A few comments on how to handle algebras and characters

1 What is an algebra? What can one do in an algebra?

An algebra is a set A on which one can perform the operations of:

- multiply an element $a \in A$ by a scalar, i.e. by a number $\lambda \in \mathbb{R}$, giving rise to a new element $\lambda \cdot a \in A$
- adding up elements $a, b \in A$ to obtain new elements $a + b \in A$
- multiplying elements $a, b \in A$ to obtain new elements $a \cdot b \in A$

Therefore, if we have a couple of elements of A, we can produce quite a few more by repeatedly applying these operations.

1. In full generality, without any extra-input, we have only two distinct elements of A: $0 = 0_A$- the neutral element related to the addition, and $1 = 1_A$- the neutral element related to the multiplication. However, given the axioms they satisfy, not a lot more new elements can be produced using the operations above. However, there is a bit:
$$\lambda \cdot 1_A \qquad (\text{for } \lambda \in \mathbb{R})$$
are all elements of A. Usually these elements are simply denoted λ and seen as "the constant elements of A".

2. What if are given/we have available yet another element $a \in A$? Then there is a lot more one can produce:
$$a \cdot a =: a^2, \quad a \cdot a^2 = a^3, \quad \text{etc},$$
then also multiply by scalars and take sums ... all together we obtain general expressions of type
$$\lambda_0 \cdot 1 + \lambda_1 \cdot a + \ldots + \lambda_n \cdot a^n \in A,$$
for all $\lambda_0, \lambda_1, \ldots, \lambda_n \in \mathbb{R}$. Please notice: it may happen, in some examples, that all elements of A are of this type. In that case one says that a **generates the algebra** A. Be aware however that, even in such cases, some of these expressions may vanish. E.g. it may happen that $a^3 = 0$ (think e.g. of the algebra $A = \mathbb{R}[T]/(T^3)$ with x being the class of T), or $1 + a + a^2 = 0$ (think e.g. of the algebra $A = \mathbb{R}[T]/(1 + T + T^2)$). In this case we talks about **relations** (satisfied by a).

3. What if we are now given two elements, $a, b \in A$? Assume that they also commute, i.e. $ab = ba$. Then, as in the previous item, one can get a lot more new elements of A, namely all polynomial expressions

$$\sum_{i=1}^{n}\sum_{j=1}^{m}\lambda_{ij}a^i b^j \in A$$

with $\lambda_{i,j} \in \mathbb{R}$ (and m and n arbitrary integers). Again, it may happen that all the elements A are of this type- case in which we say that a **and** b **generate the algebra** A. But, even in that case (and also in full generality), there may be some relations between a and b that make some of these expressions become the same although they look different. For instance, when $A = C(S^1)$ is the algebra of continuous functions on the circle, we have two interesting elements

$$f, g \in C(S^1), \quad f(x,y) = x, g(x,y) = y,$$

and they satisfy a relation, namely $f^2 + g^2 = 1$. It follows for instance that $1 + 2f - 3g + 2f^2 + 3g^2 + f^3 + fg^2 + f^2g + g^3 = 2 + 3f - 2g + f^2$ (just an example).

4. ... and one can keep on going.

Final advice: when you have an algebra A, look for "special elements"(elements that draw your attention ... this is not very precise ... it just explains to you a bit how to think before doing more precise manipulations). It would also be good to realise whether:

a). do the elements that you noticed/guess generate the algebra A?

b). are there special relations that your elements satisfy? (e.g. $T^3 = 0$ or $f^2 + g^2 = 1$ in the examples above).

All these should give you a pretty good feeling about your algebra.

2 Characters of an algebra/the spectrum?

How can one get an idea about what the spectrum of a given algebra A looks like? Again, this is about how to think about it/how to approach it. Hence the tricks I am describing here should be complemented by actual proofs. In principle, if the previous step has been carried out correctly, more than half of the job is done. Why? Well, remember what a character $\chi \in X_A$ is. It is a function $\chi : A \to \mathbb{R}$ which *is compatible with all the operations.* That implies that a lot of information about χ is superfluous. E.g., already on the elements $0_A, 1_A \in A$, we know that χ does not take arbitrary values, but

$$\chi(0_A) = 0, \quad \chi(1_A) = 1$$

(I use the index A to distinguish the zero and unit of A from the numbers $0, 1 \in \mathbb{R}$). Furthermore, in the story above, if you have an element $a \in A$, if we know the value of χ on a

$$\chi(a) =: u \in \mathbb{R},$$

then

$$\chi(a^2) = \chi(a \cdot a) = \chi(a)\chi(a) = u^2, \quad \chi(a^3) = \chi(a^2 \cdot a) = \chi(a^2)\chi(a) = u^3, \quad \text{etc}$$

and the values on all the other polynomial expressions associated to a will be determined by the number u:

$$\begin{aligned}\chi(\lambda_0 \cdot 1_A + \lambda_1 \cdot a + \ldots + \lambda_n \cdot a^n) &= \lambda_0 \cdot \chi(1) + \lambda_1 \cdot \chi(a) + \ldots + \lambda_n \cdot \chi(a^n) \\ &= \lambda_0 + \lambda_1 \cdot u + \ldots + \lambda_n \cdot u^n\end{aligned}$$

And similarly if, for your algebra, you found/noticed/guessed two distinguished elements $a, b \in A$ (as in 3. above).

Finally, you should also not forget about the special relations that the distinguished elements that you use satisfy. For instance, in the first example mentioned above, we had the scenario in which A is generated by an element $a \in A$ with the property that $a^3 = 0$. This special relation has a consequence on the character: by applying χ to it we find that $u = \chi(a)$ is not arbitrary, but it must satisfy $u^3 = 0$. Since $u \in \mathbb{R} = 0$ it follows that $u = 0$. Hence, in this case, there is only one χ, namely the one given by

$$\chi(\lambda_0 \cdot 1_A + \lambda_1 \cdot a + \ldots + \lambda_n \cdot a^n) = \lambda_0.$$

If the special relating was not $u^3 = 0$ but $u^2 - 1 = 0$, you would find two characters.

Similarly, if you exhibited two interesting elements of $a, b \in A$ then knowing the values of a character χ on a and b,

$$u := \chi(a), \ v := \chi(b) \in \mathbb{R},$$

we know the values of χ on all polynomial expression obtained from a and b. Furthermore, every special relation between a and b will have consequences on χ. For instance, if $a^2 + b^2 = 1$, it follows that the values u and v are not arbitrary but they should satisfy $u^2+v^2 = 1$. If A is generated by a and b, then any character will be completely determined by the values of u and v and, if there are no other relations between a and b other than $a^2 + b^2 = 1$ (or consequences), it looks like the spectrum of A may be S^1.

Remark: And one more advice, especially if the discussion above does not help: please do not forget the theorem of Gelfand-Naimark: for $A = C(X)$ as in the theorem, the topological spectrum is (homeomorphic to) X. So, before trying anything else, maybe it is wise to check whether you can apply that theorem directly. Just be aware that the algebra that is give to you may described differently, not as a $C(X)$, and you have to recognize that it is of that type. For instance, if you take

$$A := \{f \in C(\mathbb{R}) : f(t) = f(t+1) \quad \forall\, t \in \mathbb{R}\}$$

or

$$B := \{f \in C[0,1] : f(0) = f(1),$$

one mentioned the space $\mathbb{R}$, one mentioned the space $[0,1]$, but the both correspond to continuous functions of S^1. More precisely, each of these algebras can be identified with (they are isomorphic to) the algebra $C(S^1)$ (how?). Therefore the spectrum A, as well as of B, are both homeomorphic to the one of $C(S^1)$ which, by the Gelfabd Naimark, is S^1 itself.

I hope this helps.