Geometry
with applications and proofs

distances, voronoi-diagrams

proofs, dynamic software

conics, reflections

Advanced geometry for senior highschool
Student text and background information
Freudenthal Institute
Geometry with applications and proofs

A selection of student text of the 1995-1999 Profi-project for New Mathematics for senior Highschool

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Background information
About this book

The main parts of this book (I - III) make up a course in geometry for senior highschool, specially designed for students in the strand Nature & Technology, the strand which prepares students in their last three years of High School for studying one of the exact sciences or technology at university level.

The course was used in an experimental setting by schools in the so-called Profi-project. Its goal was basically to reintroduce ‘proof’ in a substantial way in the Dutch high school curriculum.

This book contains the greater part of the geometry course. To restrict the size of the book some parts were not included; we restricted ourselves to subjects still in the official Dutch curriculum in the Nature & Technology profile.

Readers and educators who really love mathematics and teaching may immediately try their hand at part I-III, but some background information about the materials may be handy. That’s why we include a short part zero.

Geometry in Dutch education

Geometry in Dutch Junior Highschool is almost completely related to realistic experiences and is in a way highly intuition-based. Exploring spatial objects and shapes, relating different types of images of objects and situations, calculating with proportions on similar figures, a bit of Pythagoras, computation with angles, most of the time in concrete situations, that’s where the focus is on. Argumentation is provoked, but remains in general situation-dependent and the abstraction level is quite low.

Understanding the physical world with the help of some basic mathematical tools is the main goal. It is a geometry for daily life, also preparing well (or: not too badly) for vocational schools and practical jobs.

A lot of teachers, especially at schools of the traditional gymnasium-type, nowadays include more and more provoking and proof-related problems in their geometry teaching, being sure their students can do more intellectually than the curriculum seems to allow.

This junior highschool geometry-curriculum was developed in the early nineties, as a counterbalance for the prevailing boring geometry tasks of those days. Currently there is a tendency to move part of this intuition-based (and intuition-stimulating) type of geometry its proper place in primary education; an outline of possible goals and learning-teaching trajectories has been published recently by the Freudenthal Institute. An English translation will be available in late 2004. *

The present Senior Highschool geometry course in this production fits very well in this picture; in this course a good orientation base of intuitive insight in geometry is helpful in becoming familiar with the more formal demands of mathematical proofs. We tried to link the intuitive and formal approaches without mixing them up, and even making clear to the students what the differences are. In this respect the next chapter (Geometry, classical topics & new applications, by Martin Kindt) is instructive. Important moments in the course around this theme can be found in chapter 2 of part I, Reasoning with distances, and chapter 1 of part II, Using what you Know.

Mathematical contents of the course

Part II (Thinking in circles and lines) is the closest approach the course offers to The Elements of Euclid; the title can be read as a reference to the well known ‘ruler and compass’. But no attempt has been made to cover the first six books of The Elements, where the traditional highschool geometry subjects have their origin. In the course, there is a strong focus on distance and angle related subjects; proportion, similarity and area share a relatively low presence in the text.

This is clear from the very beginning of part I, (Distances, Edges & Regions) where a famous - rather modern - division of a plane area is introduced. The division is natural in situations where there a finite number of points in the area and comparing distances to this points is important an application. The system is called ‘Voronoi-Diagrams’. Voronoi-diagrams are used in many sciences today, from archeology to astronomy and medicine. Basic geometrical ideas like perpendicular bisectors, distances, circles spring up here almost by themselves.

Other distance-related subjects can be found in part I too, for instance the so-called iso-distance lines around regions. An example is the famous 200-mile fishery zone around Iceland. Distance-optimisation of routes, in di-

verse situations, concludes part I.

In part I as a whole one may see a gradual road from application-oriented problems to more pure mathematical thinking. But in this part systematic mathematical deduction is not yet guiding the choice of problems. That changes in part II, by far the most ‘pure’ part of the course. Several ideas which originated in the distance-geometry of part I, are taken up again in a systematic way. Circles and angles (midpoint and peripheral) played a role in determining Voronoi-diagrams, special lines like perpendicular and angular bisectors did also. In part II they are placed in their proper mathematical environment, an environment ruled by clarifying descriptions and organised argumentation, where you are supposed to use only certain statements in the process of argumentation. This typically mathematical way of exploring figures and their relations has its own form of expression: the proof.

A proof should not be a virtuoso performance of a gifted teacher or student on the blackboard in front of the class. It is also not something you have to jot down in your notes, it should be found and formulated by yourself as a student. This is a heavy demand, and because of that we pay a lot of attention in part II to the problem of finding and writing down a proof. This also requires some reflection by the students on their own thinking behavior. In the current situation, students of 16 and 17 year old are involved; with them, such an approach can be realised a lot easier than with the young students who traditionally read Euclid already at 12 or 13 in the not so remote past.

Important in part II is the stimulating role of a Dynamic Geometry Software package DGS, Cabri Géomètre II. We will spend a separate paragraph to it.

Part III (Conflict lines and Reflections) is also connected with part I. A conflict line of two separate regions A and B is the line consisting of the points which have equal distances to the regions A and B. It is in a way a generalisation of the Voronoi-diagram concept of part I and of the perpendicular bisector of two points. The division of the North Sea between Norway, Germany, Holland and Great Britain is an example. Later we specialise for simple regions like points, lines and circles. The conflict lines turn out to be good old ellipse, parabola and hyperbola, as defined by Apollonius of Perga 200 years before Christ. Their properties are studied with distance-based arguments and again with DGS: tangents, director circles and lines, foci and reflection properties.

One of the deep wonders of mathematics is that, as soon as you have clarified your concepts, say from conflict lines in the North Sea to mathematical ones like ellipses and parabolas, those newly constructed ideal objects start to generate new applications by themselves. In part III several old and new acoustic and optical applications of conics are taken in.

In the current edition we did not include the analytical approach to conics. For two reasons: in the Dutch situation this approach was later left out of the curriculum almost totally because of time constraints, secondly because this approach is well known, as in a lot of textbooks it is the only way to get some grip on the subject of conics.

**A short note on axioms and deduction**

Euclid’s started *The Elements* with:

Definition 1: A point is that which has no parts.

We don’t. We start in the middle, where the problems are.

So the practice of the Voronoi-diagram was used to start argumenting in two directions; downward, looking for basic facts to support the properties of found figures, and upward, by constructing new structures with them. The two directions are called analysis and synthesis in the words of Pappos. Euclid on the other hand, and many of his followers after him, presented mathematics as building upward only. This course, especially in chapter 2 of part I, indicates that we see axioms also as objects to discover or to construct, not as given by the bearded old unknown Gods of mathematics.

A anecdote from the classroom will clarify the idea.

At a certain moment triangle inequality was introduced as a basic underpinning of the distance-concept. It expresses the shortest-route idea very well. Let us agree about the triangle inequality, we will use that as a sure base for our arguments! But no, this was not accepted by everybody.

A group of three students asked: Why the triangle inequality and not something else? I countered: well, it is just a proposal. By the way, if I propose something else, you will again probably have objections too, don’t you think? They agreed with that. So I asked: What would be your choice? After a few moments they decide for ‘Pythagoras’ as a basic tool to argue about distances. I said, that’s okay, but there is a problem here: can you base the triangle inequality on ‘Pythagoras’? Ten minutes later they called me again. Yes, they could, and showed me the proof.

The main point of this little story is not the debate over what are axioms and what are not. The main point is that students were actively involved in building up the system itself.

It took the mathematical community over 2000 years (from Euclid to Hilbert) to build a safe underpinning for geometry in a fully axiomatic-deductive way. It is an illusion to think we can teach students such a system in a few lessons. But we can make them help to become part of the thinking process. Probably we don’t come further than local organisation of some theorems and results by this approach in the highschool period. But we reached a cornerstone of mathematics anyway: building by arguments, actively done by yourself.

**Dynamic geometry software**

Part I of the course includes a computer practical; a program which can draw Voronoi-diagrams for a given set of point is used. Such a program allows us to explore prop-
The profi-project’s aftermath

The profi-project was executed in close collaboration by teachers and a designers-group at the Freudenthal Institute and overseen by a committee of university researchers. Experimental textbooks were designed, tried out in class and improved. In a later phase, textbooks were produced by commercial editors, this being the usual approach in the Netherlands. In many aspects the experimental textbooks illustrated the underlying ideas (which are in a way a senior highschool elaboration of the theory of Realistic Mathematics Education) much more clearly than the commercial books do, but on the other hand - it should be said also - the commercial books are sometimes better geared to the daily organisational problems faced by the common teacher and student.

Other activities in the Profi-project were the design of the so called ‘project tasks’. They are meant for individual or team-use by students, to help them do some independent mathematical research, related to a real life or purely mathematical problem situation. In many cases, students themselves chose Cabri as a tool in those tasks.

Each year the Mathematics B-day is organised in the Netherlands, for this group of students. It’s a team-competition on one day. Students attack (in teams of four) such a problem situation, send in their results, hoping for the honour to be one of the best teams and getting a small prize. The enthousiasm is overwhelming and the number of teams involved is still growing. Students themselves commented that working in depth for a longer period on one problem is very stimulating for them. Their sound view is not supported by current antididactical trends in education, where subjects are often split up in small digestible bits and mathematics as the activity of building structures disappears totally out of sight.

Shortly after the project ended, a major organisational change was introduced in Dutch Senior HighSchool, the much debated ‘New Second Phase’. Students were supposed to become overnight independent learners, teachers should lay down their supposed superior role of educator in front of the classroom and become counsellor; regular classroom situations were diminished heavily in time.

The contribution ‘Circle and Butterfly’ in this zeroth part of the book is a report about a regular (but small) class, fighting with the notion of proof. Our viewpoint there is very clear: learning to prove goes very well by communicating arguments in a debate in the traditional classroom, based on provoking problems; independent student work may be part of the process. The teacher is a sparring partner in the debate and a guide to help students get some order in their arguments.

The authors

Wolfgang Reuter was one of the teachers involved in the Profi-project. He put the other designers of the course at important moments with their feet on the ground where the students are. His contribution is visible in the careful working out of some task-sequences in part I and III.

A year after the project ended, Wolfgang died. Almost all his students came to the funeral. Some of them told moving stories about the way he worked with them. Martin Kindt did not only design parts of the geometry course for the project, he also gave shape to the calculus course of the project, which breathes the same air. Part of this course is available in a Swiss edition. Martin retired officially from the Freudenthal institute in 2003, but is still working for the Mathematics in Context project.

Aad Goddijn has been involved in many curriculum development projects in geometry education in both junior and senior Highschool and University level also.

*) ‘Differenzieren - Do it Yourself’ (ISBN 3-280-04020-5; Orell Füssli Verlag, Zurich). The translated title is in line with what is said above, but beware: the book is in German.
Geometry is a Greek invention, without which modern science would be impossible. (Bertrand Russell)

Modeling, abstracting, reasoning

The meaning of the word ‘classical’ depends on the context. The classic interpretation of ‘classical geometry’ is ‘Greek geometry’, as described by Euclid. In his work ‘History of Western Philosophy’, Bertrand Russell gave expression to his admiration of the phenomenal performance of the scientific culture in Greece. Maybe the above quotation gives a sufficient reason why a part of Euclidean geometry should be taught today and in the future and in the future of the future...

To be honest, I must say that Russell is also critical about the Greek approach; he considers it to be one-sided. The Greeks were principally interested in logical deduction and they hardly had an eye for empirical induction.

Lately I found a lovely booklet ‘Excursions in Geometry’ by C. Stanley Ogilvy. The first sentence of the first page says: What is Geometry? One young lady, when asked this question, answered without hesitation, “Oh, that is the subject in which we proved things”. When pressed to give an example of one of the ‘things’ proved, she was unable to do so. Why it was a good idea to prove things also eluded her. The book was written in ’69. If it were written in ’96 a young lady in my country, confronted with Stanley Ogilvy’s question, perhaps would answer: “Oh, that’s the subject in which my daddy told me that he had to prove things”.

Stanley Ogilvy very rightly observes that the traditional method of geometry education failed. The things to prove were too obvious to inspire students, the system was too formal, too cold, too bald. In the late sixties, when he published his geometrical essays, the Euclidean approach was more or less skipped in the Dutch curriculum. Alas, the alternatives such as ‘transformation geometry’ and ‘vector geometry’ did not fulfil the high expectations. Proofs disappeared gradually, the system (if there was one) was not clear for the students.

Back to the classics in a wider sense. The movies of Buster Keaton undoubtedly may be considered as classic.
I remember the famous scene in which he is standing, backwards, before a house just when the front is falling over. It was a miracle, Buster was standing in the right place, where the open window of the roof landed. The brave actor didn’t use a stand-in. Could it be because Buster had an absolute confidence in geometry? Indeed, you can exactly determine the safe position with geometry!

Make a side view as in figure I. The segment $AB$ represents the rectangle on the ground which will be the place of the fallen window. But is the total rectangle a safe area?

Of course not! A man has three dimensions and you have also to reckon with the height. Figure II shows the side view of a safe area.

An interesting question to follow this, is: ‘could the scene be made with a giant?’

As a second example of geometrical modeling I will take the story of a fishery conflict between England and Iceland (in the seventies). England had a big problem with the extension of the Icelandic fishery zone, from a width of 50 miles to a width of 200 miles. In the newspaper we found this picture:

The picture is not only provoking in a political sense, but also geometrically! For instance one can wonder:

- how to measure distances to an island from a position at sea (or vice versa)?
• how to draw the so called iso-distance-curves?
• why is the shape of the boundary of both zones rather smooth compared with the fractal-like coast of Iceland?
• moreover; why is the 200 miles curve more smooth than the 50 miles one?

These are typical geometrical questions to investigate.

I restrict myself now to the first two questions.

How to determine the distance on a map between an island and an exterior point?

You should give this as a, preferably open, question to students (of age 15 for instance).

At a certain moment they will feel the need for a definition. Let them formulate their own definition! After a discussion the class will reach an agreement.

For instance: the distance from an exterior point to the island is the length of the shortest route from that point to the coast.

This definition is descriptive, not constructive. It does not say how to find the distance, how to determine the shortest route. A primitive way is to measure some routes departing from a given point P. In most of the cases you can quickly make a rather good estimation of the nearest point, without measuring all the distances (if... you don’t have too bad an eye for measurements).

More sophisticated is the method using circles. The ‘wave front’ around P touches the island once; the smallest circle around P which has at least one common point with the island, determines the distance:

From this idea, the step to the strategy of drawing an iso-distance curve by means of a rolling circle is not a big one.

Remark: there is an interesting alternative approach of measuring distances departing from points at the coast. The iso distance curve arises as the envelope of the circles with a fixed radius and their centers on the boundary of the island.

Did the Greek geometers have no eye for the aspect of geometrical modeling?

They had; Euclid for instance wrote a book about optics (‘vision geometry’). But they made a separation between pure mathematics (the geometry of the philosopher) and practical mathematics (the geometry of the architect). There is a nice dialogue of Plato between Socrates and Protarchos about the two types of mathematics.

One of the characteristics of the philosophy of Hans Freudenthal is a complete integration between mathematics of real life and so-called pure mathematics. Mathematizing is an activity within mathematics.

In the Iceland case, the fishery conflict can be a good starting point to develop a theory about iso-distance curves of simple geometrical shapes like a quadrangle, to study the difference between convex and not-convex shapes and to make local deductions. There are also possibilities to link this subject with calculus. For instance, it is easy to understand geometrically that in the case of an island with a ‘differentiable boundary’, the shortest route from an exterior point to the island has to be perpendicular to the boundary.

Indeed, the circle which determines the distance has a common tangent line with the boundary of the island L and the tangent of the circle is perpendicular to the line segment PPf.

In our standards for the math curriculum on pre-university level, the following three important aspects are mentioned:

• Modeling: the student will get insight in the coherence between a mathematical model and its realistic
source.

- **Abstracting**: the student will learn to see that a mathematical model may lead to an autonomous mathematical theory in which the realistic source disappears to the background.
- **Reasoning**: the student will learn to reason logically from given premises and in certain situations, will learn to give a mathematical proof.

**the Dutch geometry curriculum**

The Buster Keaton problem fits very well in the curriculum for the age group 12-15 (‘geometry for all’), it is an example of ‘localization’, which is one of the four strands:

1. **Geometry of vision** (about vision lines and vision angles, shadows and projections, side views and perspective drawings)
2. **Shapes** (2 and 3-dimensional)
3. **Localization** (different types of coordinates, elementary loci)
4. **Calculations in geometry** (proportions, distances, areas, volumes, theorem of Pythagoras)

As characteristics of this ‘realistic geometry’ I will mention:

- An intuitive and informal approach
- A strong relationship with reality
- No distinction between plane and solid geometry, everything is directed at ‘grasping space’.

The Iceland problem can be extended to a rich field of geometry which I will call here ‘geometry of territories’. It fits very well within the new geometry curriculum envisaged in the nature and technology profile of pre-university level (age 16 - 18).

From 1998 we distinguish four profiles in the Dutch curriculum:

- **Culture and Society**
- **Economy and Society**
- **Nature and Health**
- **Nature and Technology**

In each of the four profiles mathematics is a compulsory subject, but only in the fourth profile is geometry a substantial part of the curriculum (besides probability and calculus). We developed (and experimented with) a new program for the ‘Nature profiles’ in the last three years. Thereby we paid attention to:

- the relationship between mathematics and the subjects of the profile (physics, chemistry, biology);
- the mathematical language (how specific should it be?)
- the role of history (mathematics was and is a human activity);
- the use of technology (graphic calculator, software such as Derive and Cabri);
- the ideas of horizontal and vertical mathematization, local deduction.

I will focus on the geometry part here. We chose the following three strands:

1. **Classical metric plane geometry** (especially: loci based on distance and angle)
2. **Conic sections** (synthetic approach)
3. **Analytic geometry** (elementary equations of loci)

The most important contextual sources in the new program are:

- Territories (conflict lines and iso distance curves)
- Mirrors (focus, normal, tangent)
- Optimization (shortest path, minimal angle)

Involving as main activities:

- Exploring (using computers)
- Modeling
- Proving (local deduction).

**Geometry of Territories**

The North Sea is divided in national territories. A point in the sea which is equidistant from England (GB) and the Netherlands (NL) is called a conflict point of both countries. All possible conflict points form a conflict line (or conflict curve).

The boundaries of the national territories at sea are parts of conflict lines. Studying a map, it is notable that there exist ‘three nation points’. For instance there is one point which is equidistant from GB, NL and DK (= Denmark).

Students can reason why: the intersection of the conflict lines (GB, NL) and (NL, DK) is a point which is on the one hand equidistant from GB and NL, on the other equidistant from NL and DK. Conclusion: the point is equidistant from GB and DK (following the first common notion of Euclid: things which are equal to the same thing are also equal to one another).

So the intersection point has to be a point of the conflict line of GB and DK.

This is a well-known scheme of reasoning, Polya speaks about the pattern of two loci.

To simplify things I will study the territories of five small islands (say points) in the ocean.

Where the ‘wave fronts’ around two islands meet each other, we have a conflict line. In this case the conflict lines are perpendicular bisectors. The fourth picture, without the circles is called a Voronoi-diagram. The territories are called Voronoi cells. A boundary between two adjacent cells is called an edge. Three edges can meet in one point (‘3 nations point’), such a point is called a vertex. The ‘islands’ are the centers of the diagram. Voronoi-diagrams (also called ‘Thiessen polygons’) are applied in a lot of disciplines: archeology, geography, in-
informatics, robotics, etc. Software exists which makes complicated Voronoi-diagrams on the computer screen and students can do a lot of explorations. After some lessons they have a rather good idea of this concept, and then we ask them typical ‘sophistic’ questions.

- Why can we be sure that the boundaries are straight lines?
- Why do three boundaries sometimes meet in one point?

Analyzing these questions we find two important reasons:

1. Every point of the perpendicular bisector of two points is equidistant from the two points;
2. Every point outside the perpendicular bisector of two points is nearer to the one or nearer to the other (depending on the side of the boundary).

How to prove 1?

In former times we used the congruence of triangles (the case SAS). In the age of transformation geometry we used the basic principle of reflection in a line. Our students, who have grown up with realistic geometry, proposed using the theorem of Pythagoras. It is worthwhile discussing about these things. You can continue to ask ‘why’, but at a certain moment you have to choose starting points, we call them ‘basic rules’. A powerful basic rule is the ‘triangle inequality’ from which it follows that a point on the same side as A of the perpendicular bisector of A and B, belongs to the territory of A.

Now the pattern of the two loci can be applied and we have a local rigorous proof. With the students we compare two directions of thinking:

In the traditional approach of geometry education we only followed the logical path. That was one of the big didactical mistakes. It is important to show the students (of all levels) the genesis of a piece of mathematics from time to time: the path of exploration. Often the history of mathematics is an excellent source of inspiration, but in
In this case I took a rather modern subject, which is really a rich one. The Voronoi theme gives rise to a lot of problems to investigate.

A few examples:
1. Given four points. Make a classification of all types of Voronoi-diagrams.
2. Given four points. One point moves along an arbitrary straight line. How does the Voronoi-diagram change? (see below)
3. 4 nations points are very rare. Can you find a criterion for a such a point? Is it possible to formulate this in terms of angles? This leads to the concept of cyclic quadrangle and the theorem of opposite angles.
4. Given three Voronoi edges, meeting in one point. Can you reconstruct the centers? How many possible solutions are there?
5. Study the Voronoi-diagrams of regular patterns. For instance: 12 points regularly lying on a circle give a star of rays. If you add a new center (the center of the circle) than a regular polygon arises:

How does the shape of the polygon change if the center moves to the ‘North’?

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1. We looked for 4 nations points in the Atlas; only on the map of the USA we found one: the common vertex of Utah, Colorado, Arizona and New Mexico.
Do the same with a row of equidistant points (the Voronoi-diagram consists of parallel strips. If you add one new center outside the row, we get an interesting figure:

You could call a part of this pattern a ‘discrete parabola. If we interpolate the row with more and more points the polygon will tend to a parabola!

We get the continuous version of the Voronoi-diagram in this case, if we study the conflict line between a straight line (‘coast’) and a point (‘very small island off the coast’). Take an arbitrary point on the coast line and draw the perpendicular bisector between the island and this point. With Cabri you can move the point along the coast and see how the perpendicular bisector envelopes a curve. The conflict line of island and coast is by definition a parabola and now the students spontaneously discover the property of the tangent of a parabola, which leads to important technical applications (parabolic mirror, telescope).

In the right picture a point $Q$ outside $P$ is drawn on the perpendicular bisector of $L$ and the foot of $P$ ($P_f$).

If a ship is at the position $Q$ it is clear that it is nearer to the coast than to $L$ (for $Q$ has equal distances $a$ to $L$ and $P_f$, and because $Q$ is outside of $P$ this distance is longer than the shortest route to the coast ($= b$)

So every point from the perpendicular bisector, except $P$, belongs to the territory of $C$ and this means that you can say that the line is a tangent of the parabola.

Analogously we find a hyperbola or an ellipse in the case where the coast line is a circle.

From both pictures above you can deduce that $d(P, M) - d(P, L)$ respectively $d(P, M) + d(P, L)$ are constant (namely the radius of the circle) and now you know that the conflict line is respectively (a portion) of a hyperbola and an ellipse.

For the students we used the concept of conflict line to introduce both types of conic section and we confronted them afterwards with the classical definitions. In both cases the perpendicular bisector of $L$ and $P_f$ is the tangent. If the point $L$ is substituted by a circle (with a radius smaller than the radius of $C$), we get the same results.

Now there is a ‘world’ of nice exercises about conflict.
lines and parabolic, hyperbolic or elliptic mirrors.

**Some conclusions**

After three years of experiments with students of age 16-18, we are very hopeful that the new geometry curriculum can be realized in a motivating way.

Our experience taught that:

- the students felt challenged by classical geometrical problems,... provided that these are either introduced by meaningful contexts or are discovered by empirical activities;
- sometimes the students are more critical with proofs than the teacher;
- the geometry software (we used Cabri, but Sketchpad seems to be a good alternative) is really a success; students enjoy the dynamic character and they don’t have difficulties with managing the program;
- students are aware of the uncertainty of a discovery by means of the computer; they experience a need to prove non-trivial results.

The geometry stuff is a lucky mixture of ‘old fashioned’ geometry about circles and conic sections and new applications (Voronoi-diagrams). Using new technology makes things much more accessible.

While the students for whom this stuff is meant, are much more mature than the students who were confronted with classical Euclidean geometry in the past, making geometrical proofs is attainable. On the other hand, these older students are less disciplined than the younger ones from the past and this may be a problem when presenting proofs. This last point seems to be the most difficult one. But remember the words of Stanly Ogilvy: to avoid the catastrophe of an uninspired and uninspiring geometry course we will beg the forgiveness of the mathematicians, skip the formalities and take our chances with the rest.

**Literature**

Given: circle with butterfly
or: how do you learn proving?

aad goddijn

What preceded?

The advanced geometry of mathematics B2-VWO explicitly contains the subject ‘proving in the plane geometry’. This article wants to give an impression of how this could work in a classroom. The class used for this article is a small 6 VWO-B group from the Gregorius College in Utrecht; the school belongs to the circle of ten schools which have been working with the experimental material of the profi-team. The advanced geometry starts with the book ‘Distances, edges and domains’ (see bibliography). This book gradually works from several applications of the concept of distance – among them Voronoi-diagrams, iso-distance lines and optimization problems – towards making proving more explicit. In the next part, ‘Thinking in circles and lines’, it is explicitly explained what a proof is, what you can use in one and how to write one down.

The new geometrical material in this part is really geared to the previous book; the theorem of the constancy of the inscribed angle on a fixed arc and the theorem of the cyclic quadrilateral are important. Since these will play an important role in the examples used later on, here is an illustration of both.

The constant angle theorem says that if $A$, $B$, $C$ and $D$ lie on one circle and $C$ and $D$ lie on the same side of line $AB$, then $\angle ACB$ and $\angle ADB$ are equal.

The twin of this theorem is the theorem of the cyclic quadrilateral. This says: if $A$, $B$, $C$ and $D$ lie in one circle and $C$ and $D$ lie on different sides of $AB$, then the angles at $C$ and $D$ are $180^\circ$ together.

With such building blocks a lot can be done in numerous proofs. Next is an example of what the learning of proving could look like in this stage.

Karin, one of the students in this class, shows that there is something special about the wings of the butterfly shown in the figure on the right.

In the proof ‘angles on the same arc’, that is the named constant angle theorem, has been referred to twice. The idea behind the proof is good, but the execution is not yet perfect: this is called similarity rather than congruence and two angles is enough.

In the course of the learning process solutions and usage of terminology become more accurate and better written. This needs to be worked on in class, but this is not the key point when it comes to learning proving. The real problem for Karin and her classmates Sigrid, Janneke, Bas, Mark, Monica, Marleen and Petra is: how do you find a proof in a still unfamiliar situation? And for their teacher Marcel Voorhoeve: how do I help them finding proofs themselves? The second half of ‘Thinking in circles and lines’ deals especially with this search – and learning how to search – for proofs.

I am sitting here in the classroom as co-author of the material used in the class and would like to see how this develops.

Form as tool

A beautiful proof is like a good sonnet: form and content
support each other. Example exercise 1 explicitly asks for a proof in a certain format, which has been seen before. Part c refers back to the proof of the concurrency of the three perpendicular bisectors of a triangle ABC. Briefly, the proof goes like this. Let the perpendicular bisectors of AB and BC intersect in M. Then \( d(A, M) = d(B, M) \) holds and also \( d(B, M) = d(C, M) \). Connect the equalities and you have that \( d(A, M) = d(C, M) \). From there, it also follows that M lies on the perpendicular bisector of AC. The characterization by equal distance of the perpendicular bisector is used, first twice from middle-and-perpendicular to equal distances and then after the connection step once from equal distances to middle-and-perpendicular. The students know this as the 1-1bis form. This form has been assimilated in a scheme in ‘Distances, edges and domains’.

In example problem 1 a lot of help is offered: it is even made clear that you should not suppose that the circle through B and C goes through the intersection S of the other two circles. This will bear fruit; later on, one of the students used in a completely different, but very difficult proof the phrase ‘you cannot assume that.’. But now first Sigrid’s solution:

The connection step is not explained, it is hardly necessary and it did not fit into the outline: the rest has been explained right above the fragment shown here, including the reverse of the cyclic quadrilateral theorem, which is used in the conclusion.

Heuristics

Such a format can be trained and practiced, but for me rules like ‘if you need to prove that three lines or circles go through one point, then use the outline of 1-1bis’ are absolutely not done. This leads to mock results. In this fashion laws are laid down where the student needs to learn to make choices and come up with his or her own plans. Moreover, such rules lead as often to nothing as they lead to real solutions. If one wants to help students finding (or choosing) the form of proofs, the support that is offered must have a more open character. It needs to improve oriented searching, but can never give a guaranteed solution strategy. Such guide rules are also called heuristics. In Anne van Streun’s dissertation ‘Heuristic math education’ he mentions the two just named properties. Van Streun offers a good overview of mathematicians and didactics, which have both coped with this subject and compares several approaches in this area; the mathematical-relevant target area is specified no further than ‘the subject matter of 4VWO’.
Still to be recommended, especially since there are many geometric examples in there for this audience, is ‘How to solve it’ from George Polya. In Polya heuristic reasoning is meant to find a solution; but the heuristic reasoning is certainly not meant to be the proof itself. Polya’s harder founded ‘Mathematical Discovery’ contains a first chapter named ‘The Pattern of Two Loci’. Some of the heuristics used in ‘Thinking in circles and lines’ can be found there.

In the remainder I will assume the view that some heuristics are very general like ‘make sure you understand the problem, then come up with a plan’ and others are more subject-specific, like the example of the three circles from above. I also would like to show more examples than to preach general theories. Due to the restricted size of the Nieuwe Wiskrant [the magazine where this article was published originally, see http://www.fi.uu.nl/wiskrant/] not all named heuristics in ‘Thinking in circles and lines’ will be discussed here. I will not limit my comments on the work of students and teachers solely to heuristics. In an active process of learning a lot of things occur at the same time.

**Recognizing**

Almost nobody had a problem with example exercise 2, but it does bring some special things to light. One of those things that novices in proving need to practice in plane geometry is recognizing several familiar configurations within a new complex figure. Herein also lies an opportunity for the teacher in the classroom to revisit what is known, or at least should be known. Marcel, the teacher in our case, gratefully used this opportunity regularly. You can doubt whether the students need to know ‘heuristics’ explicitly, but at least for the teacher it is of importance to keep a couple of heuristics in readiness as keys in a learning conversation.

Bas and Mark work together; they have recognized the theme: the right-angled triangle in the half circle, so the theorem of Thales:

\[ \angle BAC = 180° - \angle ABD = 180° - \angle CBD \]

The angles at \( D,E \) and \( F \) are 90°, thus the fourth angle of quadrilateral \( DEBF \) has to be the same. Good, but the first step of the proof, the perpendicularity of \( \angle ABD \) and \( \angle CBD \) now is unpleasantly useless. Mark observed that \( D \) could also lie somewhere else on the great half circle, then \( DEBF \) would still be a rectangle. Then why was the tangent \( BD \) to the small half circles given at all? That was a sharp insight! Here one of the facts was redundant. Normally this is not the case in this kind of geometry and this is a good occasion to point out another general heuristic: check during your work whether you used all that was given!

**Learning to note**

The next exercise was: show that \( EF \) is a tangent line of both small half circles. (By the way, in 1996 this was part of the second round of the Dutch Mathematics Olympiade.) All components of the proof are shown in the figure Monica has drawn.

She writes down the actual proof pretty briefly: the crosses, balls, squares and other things in angles and on line segments do the actual work.
Such symbols (and often a complete rainbow of felt-tips) come in very handy in the phase of searching for a proof. But it remains draft work, a neat form of noting must be worked at.

Initially students use several angle notations like $\angle ABD$, $\angle A_1$ and mix many symbols, also in the proofs presented to the public. The first two are, in combination with a sketch, acceptable, but the third (the crosses, balls, squares and so on) is not, since the indicated angles are not uniquely fixed, the symbols only indicate the equality in angles, not which angles they are.

There is a good traditional manner to improve the correctness of the writing: just let them write down a proof in detail, correct it and provide it with personal comments. It takes time, but it pays off; students often develop their own specific notations, which need comments. Here is a piece of comment given by Marcel Voorhoeve on a piece of Bas’ work. Naturally Bas knew what he meant and Marcel started from there as well, but it looked as if the direction of the logic went in the opposite direction.

The comment points out that the arrow is not being used correctly, and is used to introduce an explanation rather than a conclusion.

**Find a link**

A different specific geometrical heuristic was introduced with example exercise 3.

**example exercise 3**
Here two circles are given and two lines $l$ and $m$, which go through the intersections $A$ and $B$ of the circles.

![Diagram of two circles and lines](image)

To prove: $PQ \parallel RS$.

**Approach:** assume the idea that you need to prove parallelity through finding equal angles and look for a link. The circle and the points $A$ and/or $B$ of course play a big role.

Volunteer Mark starts his story for the class on the overhead projector after adding a few numbers with: I am going to prove that $\angle Q_2 = \angle S_2$. As far as I am concerned nothing can go wrong: the very general heuristic of ‘know what it is about’ has been applied. Because of this the story to come has a goal and a direction. This follows from earlier class conversations; it often occurs that the student tells a story in class, which is a totally dark path with lots of detours for the rest of the class. It is only a matter of time before someone -- students or teacher -- asks: what on earth are you talking about? These are enlightening moments, since the student in question is often able to say what it is about in one sentence!

The specific heuristic, which under the hood of ‘approach’ is alluded to, is really a totally different one. There is no theorem you can directly apply to show the equality of angles. Thus one needs some intermediate step, object, angle or something else. Despite the rather directive hint in the text of the assignment, which will quickly lead to sketching the help line $AB$, it will take some time before the proof can be seen clearly from the drafts. After six crossed out lines, this is what Mark’s notes say

From $Q$ we first go to intermediate stop $A$, and from there to $S$. In the notebook the right angles get as many attention as the usage of cyclic quadrilaterals, but in the explanation the leading role is for the cyclic quadrilateral. This is based on the fact that you can use the angles at $A$ as a link.

Also, there are numerous different variations here in the approach possible for the students. In essence they all use the same elements, but that is not seen right away. Someone who used Z-angles instead of F-angles may think that she found a different proof. In this case the class conversation is of great value, the teacher makes clear what the essential line is and what the necessary details are. Thus it came out that the proofs generally differed only in detail.

The strategy of finding links, of which later on an example in a different frame, has a very positive side effect: threshold reduction. A student who suddenly, after doing a lot of exercises, solving equations and working out brackets, is confronted with the proof question in this example sometimes is likely to sigh: well, I don’t know, no idea how I should do it. ‘Find a link’ therefore also means: see if you are able to write something down, even if you do not know up front whether it leads to the solution. After a while you - may - have enough pieces and even have one or two which match in order to solve your
problem. The kindness of the part of geometry on which we are working is that there is so much opportunity for these learning processes, which by the way do not all evolve consciously.

The ‘link’ is as old as proving in geometry itself. Book I of Euclid’s ‘The Elements’ contains, after a list if twenty-three definitions and five postulates, the five ‘general rules’ and the first is:

1. Things, equal to the same, are equal to each other.

In Euclid a lot of reasoning steps would need some more argumentation according to today’s mathematical standards, but this blindingly obvious platitude is called upon explicitly in crucial parts of proofs. Looked at from a logical perspective, general rule number one formulates an explicitly allowed reasoning step, but in the practice of proving in The Elements it is clearly a structuring tool. I consider it a heuristic.

????????? Yessssssss!!!!!!

While solving a proof exercise, the keystone of the vaulting sometimes falls into your hands without prior notice. This is a nice moment, a flood of bright white light is surging trough your head, chaos alters into a pattern and suddenly all lines, angles and circles are made of sparkling crystal. And the path to the proof is wide open lying in front of you.

Did such heavenly moments occur in the class? Yes, and not so rarely. To some extent Mark has that feeling for a moment if after six lines of bungling he starts over and the proof rolls out in a tight bow. There the watch suddenly has started ticking and naturally this happens more. The moment often closes a period of frustrated searching, but I do feel that the ‘good feeling’ for many students is more than just the relief ‘Oh, am I glad this is over’.

Next is such a fragment, which was audible in class. Example exercise 4 closes the link-section; right before, it has been said that you need to do more than only use one link.

In the class Janneke has been working on it for a while. In the four drawn circles she has made cyclic quadrilaterals and now she is staring at an anthill of numbered, signified and colored angle relations. She needs to show that in quadrilateral $EFGH$ the two opposite angles together make $180°$. But for Heaven’s sake how?

Suddenly a cry: ‘but those also lie on a circle!’. Those are the points $A$, $B$, $C$ and $D$. At this time the proof arises. A scratch through the mess, we are starting to rewrite and the details almost fill out themselves in the computation. This will provide a second bonus: Yeesss, it is correct!!!

Thus three stages: hard work with possible frustration, breakthrough of the insight and getting the verification conclusive.

‘Vigor, vision, verification: aspects of doing mathematics’. This was de title of the inaugural speech of Prof. F. Oort in 1968, in Amsterdam. Beautiful to see it so clearly with VWO-students!

In Janneke’s notebook (see below) the frustration phase is very easy to recognize in the upper part. From that part it is easy to see how Janneke (in the figure in the cheap experimental book of course) has numbered

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**example exercise 4**
The circles $c_1$, $c_2$, $c_3$, and $c_4$ intersect as is shown in $A$, $B$, $C$, $D$, $E$, $F$, $G$ and $H$.

To prove If $A$, $B$, $C$ and $D$ lie on one circle, then $E$, $F$, $G$ and $H$ also lie on one circle.

---

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the sub-angles. Also, here there is no visible line in the proof and no usage of the relations between the A- and C-angles. But underneath the line - the moment of insight - it goes really well, the verification is running. The first line is confusing for a moment; it still needs to be proven that ∠EHG + ∠EFG = 180°, the goal of the computation has been announced here so to speak. Look, at the end this equality returns. The fourth line (the first that starts with 360) contains the joining of the opposite angles ∠EHG and ∠EFG; underneath this the key step has been written down in full:

∠A_1 + ∠A_2 + ∠C_1 + ∠C_2 = 180°

Next the deduction is continued by manoeuvering these four angles in the right positions, after which the result follows. ‘thus ∆’ obviously means ‘thus EFGH is a cyclic quadrilateral’. Yes, yes, you need to write down after that E, F, G and H thus lie on a circle, but we won’t stumble at a trifle right now. It is something that at some point needs to be learned, that you really need to touch the finish line!

Translating

In example exercise 5 a by now familiar figure is shown.

**example exercise 5**

A familiar figure. The circumcircles of the three equilateral triangles pass through one point. You know that and you can use it.

That the three circles have a point of intersection may be used, since this has been proven. It is useful to let something like this occur explicitly in the learning path; it shows something of the structure of the course. Students are often prepared in their schoolish kindness to again prove that the three circles intersect. Beforehand it has been said - a heuristic - that sometimes you need to translate that which is to be proven into something else. The translation can be almost the same as the original. For example: isosceles triangles, this is the same as equiangular triangles. Or: three points lie on a line, then two line segments make an angle of 180° with each other. Things which are very close to each other, but give one handle more and some more flexibility.

Here is a fragment from Karin’s work; the remainder of the proof is showing that the angles at S are indeed angles of 60°. That is simple.

Transcription: You may not assume that A, S, D are on one line. If they are on one line, then ∠ASD = 180°. The explication of what still needs to be done, namely showing that ∠ASD = 180°, also helps preventing to walk into the trap of already using that form ASD one line. Later this is emphasized on the blackboard by using two different colors for AD and DS, an old-fashioned neat classroom trick for the teacher.

**Conjectures and Cabri**

In the final chapter the students have to formulate their own conjectures while they are experimenting with Cabri. These conjectures will be proven later. A special heuristic belongs to this learning method. (The following example is exactly the one I have used before to compare several dynamic geometry programs. Whoever is not familiar yet with Cabri, can look at Nieuwe Wiskrant 17(3).)

In the computer room I am sitting in front of the screen next to Petra and Mark. A circle has been drawn, a triangle lies on the circle with its vertices, so that the vertices can be dragged, while the circle remains fixed. The orthocentre of the triangle has been drawn. Now C is moving over the circle, and therefore H moves as well. Petra let H make a trail; Cabri has an option to do so. What happens? H also moves over a circle. The effect is spectacular when you see it happening and it immediately raises the question: why a circle? This question is a natural motive for looking for a proof. The figure can be seen in Petra’s work on the next page.

An important heuristic in Cabri-work (or other dynamic geometry programs) is: look at the movements on the screen and try to find something that is moving also, but has something constant to it. If you find something like this, you may have a key, maybe a link, in your hands for the proof. This is a good working approach for Cabri and I mention it in the conversation in front of the screen. Petra answers that ∠AHB is constant and shows that she also sees the constancy of ∠C. The bell interrupts and all I can say is ‘you can do this’. I feared that Petra looked at it a little bit differently: she maybe deducted the constancy of ∠AHB from the fact that H lies on a fixed circle.
through $AB$. That is assuming what you need to prove, the most deadly sin there is in mathematics.

She has worked it out on paper at home, and the result surprised me. I was wrong, or Petra changed her mind. Look at how Petra starts her argumentation:

\[
\text{if the path of } H \text{ is a circle, then } \angle AHB \text{ must be constant, so we will prove it to be.}
\]

That is the old heuristic, which Pappos named ‘analysis’: exploring the problem from the assumption that we have the solution. Next should be the synthesis-phase: constructing the proof from the given, the opposite direction of the analysis. Petra’s synthesis starts at the fixed angle $C$; a long detour of almost a page in which – how else – we encounter cyclic quadrilateral $CFHE$, leads her via

\[
\angle AHB = 180^\circ - \angle ACB
\]

to the required result.

Petra must have enjoyed this success; she closes very professionally with step 13, where it becomes clear that she acknowledges the (not at any cost necessary) case distinction.

\[
\text{if } C \text{ is on the other side, it is the same.}
\]

Translation: if $C$ is on the other side, it is the same.

Other side: down under $AB$, yes.

Finally: teacher and student

During this story I have pointed out several times that heuristics also belong to the conversation tool kit of the teacher. The heuristics then are an aid to finding coherence in the search process. The idea here is not about one heuristic being better than another, but about the direct effectiveness with respect to the content of it all. The emphasis lies on stimulating the search so that one will no longer say: I do not know that now, so I cannot do it. This is achieved in this little paradise class.

Teacher Marcel Voorhoeve also takes on other roles besides the organizational-directing one: the role of co-solver and also that of critical sounding board via questions dealing for example with half-grown proof steps and sometimes students take over that last part in conversations. The task of demonstrating on the blackboard is hardly of any importance; as far as learning to find proofs it does not seem very effective, and directing towards acceptable ways of noting down could also be learned through the students’ work.

Sieb Kemme and Wim Groen have written in Nieuwe Wiskrant 19(2) about problem solving as a trade. After their introduction, I of course started to deal with their example problem in a different manner, but also their reflections on the search process matched my approach and agree with what I have brought up in this article. Reflecting independently is something I see Sieb, Wim and myself however, naturally – or just because of age and professional knowledge – do more than the students in this 6 VWO class; in 6 VWO such things are more open for discussion than in sayn a 3 grammar school class. Here again lies a task for the teacher.

Geometrical footnotes

Sometimes to keep on solving a problem yourself and to look how you do that, remains an important exercise for those who have to teach those things. So why not add a couple of nice continuations of one of the problems presented in this article?

1. The point $S$ in example exercise 1 is the first point of Fermat, $F_1$, although the Italians will keep it calling the point of Torricelli. Reflect the triangles also to the other side of the sides; show that the three circles then also pass through one point $F_2$. Use the plagiarism-heuristic: detailed copying of a proof with some small changes.

2. In example exercise 5 $AD$, $BE$ and $CF$ all three of course go trough the point $S$ (of $F_1$). But those three segments also have the same length. This should not be difficult to prove, especially for those whose memory of a previous phase of the geometry education (transformations) is still vivid.

3. Plagiarize exercise 2 like exercise 1 plagiarizes example exercise 1.

4. In example exercise 5 it was proven that $AD$, $BE$ and $CF$ go through one point, for the case that the outer triangles are equilateral. Now put three isosceles, mutually uniformly, triangles with their bases on the three sides of $ABC$ and now prove also that $AD$, $BE$ and $CF$ go through one point. You need to forget about the circles! This exercise may be seen as more difficult.

5. Sketch a triangle (with Cabri). Sketch both Fermat-points, the center of the circumcircle and the center of the nine points circle and show that these four lie on one cir-
J. Lester has showed this remarkable relation in 1995. Heuristic: use someone else’s work from the Internet.

**Literatuur**


