Dynamic Meteorology: lecture 4

Sections 1.6, 1.7 and 1.17

Including the influence of Earth’s rotation (Coriolis effect)
Material derivative (in a rotating coordinate system on a sphere)
The influence of the Coriolis force on vertical circulations
Example 1: Tropical “Hadley” circulation
Example 2: the seabreeze (a second mode of convection)
Inertial frequency: second “eigen-frequency” of the atmosphere

3/10/2018: tutorial 3

Problem 1.12, problems in Box 1.8 + extra problems

http://www.staff.science.uu.nl/~delde102/dynmeteorology.htm
The equations* 

**momentum**

\[ \frac{d\vec{v}}{dt} = -\alpha \vec{\nabla} p - g \hat{k} - 2\vec{\Omega} \times \vec{v} + Fr \]

\[ \alpha \equiv \frac{1}{\rho} \]

Pressure gradient (1.3)  Gravity (1.3)  Coriolis (1.7)  Friction (1.8)

**mass**

\[ \frac{d\rho}{dt} = -\rho \vec{\nabla} \cdot \vec{v} \]

eq. 1.6

**energy**

\[ J dt = c_v dT + pd\alpha \]

same as eq. 1.22 (p.24)

**state**

\[ p = knT \]

eq. 1.10a

Unknowns:

\[ \vec{v}, \rho, T, p \]

*see geophysical fluid dynamics and sections 1.2, 1.3, 1.4 and 1.7 of lecture notes
Material derivative of a scalar

Scalar is a function of $x$, $y$, $z$ and $t$

\[
\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla = \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z}
\]

eq. 1.5

Material derivative  Local derivative  “Advection”  Non-linear!!!
\[ \frac{d}{dt} (...) = 0 \]

(conserved: frontal clouds)

\[ \frac{\partial}{\partial t} (...) = 0 \]

(stationary: wave clouds)
cloud advection & stationary gravity waves

\[
\frac{d}{dt}(\ldots) = 0
\]

(conserved features)

\[
\frac{\partial}{\partial t}(\ldots) = 0
\]

(stationary features)
cloud advection & stationary gravity waves

\[ \frac{d}{dt}(...) = 0 \]  (conserved features)

\[ \frac{\partial}{\partial t}(...) = 0 \]  (stationary features)
Material derivative of a vector

\[
\frac{d\vec{v}}{dt} \equiv \left( \frac{du}{dt} - \frac{uv\tan\phi}{a} + \frac{uw}{a} \right) \hat{i} + \left( \frac{dv}{dt} + \frac{u^2\tan\phi}{a} + \frac{vw}{a} \right) \hat{j} + \left( \frac{dw}{dt} - \frac{u^2 + v^2}{a} \right) \hat{k}
\]

Additional terms due to curved coordinate system!!

These terms are frequently neglected in theoretical analysis (see box 1.6)

For derivations see *geophysical fluid dynamics* and/or *Holton, chapters 1 and 2*

Section 1.7
Coriolis effect

\[
\frac{d\vec{v}}{dt} = -\alpha \vec{v} p - g\hat{k} - 2\vec{\Omega} \times \vec{v} + Fr
\]

\[
\vec{\Omega} \times \vec{v} = (w\Omega \cos \phi - v\Omega \sin \phi) \hat{i} + (u\Omega \sin \phi) \hat{j} - (u\Omega \cos \phi) \hat{k}
\]

From “scale analysis”:*  

\[
2\vec{\Omega} \times \vec{v} \approx -(2\Omega v \sin \phi) \hat{i} + (2\Omega u \sin \phi) \hat{j} = -fv\hat{i} + fu\hat{j}
\]

\textit{f is the Coriolis Parameter}

\[
f \equiv 2\Omega \sin \phi
\]

* |w|<<|v| and |w|<<|u|, see \textit{Holton, chapter 2} or \textit{geophysical fluid dynamics}
Equations in terms of potential temperature and Exner-function

\[ \frac{d\theta}{dt} = \frac{J}{\Pi} \]

- eq. 1.23 or 1.33

\[ \frac{d\vec{v}}{dt} = -\theta \vec{\nabla} \Pi - g \hat{k} - 2\vec{\Omega} \times \vec{v} \]

- eq. 1.35 + Coriolis

\[ \frac{d\Pi}{dt} = -\frac{R\Pi}{c_v} \nabla \cdot \vec{v} + \frac{RJ}{c_v \theta} \]

- eq. 1.37

Three differential equations with three unknowns!
The influence of earth’s rotation on atmospheric circulation patterns

The Hadley circulation and the sea breeze both constitute a magnificent illustration of the influence of rotation on the circulation!

What is the Hadley circulation?

What is the sea breeze?
Figure 1.15. The time-average meridional circulation in the tropics, called the Hadley circulation, is driven principally by latent heat release in large convective clouds in the Inter-Tropical Convergence Zone (ITCZ). Water vapour is forced upwards over the ITCZ. There is compensating subsidence in the subtropics, which leads to a reduction of the relative humidity, cloudless skies and drought.

Precipitation at ITCZ & Hadley circulation

Precipitation maximum at the intertropical convergence zone (ITCZ) is a footprint of the so-called Hadley circulation (see next slide)
Winter Hadley cell and Ferrel cell

Ensemble mean **zonal mean meridional velocity** (red: toward north pole) (labeled in m s\(^{-1}\)) and **zonal mean zonal velocity** (black contours, labeled in m s\(^{-1}\)) in January (1979-2017).
Winter Hadley cell

Ensemble mean Zonal mean meridional velocity (red: toward north pole) (labeled in m s\(^{-1}\)) and zonal mean zonal velocity (black contours, labeled in m s\(^{-1}\)) in January (1979-2017).

Conservation of angular momentum per unit mass \((M_a)\) *

\[
M_a = (\Omega a \cos \phi + [u]) a \cos \phi
\]

If \([u] = 0\) at \(\phi = 0\):

\[
[u] = \frac{\Omega a \sin^2 \phi}{\cos \phi} \approx \frac{\Omega y^2}{a \cos \phi}
\]

\(a = 6371\) km; \(\Omega = 7.3 \times 10^{-5}\) s\(^{-1}\)

\(\phi = 30^\circ\): \([u] = 131\) m s\(^{-1}\)

Larger than observed (about 40 m s\(^{-1}\))! Why?

Source: ERA-Interim reanalysis

* See pages 147-148
Winter Hadley cell

Ensemble mean Zonal mean meridional velocity (red: toward north pole) (labeled in m s\(^{-1}\)) and zonal mean zonal velocity (black contours, labeled in m s\(^{-1}\)) in January (1979-2017).

We will return to the Hadley circulation later in the course

Source: ERA-Interim reanalysis
Three vertical cross-sections of the lowest 2 or 3 km of the atmosphere, schematically illustrating different stages of the development of the pressure gradients and sea-breeze circulation.

The **left-hand diagram** shows the initial conditions.
The **middle diagram** shows the initiation of the circulation.
The **right-hand diagram** shows the final situation with fully-developed sea breeze circulation.

Surface pressure change (tenths of hPa) over Europe between 1000 and 1300 UTC in June.

Isolines of the north-south component of the wind velocity (in m/s) above Djakarta (Java), obtained from hourly mean values evaluated from balloon observations from May to November 1909 to 1915. Wind from the sea (sea breeze) is hatched. There is a pronounced return current with a maximum shortly before sunset and in height of 2 km. In the tropics the influence of rotation (Coriolis) is negligible.

Evidence of sea breeze in the Netherlands

Computer model (GALES, Delft) simulation of cumulus clouds on 6 July 2004. In the right lower corner is the corresponding satellite image. Forced upward motion at the sea-breeze front leads more and larger clouds (Source PhD thesis by Jerôme Schalkwijk).
Photograph made on Saturday 1 October 2016, 15:00.
Sea breeze in the Netherlands

Sea breeze over the Netherlands
8 May 1976

Map showing wind direction and temperature over the Netherlands and surrounding areas.

Graph showing temperature over time in the Netherlands.

Wind velocity vectors indicating the direction and speed of the sea breeze.

Legend for wind vectors:
- 5 m s⁻¹
Model based on equations of motion

Velocity components:

\[ u \equiv \frac{dx}{dt}; \quad v \equiv \frac{dy}{dt} \]

Pressure gradient force (per unit mass)

\[ \frac{du}{dt} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + fv \]

Coriolis force (per unit mass)

\[ \frac{dv}{dt} = -\frac{1}{\rho} \frac{\partial p}{\partial y} - fu \]

\[ f = 2\Omega \sin \phi \]

\( \rho \) : density

\( p \) : pressure

\( \Omega \) : Earth’s angular velocity

\( \phi \) : latitude
Prescribe pressure gradient

\[ \frac{\partial p}{\partial x} = A \cos(\Omega t + \varphi) \]

- **Pressure**
- **Earth's angular velocity**
- **Time**
- **Phase**
- **Amplitude**

(1 hPa per 100 km)
Equations of motion

\[
\begin{align*}
\frac{du}{dt} &= -\frac{1}{\rho} \frac{\partial p}{\partial x} + fv \\
\frac{dv}{dt} &= -\frac{1}{\rho} \frac{\partial p}{\partial y} - fu
\end{align*}
\]

Prescribe:
\[\frac{\partial p}{\partial y} = 0\]
\[\frac{\partial p}{\partial x} = A \cos(\Omega t + \varphi)\]

(“parametrisation”)
Model equations

\[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = f v - \frac{A}{\rho} \cos(\Omega t + \varphi) \]

\[ \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} = -f u \]

At the coast:

\[ \frac{\partial u}{\partial x} = 0 \quad \frac{\partial v}{\partial x} = 0 \quad \text{(reasonable assumption)} \]
Model equations

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = f v - \frac{A}{\rho} \cos(\Omega t + \varphi)
\]

\[
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} = -fu
\]

At the coast:

\[
\frac{\partial u}{\partial x} = 0 \quad \frac{\partial v}{\partial x} = 0
\]

(reasonable assumption)

Equations are linearised:

\[
\frac{\partial u}{\partial t} = fv - \frac{A}{\rho} \cos(\Omega t + \varphi)
\]

\[
\frac{\partial v}{\partial t} = -fu
\]

(“Eulerian” model)

\[
\frac{\partial^2 u}{\partial t^2} + f^2 u = \frac{A\Omega}{\rho} \sin(\Omega t + \varphi)
\]

(forced harmonic oscillator)
Equation for time-evolution of the "sea breeze"

\[ \frac{\partial^2 u}{\partial t^2} + f^2 u = \frac{A \Omega}{\rho} \sin(\Omega t + \varphi) \]

- wind velocity perpendicular to the coast
- Coriolis parameter: \( f = 2\Omega \sin \varphi \)
- earth’s angular velocity
- amplitude of the daily cycle in pressure perpendicular to the coast
- latitude
- phase

\[ \nu \]

North Sea

\[ u \]
Analytical solution

\[
\frac{\partial^2 u}{\partial t^2} + f^2 u = \frac{A\Omega}{\rho} \sin(\Omega t + \varphi)
\]

\[\varphi = 0\]

**Solution:**

\[
u(t) = C_1 \sin(ft) + C_2 \cos(ft) + \frac{A\Omega}{\rho \left(f^2 - \Omega^2\right)} \sin(\Omega t)\]
Imposing boundary conditions

\[
\frac{\partial^2 u}{\partial t^2} + f^2 u = \frac{A\Omega}{\rho} \sin(\Omega t + \varphi) \quad \varphi = 0
\]

Solution:

\[
u(t) = C_1 \sin(ft) + C_2 \cos(ft) + A\Omega \frac{\sin(\Omega t)}{\rho (f^2 - \Omega^2)}
\]

Boundary conditions: \( u = 0 \) and \( v = 0 \) at \( t = 0 \)

(for example)

\[C_2 = 0\]

\[
\frac{\partial u}{\partial t} = -\frac{A}{\rho} \quad \text{at} \ t = 0
\]

\[f = 2\Omega \sin \phi\]

\[
\frac{\partial u}{\partial t} = f v - \frac{A}{\rho} \cos(\Omega t + \varphi)
\]

\[C_1 = -\frac{A\phi}{\rho (f^2 - \Omega^2)}\]
Periods of oscillations

Therefore,

\[ u(t) = \frac{A}{\rho(f^2 - \Omega^2)} \left( \Omega \sin(\Omega t) - f \sin(ft) \right) \]

Two time-scales: \(\frac{2\pi}{\Omega}\) and \(\frac{2\pi}{f}\)

Forcing period

natural (inertial) period

\[ f = 2\Omega \sin \phi \]

\[ A \approx 1 \text{ hPa per 100 km} \]
Analytical solution

\[
    u(t) = \frac{A}{\rho \left( f^2 - \Omega^2 \right)} \left( \Omega \sin(\Omega t) - f \sin(ft) \right)
\]

\[
    \frac{\partial u}{\partial t} = fv - \frac{A}{\rho} \cos(\Omega t)
\]

\[
    v(t) = \frac{Af}{\rho \left( f^2 - \Omega^2 \right)} \left( \cos(\Omega t) - \cos(ft) \right)
\]  

(eq. 108b)

\[
    v = \frac{1}{f} \frac{\partial u}{\partial t} + \frac{A}{f \rho} \cos(\Omega t)
\]

\[A \approx 1 \text{ hPa per 100 km}\]

What happens when \( f^2 = \Omega^2 \)?
Is this a realistic scenario?
**Figure 1.33. Wind direction**, according to the **analytical solution** (eq. 1.108a,b), for latitude=52.5°N, $A=0.001$ Pa m$^{-1}$, $\varphi=0$ and $\rho=1.16$ kg m$^{-3}$, assuming that $u=v=0$ at $t=0$. A wind direction equal to 270° implies a sea breeze perpendicular to the coast. The wind direction does not follow the same daily cycle each day and changes more quickly in theory (this figure) than in practice (see next slide, figure 1.34). Rate of turning of the wind vector=20° per hour in first 12 hours, but between $t=30$ and $t=43$, this is nearly 30° per hour.
Hourly wind direction (red circles) and temperature (black squares) at Ijmuiden at the coast on May 7 and 8, 1976. Horizontal dashed line indicates the direction perpendicular to the coast; Horizontal solid line indicates the direction parallel to the coast.

In how far are the observations of the wind direction reproduced by the model?
Analytical solution

\[
\begin{align*}
\frac{\partial u}{\partial t} &= fv - \frac{A}{\rho} \cos(\Omega t) \\
v &= \frac{1}{f} \frac{\partial u}{\partial t} + \frac{A}{f \rho} \cos(\Omega t) \\
v(t) &= \frac{Af}{\rho \left(f^2 - \Omega^2\right)} \left(\cos(\Omega t) - \cos(ft)\right)
\end{align*}
\]

\(A \approx 1\ hPa\ per\ 100\ km\)

What is the maximum value of \(u\) if \(f=0\), \(\rho=1.16\ \text{kg m}^{-3}\)? Do you expect higher or lower absolute wind speeds if \(f\neq 0\).

How will the wind vector rotate in time in the northern hemisphere (clockwise or anti-clockwise), given that \(u=0\) and \(v=0\) at \(t=0\)?
Analytical solution

\[ u(t) = \frac{A}{\rho \left( f^2 - \Omega^2 \right)} \left( \Omega \sin(\Omega t) - f \sin(ft) \right) \]

\[ \frac{\partial u}{\partial t} = fv - \frac{A}{\rho} \cos(\Omega t) \quad \Rightarrow \quad v = \frac{1}{f} \frac{\partial u}{\partial t} + \frac{A}{f \rho} \cos(\Omega t) \]

\[ v(t) = \frac{Af}{\rho \left( f^2 - \Omega^2 \right)} \left( \cos(\Omega t) - \cos(ft) \right) \]

\[ A \approx 1 \text{ hPa per 100 km} \]

If \( u=0 \) and \( v=0 \) at \( t=0 \), at what earliest time is \( u=0 \) and \( v<0 \) at the latitude of Ijmuiden (52°N)? At what next point in time is this again the case? Does the time difference correspond to the frequency of the forcing?
Adding friction

\[ \frac{\partial u}{\partial t} = fv - \frac{A}{\rho} \cos(\Omega t + \varphi) - \lambda u \]

\[ \frac{\partial v}{\partial t} = -fu - \lambda v \]

“Rayleigh damping”

Write down the “stationary solution” of this system of equations and give an interpretation of this solution.

All questions in this lecture (green background) will be discussed in tutorial 4.
Dynamic Meteorology: lecture 5

Sections 1.18-1.20


9/10/2019: tutorial 4

Problem in Box 1.6 + extra problems on the sea breeze and the stability of thermal wind balance