MATHEMATICAL MODELING, PROJECT 2 (2012)

SUPPLEMENTARY NOTES

Herein, we give some supplementary notes that may be helpful in completing Project 2. These notes mainly consist of identities from multivariable calculus. For Part (1) of the project it is recommended that you work at the level of vectors and operators, and not work with the vector components individually. For example, in the expression

$$\nabla \times \boldsymbol{B} = \begin{pmatrix} \partial_y B^z - \partial_z B^y \\ \partial_z B^x - \partial_x B^z \\ \partial_x B^y - \partial_y B^x \end{pmatrix},$$

the preferred notation is that on the left side.

Let $\boldsymbol{x} = (x, y, z)$ denote coordinates in $\mathbb{R}^3$, and $\phi(\boldsymbol{x}) : \mathbb{R}^3 \to \mathbb{R}$ be any twice differentiable scalar function and $\boldsymbol{A}(\boldsymbol{x}) : \mathbb{R}^3 \to \mathbb{R}^3$ be any twice differentiable vector function. Then the following identities hold:

$$\nabla \times \nabla\phi(\boldsymbol{x}) = \text{curl grad}\,\phi = 0, \qquad \nabla \cdot \nabla \times \boldsymbol{A} = \text{div curl}\,\boldsymbol{A} = 0.$$

Obviously, you should convince yourself that these hold.

Helmholtz decomposition. Any vector field $\boldsymbol{F}(\boldsymbol{x}) : \mathbb{R}^3 \to \mathbb{R}^3$ (sufficiently smooth and rapidly decaying) can be written as

$$\boldsymbol{F}(\boldsymbol{x}) = \nabla\phi_F(\boldsymbol{x}) + \nabla \times \boldsymbol{A}_F(\boldsymbol{x})$$

for some scalar function ϕ_F and some vector function $\boldsymbol{A}_F$. In this case ϕ_F and $\boldsymbol{A}_F$ are referred to as the *scalar potential* and *vector potential* for $\boldsymbol{F}$, respectively. Note that using our identities above it follows that

$$\nabla \cdot \boldsymbol{F} = \nabla \cdot \nabla\phi_F, \qquad \nabla \times \boldsymbol{F} = \nabla \times \nabla \times \boldsymbol{A}_F.$$

If fact, using the additional identity

$$\nabla \times \nabla \times \boldsymbol{A} = \nabla(\nabla \cdot \boldsymbol{A}) + \nabla \cdot \nabla\boldsymbol{A},$$

where it is understood that the Laplace operator $\nabla^2 = \nabla \cdot \nabla$ in the last term operates on each component of $\boldsymbol{A}$ individually, the Helmholtz decomposition is unique if one also restricts $\boldsymbol{A}_F$ to the class of divergence-free vector fields ($\nabla \cdot \boldsymbol{A}_F = 0$). This is because the Laplace operator can be uniquely inverted on rapidly decaying functions.

First variational of a vector function. Let us consider a concrete example:

$$S[\boldsymbol{A}] = \iiiint \frac{1}{2}|\nabla \times \boldsymbol{A}|^2 \, dt\, dx\, dy\, dz.$$

Consider a variation $\delta\boldsymbol{A}(t, \boldsymbol{x})$. The first variational of S is

$$\delta S = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}(S[\boldsymbol{A} + \varepsilon\delta\boldsymbol{A}] - S[\boldsymbol{A}]).$$

Computing gives

$$\delta S = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \iiiint \frac{1}{2}\left[|\nabla \times \boldsymbol{A}|^2 + 2\varepsilon(\nabla \times \boldsymbol{A}) \cdot (\nabla \times \delta\boldsymbol{A}) + \varepsilon^2 |\nabla \times \delta\boldsymbol{A}|^2 - |\nabla \times \boldsymbol{A}|^2 \right] dt\, dx\, dy\, dz$$

$$= \iiiint (\nabla \times \boldsymbol{A}) \cdot (\nabla \times \delta\boldsymbol{A})\, dt\, dx\, dy\, dz.$$

Next we integrate by parts to isolate the arbitrary variation $\delta\boldsymbol{A}$. To do this, one can either work with components, which gets messy, or use the following identity:

$$\nabla \cdot (\boldsymbol{A} \times \boldsymbol{B}) = \boldsymbol{B} \cdot (\nabla \times \boldsymbol{A}) - \boldsymbol{A} \cdot (\nabla \times \boldsymbol{B}).$$

If we integrate both sides of this expression over a domain V, and apply the divergence theorem to the left side:

$$\iiint_V \nabla \cdot \boldsymbol{F}\, dV = \oiint_S \boldsymbol{F} \cdot \boldsymbol{n}\, dS,$$

where V is a volume and S the surface of V. This gives

$$0 = \iiint \nabla \cdot (\boldsymbol{A} \times \boldsymbol{B})\, dx\, dy\, dz$$

under the assumption that the vector field is zero on the boundary of V. (The same holds on an unbounded domain if the vectors decay rapidly at infinity.) Therefore,

$$\iiint \boldsymbol{B} \cdot (\nabla \times \boldsymbol{A})\, dx\, dy\, dz = \iiint \boldsymbol{A} \cdot (\nabla \times \boldsymbol{B})\, dx\, dy\, dz.$$

This defines our integration by parts formula for the first variation,

$$\delta S = \iiiint \nabla \times (\nabla \times \boldsymbol{A}) \cdot \delta\boldsymbol{A}\, dt\, dx\, dy\, dz,$$

and since the variation $\delta\boldsymbol{A}$ is arbitrary, the functional derivative of S with respect to $\boldsymbol{A}$ is the vector

$$\frac{\delta S}{\delta \boldsymbol{A}} = \nabla \times \nabla \times \boldsymbol{A}.$$