Quotients in the year 2023

0.1 Quotient topologies

Definition 0.1. A map $\pi : (X, \mathscr{T}_X) \to (Y, \mathscr{T}_Y)$ between two topological spaces is called a **topological quotient map** if it is surjective and satisfies the condition that, for $V \subset Y$, one has

$$\pi^{-1}(V) \in \mathscr{T}_X \Longleftrightarrow V \in \mathscr{T}_Y.$$

Intuitively, one may think of this situation as saying that "$(X, \mathscr{T}_X)$ sits nicely above $(Y, \mathscr{T}_Y)$"; actually, in many examples, X is a "simpler' space which unravels Y", and then π allows one to study Y via studying the simpler X. One such example would be the exponential map $e : \mathbb{R} \to S^1$, pictured in Figure 1.3/Example 1.6. Here is a precise statement that supports this intuition.

Proposition 0.2. *Given a topological quotient map $\pi : (X, \mathscr{T}_X) \to (Y, \mathscr{T}_Y)$ then, for any other topological space Z, a function $f : Y \to Z$ is continuous if and only if $f \circ \pi : X \to Z$ is.*

$$\begin{tikzcd} X \arrow[d, "\pi"'] \arrow[dr, "f\circ\pi"] & \\ Y \arrow[r, "f"'] & Z \end{tikzcd} \tag{0.1.1}$$

Proof. Recall that f being continuous means that for any $W \subset Z$ open, $f^{-1}(W)$ is open in Y. By the quotient map condition, $V := f^{-1}(W)$ is open in Y if and only if $\pi^{-1}(V) = \pi^{-1}(f^{-1}(W))$ is open in X. But $\pi^{-1}(f^{-1}(W)) = (f \circ \pi)^{-1}(W)$, hence f is continuous if and only if for any $W \subset Z$ open $(f \circ \pi)^{-1}(W)$ is open in X, i.e., if and only if $f \circ \pi$ is continuous.

Exercise 0.3. Show that the map $f : S^1 \to \mathbb{R}$, $f(\sin t, \cos t) = \sin(2023t)2^{\cos(20t)\sin(2t)^3}$ is a continuous map.

The notion of "topological quotient map" can also be looked from a different perspective- that of construction of new topologies. To that end one starts with

$$\text{a surjective map} \quad \pi : \underbrace{(X, \mathscr{T}_X)}_{\text{topological space}} \longrightarrow \underbrace{Y}_{\text{set}}$$

and the conclusion is that Y will carry a natural topology, called **the quotient topology** on Y induced by π, denoted $\pi_*(\mathscr{T}_X)$, which is the unique topology on Y that makes π into a topological quotient map, i.e.,

$$\pi_*(\mathscr{T}_X) := \{V \subset Y : \pi^{-1}(V) \in \mathscr{T}_X\}.$$

Theorem 0.4. *$\pi_*(\mathscr{T}_X)$ is indeed a topology on Y. Moreover, it is the largest topology on Y with the property that $\pi : X \to Y$ becomes continuous.*

Proof. The axioms (T1)-(T3) for $\pi_*(\mathscr{T}_X)$ follow right away from the same axioms for $\mathscr{T}_X$ and the following properties of taking pre-images of π:

$$\pi^{-1}(Y) = X,\ \pi^{-1}(\emptyset) = \emptyset,\ \pi^{-1}(V_1 \cap V_2) = \pi^{-1}(V_1 \cap V_2),\ \pi^{-1}(\cup_i V_i) = \cup_i \pi^{-1}(V_i)$$

The last part follows from the definition of continuity and of $\pi_*(\mathscr{T})$.

Example 0.5. [The projective *space*] A very good illustration of the use of quotient topologies is the construction of the projective space $\mathbb{P}^n$, as a topological space. Set theoretically, $\mathbb{P}^n$ is

$$\mathbb{P}^n = \{l \subset \mathbb{R}^{n+1} : l \text{ is a line through the origin}\},$$

the set of all lines in $\mathbb{R}^{n+1}$ through the origin. Can one relate it to a topological space that we already know? The answer is immediate once one realises one can use the basic property of lines: lines are uniquely determined by any two distinct points lying on it. In particular, to specify a line $l \in \mathbb{P}^n$ it suffices to specify a point $x \in \mathbb{R}^{n+1} \setminus \{0\}$- a property that can be encoded in the surjective map

$$\pi : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{P}^n \tag{0.1.2}$$

that sends $x = (x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} \setminus \{0\}$ to the line through the origin and x:

$$l_x := \overline{Ox} = \{(\lambda x_0, \lambda x_1, \ldots, \lambda x_n) : \lambda \in \mathbb{R}\} \in \mathbb{P}^n.$$

Using the Euclidean topology on $\mathbb{R}^{n+1} \setminus \{0\}$, the **projective space** $\mathbb{P}^n$ is now be defined as the set $\mathbb{P}^n$ endowed with the resulting quotient topology .

Example 0.6 (Gluings). Many of the cool examples of “spaces Y” can be illustrated by starting with a strip of paper and performing certain gluings to it. Mathematically the strip of paper is represented by the unit square $X = [0,1] \times [0,1]$, and the process of gluing comes with a map

$$\pi : [0,1] \times [0,1] \to Y$$

onto the outcome Y of the gluing process. The map π is the map that keeps track of the points before and after the gluing. As the square has a natural topology on it, the Euclidean topology, we are now precisely in the scenario described above, hence: the outcome Y of the gluing carries a natural topology: the resulting quotient topology.

For a more concrete example, one can look at “the Moebius band”. The quotes are used to remind you that, so far, we have been talking only about models of the Moebius band, rather than one single one. For instance, one can use the mathematically concrete model $M_{R,r}$ from (1.5.1), conveniently re-parametrised by the square by writing $u = (2s-1)r$ and $a = 2\pi t$. In other words, $M_{R,r}$ is the collection of points in $\mathbb{R}^3$ of type

$$(R + (2s-1)r\sin(\pi t))\cos(2\pi t), (R + (2s-1)r\sin(\pi t))\sin(2\pi t), (2s-1)r\cos(\pi t)) \tag{0.1.3}$$

with $(t,s) \in [0,1] \times [0,1]$, Of course, the map π sends (t,s) to this expression. Therefore, one can now endow $M_{R,r}$ with the resulting quotient topology $\pi_*(\mathscr{T}_{\text{Eucl}})$.

Exercise 0.7. On the other hand, since we have used a model that already sits inside $\mathbb{R}^3$, $M_{R,r}$ also carries the subspace topology. Prove that this coincides with the quotient topology $\pi_*(\mathscr{T}_{\text{Eucl}})$.

0.2 How to handle “gluings”: equivalence relations and quotients

To make the last class of examples mathematically precise in full generality, one still has to formalise the process of ”gluing”. This brings us to the notion of equivalence relation:

Definition 0.8. An **equivalence relation** on a set X is a subset $R \subset X \times X$ satisfying the following:

1. If $(x,y) \in R$ and $(y,z) \in R$, then $(x,z) \in R$.
2. If $(x,y) \in R$ then also $(y,x) \in R$.
3. $(x,x) \in R$ for all $x \in X$.

Given R, two elements $x, y \in X$ are said to be ***R*-equivalent**, and we write $x \sim_R y$, if $(x,y) \in R$.

One can think of such an equivalence relation R as encoding a “gluing rule” for the elements of X, so that $x \sim_R y$ if and only if x and y become the same point after the gluing.

Example 0.9. For the gluing underlying the Moebius band, $X = [0,1] \times [0,1]$, we see that the relevant relation, denoted R_{Moebius}, consists of pairs $(x,y) \in X \times X$ such that $x = y$, or one of the elements x and y is of type $(0,t)$ for some $t \in [0,1]$, while the other one is $(1, 1-t)$.

Example 0.10. In general, any surjective function $f : X \to Y$ gives rise to an equivalence relation, namely

$$R_f := \{(x,y) : f(x) = f(y)\}.$$

Thinking of R_f as encoding a “gluing rule”, it is clear what the result of the gluing should be: Y itself.

Definition 0.11. Given an equivalence relation R on a set X, a **quotient of** X **modulo** R is a pair (Y, π) consisting of a set Y together with a surjection $\pi : X \to Y$ (called the quotient map) with the property that

$$\pi(x) = \pi(y) \Longleftrightarrow (x,y) \in R. \tag{0.2.1}$$

When $(X, \mathscr{T}_X)$ is a topological space, a **topological quotient of** $(X, \mathscr{T}_X)$ **modulo** R is a topological space $(Y, \mathscr{T}_Y)$ together with a topological quotient map $\pi : X \to Y$ satisfying (3.2.1).

Example 0.12. Assume that we want to glue the end-points of the interval $[0,1]$ ... with the rather intuitively “obvious” outcome: a circle. This is now made precise by taking $X = [0,1]$, introducing the equivalence relation R given by

$$(t,s) \in R \Longleftrightarrow (t = s) \text{ or } (t = 0, s = 1) \text{ or } (t = 1, s = 0),$$

and saying that the circle S^1 together with

$$\pi := e : [0,1] \to S^1,\ t \mapsto (\cos(2\pi t), \sin(2\pi t))$$

is a quotient of $[0,1]$ modulo R. Set theoretically, this is immediate (when seeing this for the first time, the non-triviality is to digest the definitions). However, the topological version of this statement, where $[0,1]$ as well as S^1 are endowed with their Euclidean topologies, is not completely obvious: one still has to prove that π is a topological quotient map. At this point that is a rather tedious exercise; later on we will see more elegant ways to handle with such statements.

Example 0.13. It should be clear now that starting with the equivalence relation underlying the Moebius band (Example 3.9 above), the concrete model $M_{R,r}$ discussed before (as well as any other ”paper models”) are quotients of $X = [0,1] \times [0,1]$ module R_{Moebius}. As in the previous example, the same is true also topologically, but please notice that one still has to check that the map $[0,1] \times [0,1] \to M_{R,r}$ is a topological quotient map (where both the domain as well as the codomain are endowed with the Euclidean topology).

Example 0.14. Consider on $X = \mathbb{R}^3 \setminus \{0\}$ the equivalence relation R defined by

$$(x,y) \in R \Longleftrightarrow y = \lambda \cdot x \text{ for some } \lambda \in \mathbb{R}_{>0}.$$

Then the 2-sphere S^2 together with

$$\pi : \mathbb{R}^3 \setminus \{0\} \to S^2,\ \pi(x) = \frac{1}{||x||} x$$

is a quotient of X modulo R (picture this!). One can also show that this is actually a topological quotient map.

Example 0.15. Another interesting example is when $X = \mathbb{R}^{n+1} \setminus \{0\}$, with the equivalence relation R for which

$$x \sim_R y \Longleftrightarrow y = \lambda \cdot x \text{ for some } \lambda \in \mathbb{R}.$$

Equivalently, R consists of those pairs (x,y) with the property that $0 \in \mathbb{R}^{n+1}$, x and y are collinear. It should be clear now that $\mathbb{P}^n$ together with π from (3.1.2) is a quotient of $\mathbb{R}^{n+1} \setminus \{0\}$ modulo R.

The last example indicates how to proceed to build up a "canonical" quotient in general. More precisely, for any equivalence relation R on a set X:

- for $x \in X$ we define the ***R*-orbit through** x (also called the equivalence class of x w.r.t. to R) as the subset $R(x)$ of X consisting of elements that are R-equivalent to x:

$$R(x) := \{y \in X : (x,y) \in R\}.$$

- we define the **abstract quotient of** X **modulo** R as the collection of all such R-orbits:

$$X/R := \{R(x) : x \in X\},$$

and the abstract quotient map

$$\pi_R : X \to X/R, \quad \pi_R(x) := R(x).$$

- when X is a topological space, X/R endowed with the quotient topology is called the **abstract topological quotient of** X **modulo** R.

We leave it as an exercise to show that, in general, for $x, y \in X$, one has:

$$\boxed{R(x) = R(y) \Longleftrightarrow (x,y) \in R.} \tag{0.2.2}$$

In other words, the abstract quotient $(X/R, \pi_R)$ is always a quotient of X modulo R in the sense of Definition 3.11. From the point of view of gluings the abstract quotient has the great advantage that it provides a model for the result of the gluing that does not depend on our influence on how we actually perform the gluing- think e.g. of "the Moebius band" and the various ways one can twist the strip before gluing the ends. Of course, it is important to notice the "uniqueness up to isomorphisms" of the result of the gluing- and that is Proposition 3.18 bellow. However, we first discuss some examples.

Example 0.16. On $X = \mathbb{Z}$, any integer $n \geq 1$ gives rise to the equivalence relation R_n on $\mathbb{Z}$ of being "congruent modulo n":

$$(x,y) \in R_n \Longleftrightarrow x \equiv y \pmod{n}.$$

Then the R_n-orbit of an integer $k \in \mathbb{Z}$ is

$$R_n(k) = \{\ldots, k-2n, k, k-n, k+n, k+2n, \ldots\} \ \text{(usually denoted } (k \bmod n) \text{ or just } \hat{k}\text{)}.$$

Of course, $\mathbb{Z}/R_n$ is the usual set $\mathbb{Z}_n$ of integers modulo n.

Example 0.17. We can now finally make sense of **THE Moebius band**: it is the (topological) abstract quotient of $X = [0,1] \times [0,1]$ modulo the equivalence relation R_{Moebius} described in Examples 3.9.

Proposition 0.18. *Given an equivalence relation R on a set X then any quotient of X modulo R (cf. Definition 3.11) can be identified with the abstract quotient. More precisely:*

1. *for any quotient (Y, π) of X modulo R,*

$$\iota : X/R \to Y, \quad R(x) \mapsto \pi(x)$$

defines a bijection between X/R and Y. It can also be described as the unique map satisfying $\pi = \iota \circ \pi_R$

$$\begin{tikzcd} X \arrow[d, "\pi_R"'] \arrow[dr, "\pi"] & \\ X/R \arrow[r, "\sim", "\iota"'] & Y \end{tikzcd}$$

2. *similarly in the topological context, when $(X, \mathscr{T}_X)$ is a topological space and we are looking at topological quotients: then ι will be a homeomorphism.*

Proof. Notice that the commutativity of the diagram from the statement, i.e. the equality $\pi = \iota \circ \pi_R$, forces ι to be defined as $R(x) \mapsto \pi(x)$. The only thing one has to take care of is to ensure that this is well-defined: if $R(x) = R(x')$ we have to ensure that $\pi(x) = \pi(x')$, but this follows right away from the direct implication in (3.2.2) and the fact that π is a quotient map. The reverse implication in (3.2.2) shows that ι is injective, while the surjectivity of ι follows from the one of π.

In the topological context one still has to check that

- ι is a continuous map: this be seen as an immediate consequence of Proposition 3.2 applied to π_R and $f = \iota$.
- ι^{-1} is continuous as well: this is a consequence of the same proposition but applied now to π and $f = \iota^{-1}$:

$$\begin{tikzcd} X \arrow[d, "\pi"'] \arrow[dr, "\pi_R"] & \\ Y \arrow[r, "\iota^{-1}"'] & X/R \end{tikzcd} \quad , \quad \pi_R = \iota^{-1} \circ \pi.$$

Corollary 0.19. *Assuming that (Y, π) is a topological quotient of the topological space X modulo R then, for any topological space Z, $f \mapsto f \circ \pi$ described in diagram (3.1.1) defines a 1-1 correspondence*

$$\left\{ \begin{array}{c} \text{continuous maps} \\ f : Y \to Z \end{array} \right\} \overset{1-1}{\longleftrightarrow} \left\{ \begin{array}{c} \text{continuous maps} \\ \tilde{f} : X \to Z \\ \text{satisfying } \tilde{f}(x) = \tilde{f}(x') \text{ whenever } (x, x') \in R \end{array} \right\}.$$

Example 0.20. Returning to the equivalence relation on $X = \mathbb{R}^3 \setminus \{0\}$ discussed in Example 3.14, we see that the R-orbit of an arbitrary $x \in X$ is

$$R(x) = \{\lambda \cdot x : \lambda \in \mathbb{R}_{>0}\},$$

i.e. precisely the half line from the origin passing through x. Therefore, the abstract quotient X/R is precisely the collection of all such half lines. The proposition tells us that the model S^2 discussed in Example 3.14 can be identified with X/R; that identification has a simple geometric interpretation because a half line is uniquely determined by its intersection with the sphere.

Example 0.21. The previous proposition applied to the various concrete models of the Moebius band sitting inside the space R^3 tells us that any such concrete model is isomorphic to the (abstract) Moebius band

$$M_{\text{abs}} := [0,1] \times [0,1]/R_{\text{Moebius}}.$$

The concrete model itself can be seen as describing an embedding of M_{abs} in $\mathbb{R}^3$. For instance, for the concrete models $M_{R,r}$, the embedding sends the R_{Moebius}-orbit through a point $(t,s) \in [0,1] \times [0,1]$ to the point of coordinates (3.1.3).

Example 0.22 (the torus). While the Moebius band served as an example throughout the development of the theory, let us make some similar remarks in the case of "the torus", seen as the result of gluing the opposite edges of a square. As for the Moebius band, there are many different "shapes" that can be obtained as the result of such a gluing, some of which are shown in Figure 3.1. When saying "torus" we would like to think about the intrinsic space itself, and make sense of "shapes" as different ways to embed the torus into $\mathbb{R}^3$. The way to achieve this can then be described as follows:

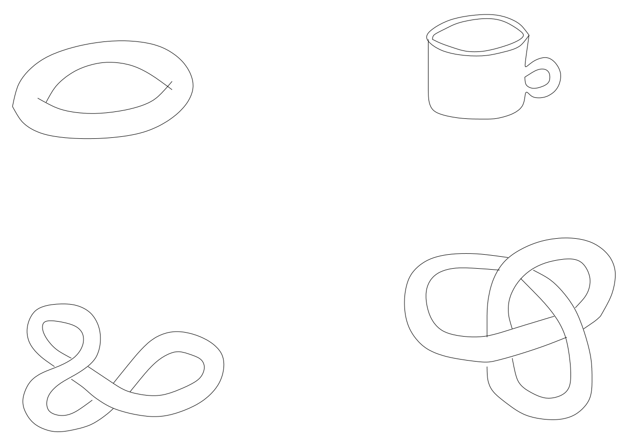

Fig. 0.1

1. Describe the relevant equivalence relation R_{torus}: it is the relation on $X = [0,1] \times [0,1]$ for which (t,s) is R_{torus}-equivalent to $(t'.s')$ provided $(t,s) = (t',s')$, or $\{t,t'\} = \{0,1\}$ and $s' = s$, or $\{s,s'\} = \{0,1\}$ and $t = t'$.

2. Consider the abstract torus, defined as the abstract quotient

$$T_{\text{abs}} := [0,1] \times [0,1]/R_{\text{torus}}.$$

This defines T_{abs} both as a set as well as a topological space.

3. Embed the abstract torus: realizing the torus more concretely in $\mathbb{R}^3$, i.e. finding explicit models of it, translates now into the question of describing embeddings

$$f : T_{\text{abs}} \to \mathbb{R}^3.$$

By Corollary 3.19, continuous f's correspond to continuous maps

$$\tilde{f} : [0,1] \times [0,1] \to \mathbb{R}^3$$

with the property that $\tilde{f}(t,s) = \tilde{f}(t',s')$ whenever (t,s) and (t',s') are R_{torus}-equivalent. The injectivity of f is equivalent to the condition that the last equality holds if (t,s) and (t',s') are R_{torus}-equivalent. As we shall see when discussing compactness, ensuring that f is continuous and injective is enough to ensure that f is an embedding. Such $\tilde{f}$'s arise from the explicit realizations of the torus (see Chapter 1):

$$\tilde{f}(t,s) = (R + rcos(2\pi t))cos(2\pi s), (R + rcos(2\pi t))sin(2\pi s), rsin(2\pi t)).$$

Exercise 0.23. Do the same for the Klein bottle and $\mathbb{P}^2$.

Some conclusions on "gluings"

Let us conclude with some general remarks on how to handle gluings in the more general context. Therefore, we assume we start with a concrete topological space X (usually a subset of an Euclidean space) and we want to perform some gluing on X. The main (possible) steps are:

1. the gluing itself is encoded in an equivalence relation R on X;

2. the outcome of the gluing is a quotient of X modulo R, in the sense described in Definition 3.11;

3. Proposition 3.18 ensures that the outcome is actually unique (up to isomorphisms);

4. to describe the outcome one can always use the abstract quotient X/R. It has the advantage that it is completely canonical (unlike other concrete models, it does not depend on the way we implement our intuition).

5. on the other hand, one can always try to build concrete models (Y,π) for the result of the gluing, using ones intuition. Often such models (Y,π) can be found sitting inside some Euclidean space,

   $$Y \subset \mathbb{R}^n.$$

   Notice that, set theoretically, finding such a model inside $\mathbb{R}^n$ is equivalent to finding a map

   $$f : X \to \mathbb{R}^n$$

   with the property that $f(x) = f(x') \Longleftrightarrow (x,x') \in R$. In more detail: f is $\pi : X \to Y$ combined with the inclusion $Y \subset \mathbb{R}^n$, while Y can be recovered as the image of f. The map f can also be seen as encoding an injection

   $$X/R \to \mathbb{R}^n, \quad R(x) \mapsto f(x).$$

   However, topologically, one faces the following problem:

   - The reason that we even care that our model Y sits inside $\mathbb{R}^n$ is that it makes the model more intuitive and, in particular, we think of Y as carrying the resulting Euclidean topology inherited from the inclusion $Y \subset \mathbb{R}^n$.
   - However, as a quotient of X, it also carries the quotient topology, which makes (Y,π) into a topological quotient of X.

   Are the two the same? The question can be rephrased in terms of the injection $X/R \to \mathbb{R}^N$ as: is the injection an embedding? The answer is not always "yes". The following describes a general situation when the answer is positive. The proof will be discussed later one, after we discuss compactness.

Proposition 0.24. *Let R be an equivalence relation on a set X and let (Y,π) be a quotient of X modulo R. Assume now that X is a closed and bounded subset of some Euclidean space $\mathbb{R}^k$ and we endow X with the resulting Euclidean topology. We also assume that Y is a subset of some other Euclidean space $\mathbb{R}^n$ and that the map π is continuous as a map from X to $\mathbb{R}^n$.*

Then, on Y, the Euclidean topology arising from the inclusion $Y \subset \mathbb{R}^n$ coincides with the quotient topology making (Y,π) into a topological quotient of X modulo R.

Here is yet another formulation of the same proposition (just use the one that is most appealing to you!).

Proposition 0.25. *Assume that $X \subset \mathbb{R}^k$, viewed as a topological space space with the induced (Euclidean) topology. Let R be an equivalence relation on X and assume that we find a topological quotient (Y,π) of X modulo R, where Y is a subset of some Euclidean space $\mathbb{R}^n$. If:*

1. *the map π is continuous as a map from X to $\mathbb{R}^n$,*
2. *X is closed and bounded in $\mathbb{R}^k$,*

then the topology on Y coincides with the Euclidean topology inherited from the inclusion $Y \subset \mathbb{R}^n$.

0.3 Special classes of quotients I: collapsing a subspace, cones, suspensions

An interesting and rather large class of quotient spaces are quotients obtained by collapsing a subspace to a point.

Definition 0.26. Let X be a topological space and let $A \subset X$. We define X/A as the topological space obtained from X by collapsing A to a point (i.e. by identifying to each other all the points of A). Equivalently,

$$X/A = X/R_A,$$

where R_A is the equivalence relation on X defined by

$$R_A = \{(x,y) : x = y \text{ or } x,y \in A\}.$$

Here are some more specific versions of this construction. Let X be a topological space.

- **The cylinder on** X is defined as

$$\boxed{\mathrm{Cyl}(X) := X \times [0,1]}$$

endowed with the product topology (and the unit interval is endowed with the Euclidean topology). It contains two interesting copies of X: $X \times \{1\}$ and $X \times \{0\}$.
- **The cone on** X is defined as the quotient obtained from $\mathrm{Cyl}(X)$ by collapsing $X \times \{1\}$ to a point:

$$\boxed{\mathrm{Cone}(X) := X \times [0,1]/(X \times \{1\})}$$

(endowed with the quotient topology). Intuitively, it looks like a cone with basis X.The cone contains the copy $X \times \{0\}$ of X (the basis of the cone).
- **The suspension of** X is defined as the quotient obtained from $\mathrm{Cone}(X)$ by collapsing the basis $X \times \{0\}$ to a point:

$$\mathrm{S}(X) := \mathrm{Cone}(X)/(X \times \{0\}).$$

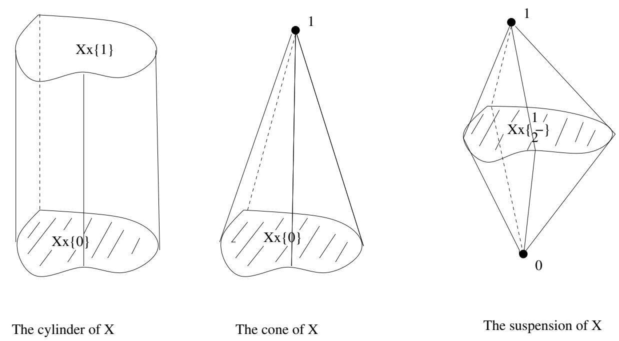

Fig. 0.2

Example 0.27. The general constructions of quotients, such as the quotient by collapsing a subspace to a point, the cone construction and the suspension construction, are nicely illustrated by the various relations between the closed unit balls $D^n \subset \mathbb{R}^n$ and the unit spheres $S^n \subset \mathbb{R}^{n+1}$. We mention here the following:

(a) D^n is homeomorphic to $\mathrm{Cone}(S^{n-1})$- the cone of S^{n-1}.
(b) S^n is homeomorphic to $\mathrm{S}(S^{n-1})$- the suspension of S^{n-1}.
(c) S^n is homeomorphic to D^n/S^{n-1}- the space obtained from D^n by collapsing its boundary to a point.

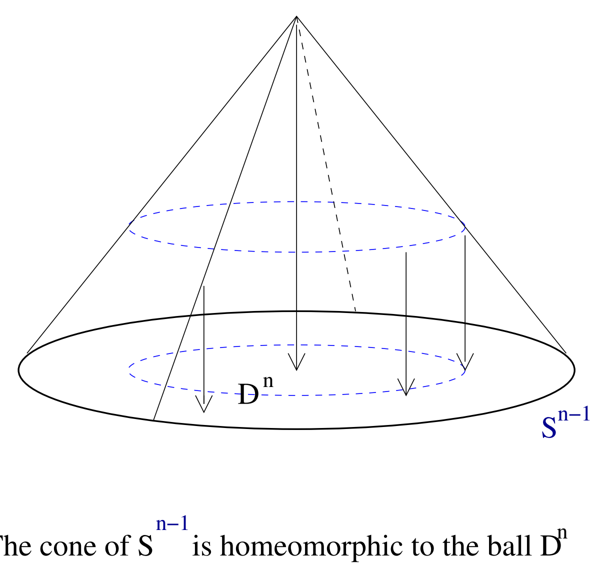

Fig. 0.3

Proof. The first homeomorphism is indicated in Figure 3.3 (project the cone down to the disk). It is not difficult to make this precise: we have a map

$$\tilde{f} : S^{n-1} \times [0,1] \to D^n, \tilde{f}(x,t) = (1-t)x.$$

This is clearly continuous and surjective, and it has the property that

$$\tilde{f}(x,t) = \tilde{f}(x',t') \Longleftrightarrow (x,t) = (x',t') \text{ or } t = 1,$$

which is precisely the equivalence relation corresponding to the quotient defining the cone. Hence we obtain a continuous bijective map

$$f : \mathrm{Cone}(S^{n-1}) = S^{n-1} \times [0,1]/(S^{n-1} \times \{1\}) \to D^n.$$

After we will discuss the notion of compactness, we will be able to conclude that also f^{-1} is continuous, hence f is a homeomorphism. Note that this f sends $S^{n-1} \times \{1\}$ to the boundary of D^n, hence (b) will follow from (c). In turn, (c) is clear on the picture (see Figure 1.11 in the previous Chapter); the map from D^n to S^n indicated on the picture can be written explicitly as

$$\tilde{g} : D^n \to S^n, x \mapsto (\frac{x_1}{||x||} sin(\pi||x||), \ldots, \frac{x_n}{||x||} sin(\pi||x||), cos(\pi||x||))$$

(well defined for $x \neq 0$) and which sends 0 to the north pole $(0, \ldots, 0, 1)$.

One can check directly that

$$\tilde{g}(x) = \tilde{g}(x') \Longleftrightarrow x = x' \text{ or } x, x' \in S^{n-1},$$

which is the equivalence relation corresponding to the quotient D^n/S^{n-1}. We deduce that we have a bijective continuous map:

$$g : D^n/S^{n-1} \to S^n$$

but, again, we leave it to after the discussion of compactness the final conclusion that g is a homeomorphism.

0.4 Special classes of quotients II: quotients modulo group actions

In this section we discuss quotients by group actions. Let X be a topological space. We denote by $\mathrm{Homeo}(X)$ the set of all homeomorphisms from X to X. Together with composition of maps, this is a group. Let Γ be another group, whose operation is denoted multiplicatively.

Definition 0.28. An **action** of the group Γ on the topological space X is a group homomorphism

$$\phi : \Gamma \to \mathrm{Homeo}(X),\ \gamma \mapsto \phi_\gamma.$$

Hence, for each $\gamma \in \Gamma$, one has a homeomorphism ϕ_γ of X ("the action of γ on X"), so that

$$\phi_{\gamma\gamma'} = \phi_\gamma \circ \phi_{\gamma'}\ \ \forall\, \gamma, \gamma' \in X.$$

Sometimes $\phi_\gamma(x)$ is also denoted $\gamma(x)$, or simply $\gamma \cdot x$, and one looks at the action as a map

$$\Gamma \times X \to X,\ (\gamma, x) \to \gamma \cdot x.$$

The action induces an equivalence relation R_Γ on X defined by:

$$(x,y) \in R_\Gamma \Longleftrightarrow\ \exists\, \gamma \in \Gamma \text{ s.t. } y = \gamma \cdot x.$$

The resulting topological quotient is called **the quotient of** X **by the action of** Γ, and is denoted by X/Γ. Note that the R_Γ-orbit through an element $x \in X$ is precisely its Γ-orbit:

$$\Gamma \cdot x := \{\gamma \cdot x : \gamma \in \Gamma\}.$$

Hence X/Γ consists of all such orbits, and the quotient map sends x to Γx.

Example 0.29. The additive group $\mathbb{Z}$ acts on $\mathbb{R}$ by

$$\mathbb{Z} \times \mathbb{R} \to \mathbb{R},\ (n,r) \mapsto \phi_n(r) = n \cdot r := n + r.$$

The resulting quotient is (homeomorphic to) S^1. More precisely, one uses Corollary 3.19 again to see that the map $\tilde{f} : \mathbb{R} \to S^1$, $t \mapsto (cos(2\pi t), sin(2\pi t))$ induces a continuous bijection $f : \mathbb{R}/\mathbb{Z} \to S^1$; then one proves directly (e.g. using sequences) that f is actually a homeomorphism, or one waits again until compactness and its basic properties are discussed.

Here is a fortunate case in which Hausdorffness is preserved when passing to quotients.

Theorem 0.30. *If X is a Hausdorff space and Γ is a finite group acting on X, then the quotient X/Γ is Hausdorff.*

Proof. Let $\Gamma x, \Gamma y \in X/\Gamma$ be two distinct points ($x, y \in X$). That they are distinct means that, for each $\gamma \in \Gamma$, $x \neq \gamma y$. Hence, for each $\gamma \in \Gamma$, we find disjoint opens $U_\gamma, V_\gamma \subset X$ containing x, and γy, respectively. Note that

$$W_\gamma = \phi_\gamma^{-1}(V_\gamma)$$

is an open containing y, and what we know is that

$$U_\gamma \cap \phi_\gamma(W_\gamma) = \emptyset.$$

Since Γ is finite, $U := \cap_\gamma U_\gamma$, $V := \cap_\gamma W_\gamma$ will be open neighborhoods of x and y, respectively, with the property that

$$U \cap \phi_a(e) = \emptyset,\ \ \forall\, a \in \Gamma.$$

Using the quotient map $\pi : X \to X/\Gamma$, we consider $\pi(U), \pi(e)$, and we claim that they are disjoint opens in X/Γ separating Γx and Γy. That they are disjoint follows from the previous property of U and V. To see that $\pi(U)$ is open, we have to check that $\pi^{-1}(\pi(U))$ is open, but

$$\pi^{-1}(\pi(U)) = \cup_{\gamma \in \Gamma} \phi_\gamma(U)$$

(check this!) is a union of opens, hence open. Similarly, $\pi(e)$ is open. Clearly, $\Gamma x = \pi(x) \in \pi(U)$ and $\Gamma y = \pi(y) \in \pi(e)$.

0.5 The projective space $\mathbb{P}^n$ again, from several perspectives

A very good illustration of the use of quotient topologies is the construction of the projective space, as a topological space. Recall that, set theoretically, $\mathbb{P}^n$ is the set of all lines through the origin in $\mathbb{R}^{n+1}$:

$$\mathbb{P}^n = \{l \subset \mathbb{R}^{n+1} : l - \text{one dimensional vector subspace}\}.$$

To realize it as a topological space, we relate it to topological spaces that we already know. There are several different perspectives.

As a quotient of $\mathbb{R}^{n+1} - \{0\}$:

The first approach is the one discussed already in Example 3.5, making use of the canonical map

$$\pi : \mathbb{R}^{n+1} \to \mathbb{P}^n, x \mapsto l_x,$$

where l_x is the line through the origin and x:

$$l_x = \mathbb{R}x = \{\lambda x : \lambda \in \mathbb{R}\} \subset \mathbb{R}^{n+1}.$$

Furthermore, as we pointed out Example 3.15 (and served as inspiration for the definitions that followed), $\mathbb{P}^n$ can also be seen as the abstract quotient modulo the equivalence relation on $\mathbb{R}^{n+1} \setminus \{0\}$ given by

$$x \sim_R y \Longleftrightarrow y = \lambda \cdot x \text{ for some } \lambda \in \mathbb{R}.$$

What we can add here is the remark that this equivalence relation comes from a group action: the group $\Gamma = \mathbb{R}^*$ (with respect to the multiplication) actiong on $X = \mathbb{R}^{n+1} \setminus \{0\}$ by scalar multiplication

$$\phi_\lambda(x) = \lambda x \ \text{ for } \lambda \in \mathbb{R}^*, x \in \mathbb{R}^{n+1} - \{0\}.$$

Therefore, the projective space becomes

$$\boxed{\mathbb{P}^n = (\mathbb{R}^{n+1} - \{0\})/\mathbb{R}^*.}$$

As a quotient of S^n:

This is based on another simple remark: a line in $\mathbb{R}^{n+1}$ through the origin is uniquely determined by its intersection with the unit sphere $S^n \subset \mathbb{R}^{n+1}$- which is a set consisting of two antipodal points (the first picture in Figure 3.4). This indicates that $\mathbb{P}^n$ can be obtained from S^n by identifying (gluing) its antipodal points. Again, this is a quotient that arises from a group action: the group $\mathbb{Z}_2$ acting on S^n. Using the multiplicative description $\mathbb{Z}_2 = \{1, -1\}$, the action is: ϕ_1 is the identity map, while ϕ_{-1} is the map sending $x \in S^n$ to its antipodal point $-x$. Hence the discussion indicates:

Proposition 0.31. *$\mathbb{P}^n$ is homeomorphic to $S^n/\mathbb{Z}_2$.*

Proof. The conclusion of the previous discussion is that there is a set-theoretical bijection:

$$\phi : S^n/\mathbb{Z}_2 \to \mathbb{P}^n,$$

which sends the $\mathbb{Z}_2$-orbit of $x \in S^n$ to the line l_x through x, with the inverse

$$\psi : \mathbb{P}^n \to S^n/\mathbb{Z}_2$$

which sends the line l to $S^n \cap l$ (a $\mathbb{Z}_2$-orbit!). We have to check that they are continuous. We use Proposition 3.2 and its corollary. To see that ϕ is continuous, we have to check that the composition with the quotient map $S^n \to S^n/\mathbb{Z}_2$ is continuous. But this composition is precisely the restriction of the quotient map $\mathbb{R}^{n+1} - \{0\} \to \mathbb{P}^n$ to S^n, hence is continuous. In conclusion, ϕ is continuous.

To see that ψ is continuous, we have to check that its composition with the quotient map $\mathbb{R}^{n+1} - \{0\} \to \mathbb{P}^n$ is continuous. But this composition- which is a map from $\mathbb{R}^{n+1} - \{0\}$ to $S^n/\mathbb{Z}_2$ can be written as the composition of two other maps which we know to be continuous:

- The map $\mathbb{R}^{n+1} - \{0\} \to S^n$ sending x to $x/||x||$.
- The quotient map $S^n \to S^n/\mathbb{Z}_2$.

In conclusion ψ is continuous.

Corollary 0.32. *The projective space $\mathbb{P}^n$ is Hausdorff.*

As a quotient of D^n:

Again, the starting remark is very simple: the orbits of the action of $\mathbb{Z}_2$ on S^n always intersect the upper hemisphere S^n_+ (for notations, see Section 1.4 in the first chapter). Moreover, such an orbit either lies entirely in the boundary of S^n_+, or intersects its interior in a unique point. See the second picture in Figure 3.4. This indicates that $\mathbb{P}^n$ can be obtained from S^n_+ by gluing the antipodal points that belong to its boundary. On the other hand, the orthogonal projection onto the horizontal hyperplane defines a homeomorphism between S^n_+ and D^n (see Figure 3.4). Passing to D^n, we obtain an equivalence relation R on D^n given by:

$$(x,y) \in R \Longleftrightarrow (x = y) \text{ or } (x, y \in S^{n-1} \text{ and } x = -y),$$

and we have done a part of the following:

Exercise 0.33. Show that $\mathbb{P}^n$ is homeomorphic to D^n/R. What happens when $n = 1$?

Corollary 0.34. *$\mathbb{P}^n$ for $n = 2$ is homeomorphic to the projective plane as defined in Chapter 1 (Section 1.8), i.e. obtained from the square by gluing the opposite sides as indicated in Figure 3.5.*

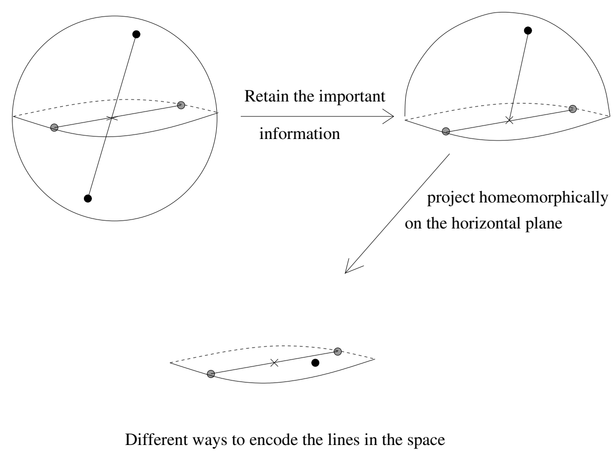

Fig. 0.4

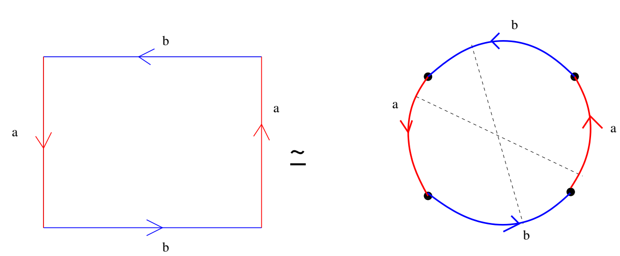

Fig. 0.5