MATHEMATICAL MODELING, PROJECT 1 (2012)

ALGAE MODEL WITH ZOOPLANKTON PREDATION

The term *Zooplankton* refers to a tremendously diverse class of tiny aquatic creatures. A few of them look like this:

We like zooplankton because they eat *Algae*, keeping our swimming waters clean. They exist at the foundation of the aquatic food chain and also play an important role in removing CO_2 from the environment. In this project we will study a number of mathematical models for the population dynamics of algae and zooplankton.

Our models will ignore spatial effects, so we consider just the concentrations $A(t)$ of algae and $Z(t)$ of zooplankton in a well-mixed lake, given in milligrams per liter water (mg/L).

The parameters used in the problem are defined in the following table. All parameters are assumed to be positive. Additionally, the concentrations of algae and zooplankton will always take values in the first quadrant: $A(t) > 0$ and $Z(t) > 0$.

symbol	units	value	meaning
A	mg/L		Concentration of algae
e	1		Efficiency of food-growth conversion for Z
g	$\text{day}^{-1}\text{mg}^{-1}\text{L}$	0.4	Grazing rate of Z
h	mg/L	0.6	Half-concentration of A at which Z get 'hungry'
K	mg/L	10	Carrying capacity for A
m	day^{-1}	0.15	Death rate for Z
r	day^{-1}	0.5	Peak growth rate for A
Z	mg/L		Concentration of zooplankton

The goal of the project is to study a number of increasingly sophisticated (yet still elementary) models of algae-zooplankton dynamics (A-Z dynamics) using the techniques we have learned during the first part of the course. The format of the project will be a short report that addresses *at least* all of the following questions. The report should be well-written and coherent (not just a list of answers) and will account for 20% of your grade. Depending on the font size and number and size of plots, I would expect it to be under 10 pages in length. You may submit a single report for a group of at most three students. These should be placed in my mailbox, on the 4th floor of Science Park 904, before 9.00 on 29 October.

(1) **Basic model.** To begin with we consider the following model, expressed in reaction notation:

$$\begin{aligned} A &\xrightarrow{r} 2A, \\ A + Z &\xrightarrow{g} 2Z, \\ Z &\xrightarrow{m} D. \end{aligned}$$

The first reaction describes the growth of algae in the absence of zooplankton. The second reaction describes the growth of the zooplankton population due to grazing on algae. The third reaction describes the passing of zooplankton on to the underworld (i.e. death), denoted by $D(t)$. We are not interested in the equation for D. The numbers above the arrows are the reaction constants.

(a) Derive the dynamical equations for A and Z.

(b) Excluding the initial conditions, what is the effective dimension of the parameter space? You may nondimensionalize A, Z and t to eliminate excess parameters.

(c) Determine any equilibria and discuss their stability.

(d) Determine a conservation law of the following form (α_j need to be determined), and discuss the stability of the equilibria in light of this:

$$H = \alpha_1 \ln A + \alpha_2 \ln Z + \alpha_3 A + \alpha_4 Z.$$

(e) Plot a few solutions of this model in the phase plane. Why might the model not be a good model for A-Z dynamics.

(2) Saturation model. In the previous model, the growth of A is only limited by grazing by Z. A more realistic model accounts for the fact that the lake can support only up to a limited amount of A (otherwise there are not enough nutrients, or sunlight does not reach deep enough for photosynthesis, etc.) To account for this, the term rA in the previous model needs to be replaced by a logistic term

$$rA\left(1-\frac{A}{K}\right),$$

where K is the *carrying capacity* of the lake for algae.

(a) Determine a set of kinetic reactions for this new model.

(b) We will primarily be interested in how the solutions depend on the growth rate r and the carrying capacity K. Rewrite the equations in dimensionless form, so that the new dimensionless groups $\bar{r}$ and $\bar{K}$ appear in place of r and K. (You may ignore the initial conditions when doing this.)

(c) Examine the equilibria and their dependence on $\bar{r}$ and $\bar{K}$.

(3) **Saturation model with overexploitation.** Another model improvement accounts for variation in the rate of consumption. For instance, if the population of A is too low, Z may not be interested, and the Z have some maximum rate of consumption. We modify the product AZ (everywhere it appears) to take the form

$$\frac{AZ}{A+h},$$

where h is a constant parameter.

(a) Nondimensionalize this model as follows: First, introduce a rescaling $t=\bar{T}\tau$, $A=\bar{A}a$, $Z=\bar{Z}z$. Next divide through the differential equation for a by $r\bar{A}$ and form the nondimensional groups. Similarly divide the differential equation for z by, say, $\bar{Z}/\bar{T}$ to form nondimensional groups. Finally, choose the scalings $\bar{A}$, $\bar{Z}$, and $\bar{T}$ such that the only free parameters left in the model are $\bar{h}\propto h$, $\bar{m}\propto m$, and $\bar{r}\propto r$. We will fix $\bar{m}=1/2$ throughout this problem. (The notation $\bar{h}\propto h$ means "$\bar{h}$ is proportional to h", that is, there is some C, independent of h, such that $\bar{h}=Ch$.)

(b) Taking $\bar{r}=1$, determine the equilibria and their stability. How does the stability depend on $\bar{h}\in(0,1)$? Make a bifurcation plot of the locations of any nontrivial equilibria (in phase space) as functions of $\bar{h}$. Use a solid line in the regime for which the equilibrium is stable and a dashed line where it is unstable.

(c) Plot some solutions of the system (phase space and as functions of time) for stable and unstable values of $\bar{h}$. Describe the solutions.

(d) Next we want to study the asymptotic behavior of this model for large values of $\bar{r}$ (rapid algae growth). We do this by taking $\bar{r} = \varepsilon^{-1}$, and analyzing the behavior for lowest order in ε. Calculate one term expansions of the solutions and compare these with numerical solutions. What conclusions can you draw about the dependence of the solutions on $\bar{r}$.

(**Hints for problem 3):** To gain some insight into this problem, it may help to plot the nullclines of a and z and directions of the vector field on these lines and within the areas they delineate. For the last part, you may also introduce a (further) rescaling of time $\tau \mapsto \sigma = \tau/\varepsilon$ and identify 'fast' and 'slow' variables.

(4) Generally compare and contrast the three models. Under what conditions is each model realistic? How could you make use of observations of real A and Z concentrations and their time dependence to choose a model?