Moduli Spaces
Spring 2014
Carel Faber

In the first lecture (February 6), I started with some remarks concerning the classification problem in algebraic geometry, following Hartshorne I.8, but focusing on curves. Next, I mentioned some results about M_g, the moduli space of nonsingular curves of genus g, one of the most important examples of moduli spaces. (No proofs.) To study M_g without knowing curves does not make much sense; at the end of the lecture, I formulated the Riemann-Roch theorem.

In the second lecture (February 13), I went through most of the proof of R-R in Fulton's Algebraic Curves.

In the third lecture (February 20), after discussing some details related to the second lecture, I proved Clifford's theorem and stated when equality holds. Then I discussed Hartshorne's proof of R-R (which uses cohomology and abstract duality theory). In the third hour, I defined smooth curves of genus g over noetherian schemes, the contravariant functor $\mathcal{M}_g$, and what it means that there exists a coarse moduli space M_g for (smooth) curves of genus g (following Mumford's Geometric Invariant Theory, §5.2). I mentioned that $\mathcal{M}_g$ is not representable (due to the existence of curves with nontrivial automorphisms) and briefly discussed two 'fixes' for this 'problem'.

In the fourth lecture (February 27), I discussed some standard results about curves: the Hurwitz theorem; any curve can be embedded in $\mathbf{P}^3$ and admits a birational morphism to a plane curve with at worst ordinary nodes; for $g \geq 2$, the canonical system is base-point-free; it embeds the curve exactly in the non-hyperelliptic case; geometric R-R (as in [ACGH]). Then I gave the standard construction of the coarse moduli space H_g of hyperelliptic curves of genus g in characteristic $\neq 2$ (assuming it exists). In particular, $\dim H_g = 2g - 1$. Next, I explained that M_3 is the disjoint union (as sets) of the 6-dimensional moduli space Q of nonsingular plane quartics (an open in $\mathbf{P}^{14}$ modulo PGL(3)) and H_3. However, M_g is irreducible (fact). So we want a construction of M_3 reflecting this, telling us in particular how a hyperelliptic curve can be seen as a limit of plane quartic curves; I briefly explained this. I also discussed the three standard types of smooth curves of genus 4. In the third hour, I discussed flat and smooth morphisms. I then began the discussion of Mumford's construction of M_g ([GIT, 5.2]): ν-canonical curves ($\nu \geq 3$); Grothendieck's theorem that the Hilbert scheme exists; the three conditions that are imposed to obtain a locally closed subscheme H_ν of the Hilbert scheme of curves with the correct Hilbert polynomial; the quotient of H_ν by the projective linear group will be the moduli space.

In the fifth lecture (March 6), I continued with Mumford's construction. I recalled the definition of smoothness in SGA 1, Exposé II and that this is an open condition. Next, the condition $h^0(s) = 1$ defines the open set (by semicontinuity) over which the fibers are connected. In the second and third hour, I gave the definition of stable n-pointed curves and discussed their dual graphs and the corresponding stratification of $\overline{M}_{g,n}$ (by "topological type"): the closure of a stratum is a union of strata; the codimension of a stratum equals the number of nodes; strata of codimension 1; strata classes in rational cohomology (or in Chow groups); the usual fundamental classes and the Q-classes (or stack classes); the behaviour of Q-classes under transversal intersections.

In the sixth lecture (March 13), I discussed deformations, following [ACG]: definitions, first-order deformations, Kodaira-Spencer class, arbitrary deformations, K-S homomorphism, Kuranishi families; Schiffer variations; at $h^1(C, T_C)$ general points, they generate $H^1(C, T_C)$ (C a smooth curve of genus g); Schiffer variations can be integrated. In the second half, I discussed some of the difficulties caused by the fact that stable n-pointed curves can have a nontrivial automorphism group (which is finite): the functor is not represented by a scheme; over a non-algebraically closed field k, isomorphism classes of curves over k do not correspond one-to-one to k-points of M_g. Over finite fields k, points of $M_g(k)$ come at least from curves over k; and if we count curves over k up to k-isomorphism with the reciprocal of the order of the k-automorphism group, we obtain the number of points of $M_g(k)$. I illustrated this with counting $|H_g(k)|$ when $|k| = q$ is odd (using N_d, the number of monic squarefree polynomials in one variable of degree d: $N_d = q^d - q^{d-1}$ when $d \geq 2$). Answer: $|H_g(k)| = q^{2g-1}$. Also, $|M_{1,1}(k)| = q$.

[HAG] Hartshorne: Algebraic Geometry.
[FAC] Fulton: Algebraic Curves.
[ACGH] Arbarello, Cornalba, Griffiths, Harris: Geometry of Algebraic Curves, Volume I.
[ACG] Arbarello, Cornalba, Griffiths: Geometry of Algebraic Curves, Volume II.
[GIT] Mumford: Geometric Invariant Theory.

Many thanks to Valentijn Karemaker and Ralph Klaasse: the description above of Lectures 4–6 is based on their notes.

Do not hesitate to contact me when you have questions: `C.F.Faber@uu.nl`