Mathematical Modelling

Lecture 1

Dimensional analysis

Preliminaries

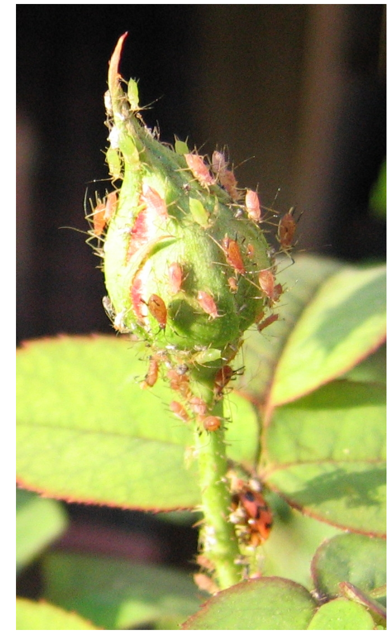
- You
- Me
- Modelling
  - focus is applied mathematics, techniques for constructing models in various applications
  - no real theory of modelling: experience; this course: a taste for how models are constructed
  - ODEs and PDEs (limited knowledge)
- Scoring: based on 3/4 projects (coherent problem set, a bit of computing, short report)
- announcement: no classes next week

Apples and oranges (pears? cats!)
Dimensional analysis

- (Don't let it fool you!)
- DA may suggest the form of a functional relation, sanity check
- Allows us to put a problem in canonical form
- Transform a solution for a specific case
- Identify a minimum set of parameters

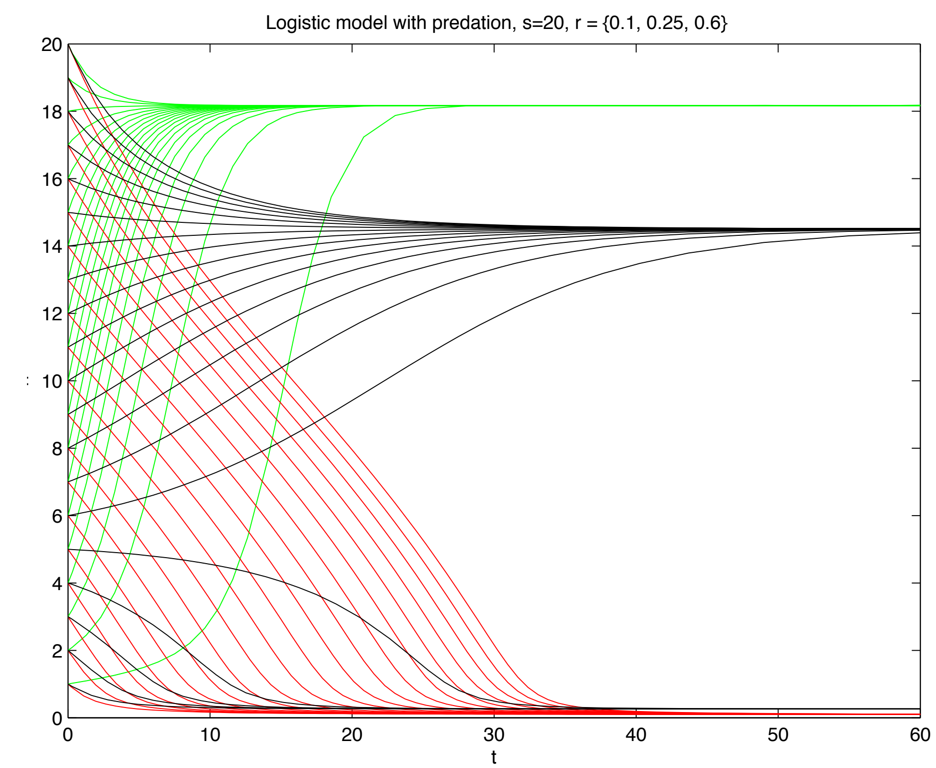
Aphids and Ladybugs

bifurcation analysis

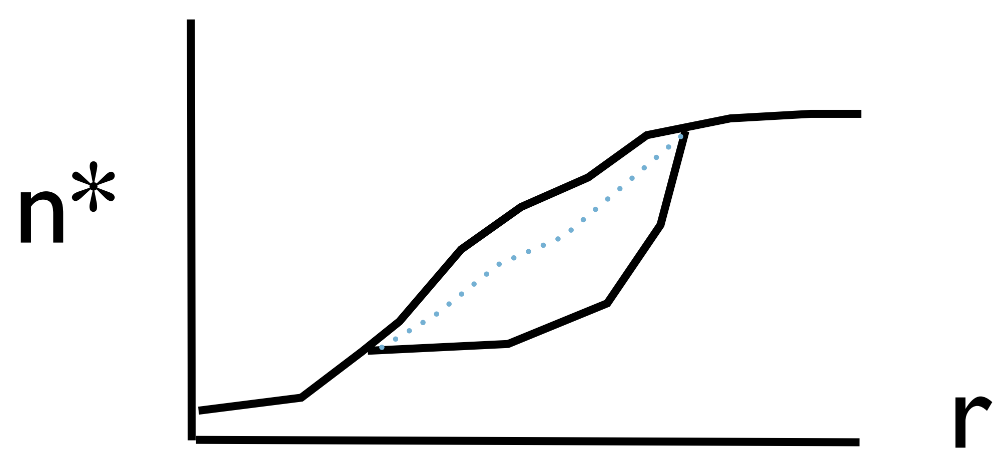


- Aphid reproduction...
- Logistic growth with predation

$$\frac{dn}{dt} = rn(1 - \frac{n}{N}) - \frac{n^2}{c + n^2}$$



- 3D parameter space (r, N, c)
- stable steady state populations



Apples and oranges (pears? cats!)
Dimensional analysis

The main idea: you can't add apples and oranges (or apples and cats)

- the equations that express a physical phenomenon should hold independent of the measurement units used:

1. both sides of an equality, inequality, sum, difference must have the same units
2. ratios, products, rational powers may mix units (km/hr)
3. arguments to transcendental functions must be dimensionless

- fundamental units: length (L), time (T), mass (M), charge (Q), temperature (θ)
- denote by [A] the fundamental dimensions of the quantity A (if X denotes length, then [X] = L.
- (see Table 1.1 in the book)

Example: projectile problem

- Dimensional model: $$\frac{d^2x}{dt^2} = -\frac{gR^2}{(R+x)^2}, \quad x(0) = 0, \quad \frac{dx}{dt}(0) = v_0$$
- x = height above ground, g = gravitational acceleration, R radius of earth
- $x + R \approx R \qquad \dfrac{d^2x}{dt^2} \approx -g, \quad \Rightarrow \quad x(t) \approx -\dfrac{g}{2}t^2 + v_0 t$
- $x_{\max} \approx \dfrac{v_0}{2g}$

Example 1: projectile problem

- Dimensional analysis:
- Functional dependence presumed known: $x_M = f(m, g, v_0)$
- Dimensions: $[x_M] = L, \quad [g] = LT^{-2}, \quad [m] = M, \quad [v_0] = LT^{-1}$
- Dimensional equation: $[x_M] = [m^a \, v_0^b \, g^c]$
- Equating dimensions:

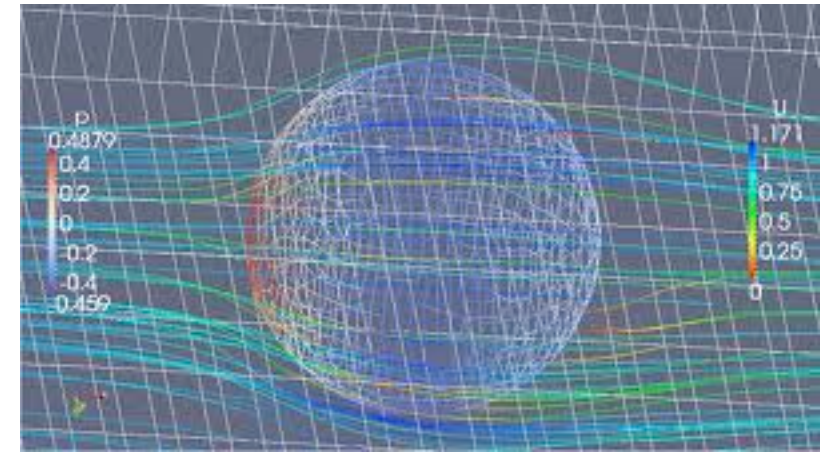
$$L = M^a (L/T)^b (L/T^2)^c$$
$$= M^a L^{b+c} T^{-b-2c}$$

$$L : \; b + c = 1$$
$$T : \; -b - 2c = 0$$
$$M : \; a = 0$$

$$\{a = 0, b = 2, c = -1\}$$

- The only possibility is $x_M = \alpha \dfrac{v_0^2}{g}$
- Parameter determined by experiment, check scaling law

Example 2: drag on a sphere



- Functional dependence presumed known:

$$D_F = f(R, v, \rho, \mu)$$

- Dimensions: $$[D_F] = MLT^{-2}, \quad [R] = L, \quad [v] = LT^{-1},$$ $$[\rho] = ML^{-3}, \quad [\mu] = M(LT)^{-1}$$

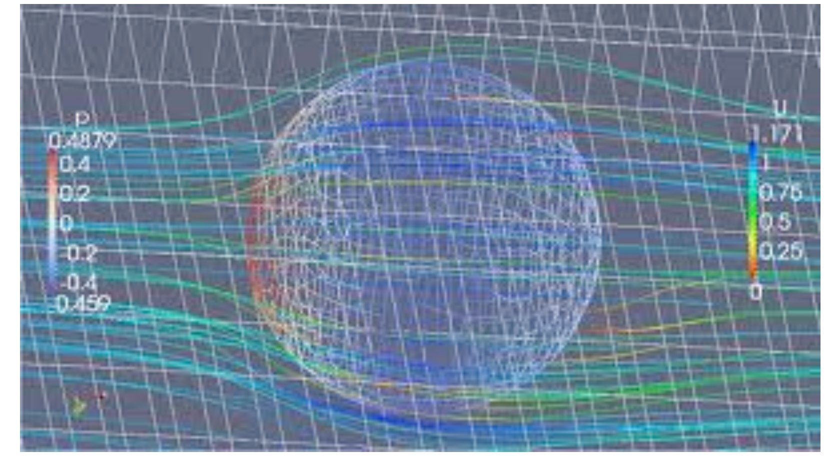
- Dimensional equation: $[D_F] = [R^a\, v^b\, \rho^c \mu^d]$

- Equating dimensions: $$\begin{aligned} MLT^{-2} &= L^a (L/T)^b (M/L^3)^c (M/LT)^d \\ &= L^{a+b-3c-d} T^{-b-d} M^{c+d} \end{aligned}$$

$$\begin{aligned} L &: \; a + b - 3c - d = 1 \\ T &: \; -b - d = -2 \qquad\qquad \{a = 2 - d, b = 2 - d, c = 1 - d\} \\ M &: \; c + d = 1 \end{aligned}$$

- A solution is: $D_F = \alpha R^{2-d} v^{2-d} \rho^{1-d} \mu^d = \alpha \rho R^2 v^2 \left(\dfrac{\mu}{R v \rho} \right)^d$

Example 2: drag on a sphere



- "Dimensionless group": $\Pi = \dfrac{\mu}{Rv\rho}, \quad [\Pi] = 1$
- Since d is arbitrary, the general form is

$$D_F = \rho R^2 v^2 F\left(\Pi\right)$$

- Here, $[D_F] = [\rho R^2 v^2]$
- Solution is nonunique. Alternative is

$$D_F = \frac{\mu^2}{\rho} H\left(\Pi\right)$$

- But these must be functionally dependent

$$H(\Pi) = \frac{1}{\Pi^2} F(\Pi)$$

The Buckingham-Π theorem

- Given a functional dependence $q = f(p_1, p_2, \dots, p_n)$
- With dimensions: $[q] = L^{\ell_0} T^{t_0} M^{m_0}$
  $[p_i] = L^{\ell_1} T^{t_1} M^{m_1}$
- Dimensional analysis: $[q] = [p_1^{a_1} p_2^{a_2} \cdots p_n^{a_n}]$

$$\begin{array}{ll} L: & \ell_1 a_1 + \ell_2 a_2 + \dots \ell_n a_n = \ell_0 a_0 \\ T: & t_1 a_1 + t_2 a_2 + \dots t_n a_n = t_0 a_0 \\ M: & m_1 a_1 + m_2 a_2 + \dots m_n a_n = m_0 a_0 \end{array}$$

- Linear system (m x n): $A\vec{a} = \vec{b}$ (dimensionally (in-)complete)
- Homogeneous problem: $A\vec{a} = 0 \quad \Rightarrow \quad K(A) = \mathrm{span}\{\vec{a}_1, \dots, \vec{a}_k\}$
- General solution: $\vec{a} = \vec{a}_p + \gamma_1 \vec{a}_1 + \cdots + \gamma_k \vec{a}_k$
- Dimensionless groups $\vec{a}_i = (\alpha_i, \beta_i, \dots, \omega_i) \quad \Rightarrow \quad \Pi_i = p_1^{\alpha_i} p_2^{\beta_i} \cdots p_n^{\omega_i}$
- Dimensional group: $\vec{a}_p = (\alpha_p, \beta_p, \dots, \omega_p) \quad \Rightarrow \quad Q = p_1^{\alpha_p} p_2^{\beta_p} \cdots p_n^{\omega_p}$
- General form: $q = Q\, F(\Pi_1, \dots, \Pi_k)$

The Buckingham-Π theorem

- Assuming the formula $q = f(p_1, p_2, \ldots, p_n)$ is dimensionally homogeneous and dimensionally complete, then it is possible to reduce it to one of the form $q = Q\, F(\Pi_1, \ldots, \Pi_k)$ where $\Pi_1, \ldots, \Pi_k$ are independent dimensionless products of $p_1, p_2, \ldots, p_n$. The quantity Q is a dimensional product of $p_1, p_2, \ldots, p_n$ with the same dimensions as q.

Nondimensionalization

- Change of variables $x = X\xi, \quad t = T\tau$
- Identify the dimensionless groups (by inspection, division)
- Use dimensionless groups to determine scaling:
  - Rule of Thumb 1: us the DGs that appear in the ICs
  - Rule of Thumb 2: use DGs that are robust wrt scaling
- Example: Projectile problem

$$\frac{d^2x}{dt^2} = -\frac{gR^2}{(R+x)^2}, \quad x(0) = 0, \quad \frac{dx}{dt}(0) = v_0$$

$$\frac{d}{dt} = \frac{d}{d\tau}\frac{d\tau}{dt} = \frac{1}{T}\frac{d}{d\tau}$$

$$\frac{d^2}{dt^2} = \frac{1}{T^2}\frac{d^2}{d\tau^2}$$

$$\frac{1}{T^2}\frac{d^2}{d\tau^2}(X\xi) = -\frac{gR^2}{(R+X\xi)^2}, \quad \xi(0) = 0, \quad \frac{1}{T}\frac{d}{dt}X\xi(0) = v_0$$

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$$\left(\frac{X}{gT^2}\right)\frac{d^2\xi}{d\tau^2} = -\frac{1}{\left[1+\left(\frac{X}{R}\right)\xi\right]^2}, \quad \xi(0) = 0, \quad \left(\frac{X}{v_0 T}\right)\frac{d\xi}{d\tau}(0) = 1$$

$$\Pi_1 = \frac{X}{gT^2}, \quad \Pi_2 = \frac{X}{R}, \quad \Pi_3 = \frac{X}{v_0 T}$$

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$$\Pi_1 = \frac{X}{gT^2}, \quad \Pi_2 = \frac{X}{R}, \quad \Pi_3 = \frac{X}{v_0 T}$$

$$\text{RoT1:} \quad X = v_0 T$$

$$\text{RoT2:} \quad \Pi_2 \ll 1 \quad \Rightarrow \quad X = gT^2 = \frac{v_0^2}{g} \qquad T = \frac{v_0}{g}$$

Nondimensionalization

- Change of variables $x = X\xi, \quad t = T\tau$
- Identify the dimensionless groups (by inspection, division)
- Use dimensionless groups to determine scaling:
  - Rule of Thumb 1: use the DGs that appear in the ICs
  - Rule of Thumb 2: use DGs that are robust wrt scaling
- Example: Projectile problem

$$\frac{d^2\xi}{d\tau^2} = -\frac{1}{(1+\varepsilon\xi)^2}, \quad \xi(0) = 0, \quad \frac{d\xi}{d\tau}(0) = 1$$

$$\varepsilon = \frac{v_0^2}{gR}$$