Models and Simulation: Examples

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Abstract

This document contains examples discussed in the lectures.

Contents

1 Economic model with adaptive expectations

Every year a farmer has to decide how much of each crop to plant based on his available resources and what he expects the prices to be. The choices made by all farmers determines the total supply of a given produce. This supply, coupled with the demand of buyers, ultimately determines the price. But the farmer has to predict the price in order to decide on the right distribution of his resources. [1]

Let p_n denote the (mean) *market price* of a commodity, say corn, in year n, and suppose that the demand is a linear function of price:

$$D(p) = a - d\,p, \tag{1}$$

where $a > 0$ and $d > 0$ are positive constants.

It is common in economics to assume a *representative agent*, that is, a single meta-farmer whose choice represents the combined behavior of all farmers. By some means, which we will define later, the meta-farmer determines an *expected price* P_n for year n, and this price determines how much corn he plants and subsequently, the supply of corn on the market. The formula for supply is

$$S(P) = c + \arctan(\lambda(P - \bar{p})), \tag{2}$$

where c, λ and $\bar{p}$ are all positive constants.

Because corn is a perishable commodity, we assume that the price p_n adjusts such that the market *clears*, that is, all of the corn is sold in year n and supply equals demand:

$$D(p_n) = S(P_n). \tag{3}$$

Substituting (1) and (2) into (3) and solving for p_n we obtain an equation for the market price as a function of the expected price

$$p_n = \frac{1}{d}\left[a - c - \arctan(\lambda(P_n - \bar{p}))\right].$$

The model is complete if we can now relate the meta-farmer's expected price P_n to the actual price p_n, and there are various economic theories about how our farmer does this.

At one extreme, we assume our meta-farmer is omniscient. In economics, one says he uses *rational expectations*. In this case, the farmer's expectation is precisely the market price

$$P_n = p_n. \tag{4}$$

[1] Example from C. Hommes, *Behavioral Rationality and Heterogeneous Expectations in Complex Systems*, Springer-Verlag, 2013.

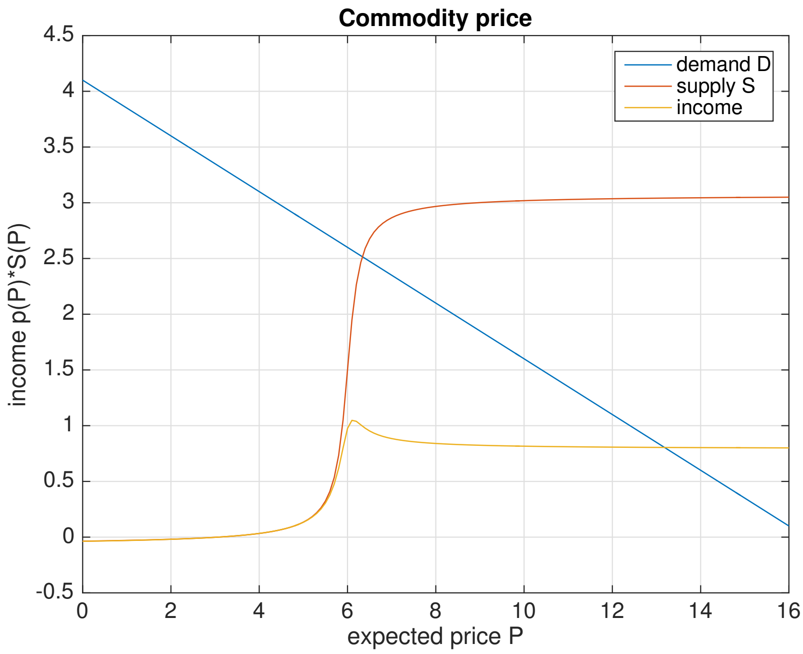

Figure 1: Supply curve $S(P)$, Demand curve $D(p)$ and revenue curve $p(P) \cdot S(P)$ as functions of the market price p and expected price P.

This seems fanciful, but economists use an evolutionary argument: any farmer whose expectations differ consistently from the rational expectation will eventually be driven out of the market by competition.

The simplest approach is called the *naive expectation*. In this case, the farmer just expects the price this year will be the same as the market price last year:

$$P_n = p_{n-1}. \tag{5}$$

A point of discussion in economics is whether seemingly random market fluctuations are purely caused by external forces, or if such behavior can follow from the internal market dynamics. Figure 1 shows the supply and demand curves for parameter choice: $a = 4.1$, $d = 0.25$, $c = 1.5$, $\lambda = 4.8$ and $\bar{p} = 6$. The intersection of the supply and demand curves is the rational expectation equilibrium $P_n = p_n$. We also plot the revenue curve, which is the product of the realized market price $p(P)$ times the supply $S(P)$ (equal to demand) for a given expected price, showing that the rational expectation equilibrium is a profitable choice. In Figure 2 we see the time series of points P_n over a long period (100 years). The price oscillates around the rational expectation equilibrium. This oscillatory behavior is observed in real commodity markets, and is known as the "hog cycle" (referring to pork prices).

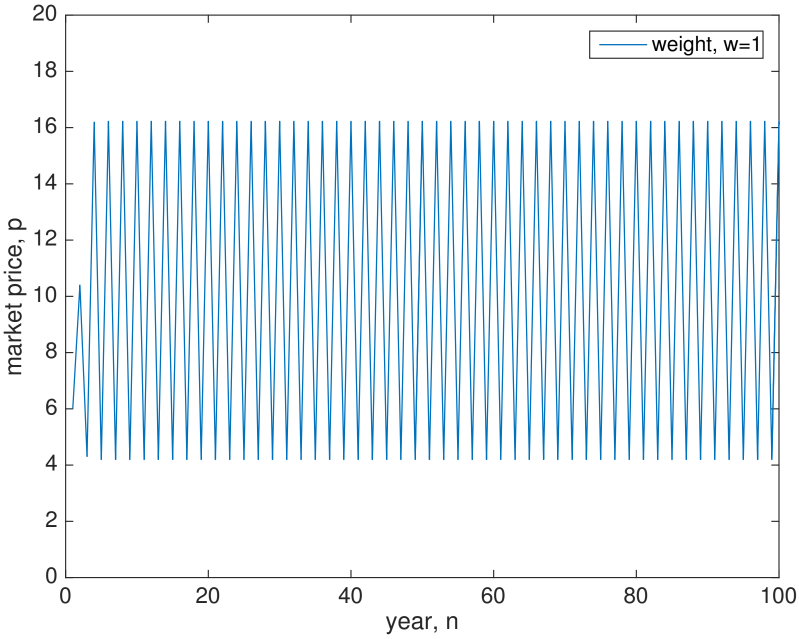

Figure 2: The "hog cycle" oscillates around the optimum price as shown, independent of the initial expected price P_0.

2 Classical mechanical model of a solar system

A solar system consists of a sun and and a number of planets or other bodies interacting in a gravitational potential. Classical mechanics is grounded in Newton's laws of motion. The most important of these is the second law, which is often written:

$$F = m \cdot a,$$

where F represents a force, m a mass, and a the acceleration imparted to the mass by the force. This is the essence of the second law, that the effect of applying a force to a mass is to induce an acceleration. A more useful form of the second law is

$$\sum_i F_i = \frac{d}{dt}(mv),$$

which now states that the sum of the applied forces is equal to the change in *momentum*. Momentum is the product of mass and velocity.

Many forces in physics can be expressed as the gradient of a potential field. That is, the force applied at a point $X = (x, y, z)^T \in \mathbf{R}^3$ is given by

$$F(X) = -\nabla U(X) = -\begin{pmatrix} \partial U/\partial x \\ \partial U/\partial y \\ \partial U/\partial z \end{pmatrix}, \qquad U : \mathbf{R}^3 \to \mathbf{R}.$$

The potential $U(X)$ is a scalar valued function, its gradient is a vector.

For a solar system, the gravitational potential is the sum of a pairwise potential between every pair of bodies. Let $X_i(t) = (x_i(t), y_i(t), z_i(t))^T \in \mathbf{R}^3$, $i = 1, \ldots, k$, be the position of the ith body (a *body* is either a planet or the sun). Further let the mass of body i be denoted M_i. The pairwise potential felt by body i due to body j (and vice versa) is

$$U_{ij}(X_i, X_j) = -\frac{GM_iM_j}{\|X_i - X_j\|},$$

where G is the gravitational constant. The associated forces

$$F_{ij} = \frac{\partial}{\partial X_i} U_{ij} = \frac{-GM_iM_j}{\|X_i - X_j\|^3}(X_i - X_j) = -F_{ji}$$

where F_{ij} is the force body j exerts on body i.

The second law of Newton now reads

$$M_i\ddot{X}_i = -G \sum_{j \neq i}^{k} \frac{M_iM_j}{\|X_i - X_j\|^3}(X_i - X_j), \quad i = 1, \ldots, k.$$

We can introduce the velocity $V_i = \dot{X}_i$ and write this in first order form (note that the M_i's cancel in the above)

$$\dot{X}_i = V_i, \quad \dot{V}_i = -G \sum_{j \neq i}^{k} \frac{M_j}{\|X_i - X_j\|^3} (X_i - X_j), \quad i = 1, \ldots, k.$$

Note that to have a unique solution, it is necessary to know both the positions $X_i(0)$ and velocities $V_i(0)$ at time $t = 0$.