Inleiding Topologie- preparing for the exam

Keep in mind: Closures and interiors are best computed using Lemma 2.37 (that is what the lemma is for!). Especially when you are asked to explain/provide the details for your answer. You should make sure you can use that lemma (you have the opportunity to use it several times in the exercises below).

A. Long exercise taking you through various basic concepts of topology: On $X = \mathbb{R}$ we consider the family $\mathcal{S}$ of subsets consisting of:

- all the intervals of type (m, M) with $m < M < 0$,
- all intervals of type (m, M) with $0 < m < M$
- the interval $[-1, 1)$.

Let $\mathcal{T}$ be the smallest topology on $\mathbb{R}$ containing $\mathcal{S}$. Please do the following:

1. show that $\mathcal{S}$ is not a topology basis.
2. describe a basis for the topological space $(\mathbb{R}, \mathcal{T})$.
3. is $(\mathbb{R}, \mathcal{T})$ Hausdorff?
4. is $(\mathbb{R}, \mathcal{T})$ second countable?
5. find an interval of type $[a, b]$ whose closure inside $(\mathbb{R}, \mathcal{T})$ is not an interval.
6. find an interval of type (a, b) whose interior inside $(\mathbb{R}, \mathcal{T})$ is not an interval.
7. find an interval of type $[a, b]$ with the property that, together with the topology induced from $(\mathbb{R}, \mathcal{T})$, is not compact.
8. find an interval of type (a, b) with the property that, together with the topology induced from $(\mathbb{R}, \mathcal{T})$, is not connected.
9. consider

$$f : (\mathbb{R}, \mathcal{T}) \to (\mathbb{R}, \mathcal{T}_{\text{Eucl}}), \ f(x) = \begin{cases} 0 & \text{if } x < -1 \\ 1 & \text{if } x \geq -1 \end{cases}$$

   Is f continuous?

10. Is f sequentially continuous?

B. An exercise regarding the manipulation with closures and interiors: Show that for any two subsets A and B of a topological space X one has:

$$\overline{A \cup B} = \overline{A} \cup \overline{B}, \quad \overline{X \setminus A} = X \setminus \mathring{A}\,.$$

C. An exercise involving product topology and the use of Lemma 2.37: Compute the interior, the closure and the boundary of

$$A = (0, 1] \times [0, 1)$$

in the topological space $X = \mathbb{R} \times \mathbb{R}$ endowed with the product topology $\mathcal{T}_l \times \mathcal{T}_l$.

D. An exercise with group actions and quotients: Let $X = (-1, \infty)$.

1. Find all the numbers $a, b \in \mathbb{R}$ with the property that

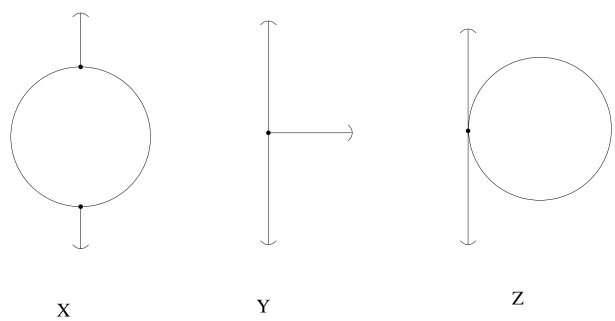
$$n \cdot t = \phi_n(t) = 2^n t + a^n + b$$

defines an action of the group $(\mathbb{Z}, +)$ on X.

2. For the a and b that you found, show that the resulting quotient space $X/\mathbb{Z}$ is homeomorphic to S^1.

F. An exercise with 1-point compactification: Let X, Y and Z be the spaces drawn in Figure 1.

1. Show that any two of them are not homeomorphic.
2. Compute their one-point compactifications X^+, Y^+ and Z^+.
3. Which two of the spaces X^+, Y^+ and Z^+ are homeomorphic and which are not?



Figuur 1:

G. An exercise involving the algebra structure on $C(X)$:

1. Show that for any continuous map $F : X \to Y$,

$$F^* : C(Y) \to C(X), \quad F^*(f) = f \circ F$$

   is a homomorphism of algebras.

2. Show that, similarly, any algebra homomorphism $G : A \to B$ gives rise to a continuous map

$$G^* : X_B \to X_A.$$

3. Using Gelfand-Naimark deduce that, for any two compact Hausdorff spaces X and Y, any algebra homomorphism $G : C(Y) \to C(X)$ is of type F^* for some $F : X \to Y$ continuous.

H. One more exercise with the spectrum: Assume that A is an algebra. Recall that each $a \in A$ gives rise to a map

$$F_a : X_A \to \mathbb{R}, \quad F_a(\chi) = \chi(a).$$

1. Show that for any $a, b \in A$ one has:

$$F_{a^2+2ab+5b^3} = F_a^2 + 2F_aF_b + 5F_b^3.$$

2. Generalise the previous item.

3. Deduce that if A is generated by two elements a and b in the sense that any element of A can be written as a polynomial expression in a and b, then the topology of the spectrum X_A coincides with the smallest topology on X_A with the property that f_a and f_b are continuous.

4. Assume now that A is the subset of $C(S^1)$ consisting of those functions which are obtained by restricting to S^1 polynomial functions in two variables. Describe two elements that generate A.

5. Show that the topological spectrum of A is homeomorphic to S^1.