $d'(x_n, x) \to 0$

i.e. $(\forall)\, \varepsilon > 0$ $\exists\, n_\varepsilon$ s.t. $d'(x_n, x) < \varepsilon$ $(\forall)\, n \geq n_\varepsilon$

fix ε. To prove.

Use B: since $B_{d'}(x, \varepsilon) \in \mathcal{T}_{d'}$ ⟹ it is also in $\mathcal{T}_d$ ⟹ $\exists\, \varepsilon_1 > 0$ s.t. $B_d(x, \varepsilon_1) \subseteq B_{d'}(x, \varepsilon)$

But $d(x_n, x) \to 0$ ⟹ $\exists\, n_\varepsilon$ s.t. $d(x_n, x) < \varepsilon_1$ $(\forall)\, n \geq n_\varepsilon$ ⟹ $x_n \in B_d(x, \varepsilon_1)$ hence in $B_{d'}(x, \varepsilon)$ $(\forall)\, n \geq n_\varepsilon$. i.e. ☑