Exercises (WISB134)

April 4, 2016

1 Exercises for Chapter 1

Exercises for Chapter 1 consist of a Mathematica tutorial and accompanying exercises from Chapter 0 of the book by Lynch.

- Install Mathematica. See the website: `https://ict.science.uu.nl/index.php/Mathematica_%28NL%29`.
- Work through the tutorial in Chapter 0 of the book by Lynch. See `http://link.springer.com.proxy.library.uu.nl/content/pdf/10.1007%2F978-0-8176-4586-1_1.pdf`.
- Work exercises 0.2, 0.4–0.7, 0.10 at the end of Chapter 0 (of Lynch).

2 Exercises for Chapter 2

Exercise 2.1

For investments in which one expects a growth factor between $\kappa = 1.01$ and $\kappa = 1.2$, economists use the approximation

$$T = \frac{72}{100(\kappa - 1)},$$

where T is the number of years in which the investment doubles, and $100(\kappa - 1)$ is the yearly interest rate. Suppose an economist asks you for which values of T and κ the approximation is accurate. What would your answer be?

Exercise 2.2

Find the the general solution of:

(a) $\frac{dy}{dx} = \frac{1+2y}{4-x}$

(b) $\frac{dy}{dx} = -\frac{\cos x}{\sin y}$

(c) $\frac{dy}{dx} = -(1 + y^2) \cdot \ln x$

(d) $\frac{dy}{dx} = \frac{y}{x^2-3x+2}$

(e) $\frac{dy}{dx} = \frac{yx}{x^2-3x+2}$

Exercise 2.3

The lifespan of a lightbulb filament. We consider the filament in an incandescent lightbulb as a thin, straight cylindrical wire. The wire maintains a high constant temperature due to the flow of electric current. We further assume that the length ℓ of the wire is constant. As a consequence of the high temperature, there is a continual evaporation of material from the wire. In each time unit the amount of material that evaporates is proportional to the exposed surface area of the wire. A time $t = 0$ the wire has thickness d_0. The wire breaks when it becomes thinner than $0.1d_0$.

(a) Derive a differential equation for the wire thickness d.

(b) Solve this differential equation.

(c) Determine the lifespan of the wire given that its thickness at time $t = 0$ is 0.1mm and at time $t = 100$ is 0.09mm.

Exercise 2.4

A mothball is a spherical ball of camphor that evaporates quickly. The mass of the mothball decays due to evaporation, and the amount lost per time unit is proportional to the exposed surface area of the sphere. The mass G (in grams) is a function of time t (in weeks).

(a) Show that there exists a positive constant c such that:

$$G' = -cG^{2/3}.$$

(b) Suppose the mothball original weighs 10 grams and after ten weeks only 1 gram. After how many weeks is the ball completely evaporated?

Exercise 2.5

Consider a lake with a constant volume V in km^3 which contains a certain quantity of chemical waste. Assume the waste is always uniformly distributed in the lake water. At time t the concentration of waste is give by $c(t)$. A polluted river empties into the lake. The concentration of waste in this river is k, and it delivers an amount s of polluted water into the lake per time unit. Another river removes water at the same rate. Derive a differential equation for $c(t)$ and solve it.
Compute $\lim_{t\to\infty} c(t)$.

Exercise 2.6

(a) Classify the values of t_0 and x_0 for which the following ODE has a unique solution:

$$\dot{x} = \sqrt{x}, \quad x(t_0) = x_0.$$

(b) What is the first time $t > 0$ for which the solutions of the following initial value problems cease to exist?

(i) $\dot{x} = x^3$, $x(0) = 1$;

(ii) $\dot{x} = x^2 + 2x + 1$, $x(0) = -\frac{1}{2}$.

Exercise 2.7

We iterate the function $f : x \mapsto \kappa\, x$ on the whole real line $\mathbf{R}$, (hence allowing also negative values of x, and also κ may be any real number. How does the recursively defined sequence $x_{n+1} = f(x_n)$ behave as $n \to \infty$? Answer this question for fixed κ, and as determine how many 'different' cases there are. How important is the choice of x_0?

Exercise 2.8

Consider a function f defined on an open interval of $\mathbf{R}$ and twice continuously differentiable there. The Newton process attempts to determine a zero α of f as follows. Choose a number x_0 as an approximation to α. Construct a correction to x_0: $\alpha = x_0 - h$. Hence,

$$0 = f(\alpha) = f(x_0 - h) = f(x_0) - hf'(x_0) + \frac{1}{2}h^2 f''(\xi).$$

Here we have written the Taylor series for f about x_0, where ξ is a number between x_0 and $x_0 - h$. If h is small, the h^2-term will be much smaller than the other terms, $f(x_0)$ and $hf'(x_0)$. If we neglect the h^2-term, then we can compute h: $h = f(x_0)/f'(x_0)$. Of course, this is not completely correct, but $x_1 := x_0 - f(x_0)/f'(x_0)$ will probably be a much better approximation of α than x_0. Iterating this approximation process leads to the Newton process:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad \text{voor } n = 0, 1, 2, \ldots.$$

(a) Show that the recursion in Exercise 2.7 represents the Newton process for solving $f(x) := x^2 - A = 0$.

(b) Interpret the Newton process as finding the intersection of the tangent to the graph of f at $(x_n, f(x_n))$ with the x-axis: x_{n+1} is the intersection.

(c) For $h_n := x_n - \alpha$ it follows that $\alpha = x_n - h_n$ and $0 = f(x_n) - h_n f'(x_n) + \frac{1}{2}h_n^2 f''(\xi)$. Hence

$$h_{n+1} = h_n - \frac{f(x_n)}{f'(x_n)} = \frac{1}{2}\, h_n^2\, \frac{f''(\xi)}{f'(x_n)}.$$

Check this. Write $B := \frac{1}{2}f''(\alpha)/f'(\alpha)$. Suppose that $x_n \approx \alpha$. Check that then $\frac{1}{2}f''(\xi)/f'(x_n) \approx B$ and conclude that $Bh_{n+1} \approx (Bh_n)^2$. Explain the 'super-fast' convergence of Exercise 2.7.

(d) Consider next the recursion

$$x_{n+1} = \frac{1}{3}\left(2x_n + \frac{A}{x_n^2}\right).$$

Prove that $\lim_{n\to\infty} x_n = \sqrt[3]{A}$.

Exercise 2.9

Apply Euler's method to solve the Lotka-Volterra system

$$\frac{dq}{dt} = rq(1-p), \qquad \frac{dp}{dt} = p(q-1), \qquad q_0 = p_0 = \frac{1}{2}, \quad t \in [0, 10],$$

with $r = 2$ and step sizes $\tau = 0.1$, $\tau = 0.01$, $\tau = 0.001$. Plot the solutions in *phase space*: that is, plot the points (q_n, p_n), $n = 0, 1, 2, \ldots$ as a sequence of points in the plane $\mathbf{R}^2$. One can prove using analysis that all solutions $(q(t), p(t))$ of the Lotka-Volterra system are closed curves in $\mathbf{R}^2$. In other words, there is some T (dependent on the initial condition) for which $q(t+T) = q(t)$ and $p(t+T) = p(t)$ for all t. Consequently, any solution remains in bounded subset of $\mathbf{R}^2$. Compare this fact with what you observe for Euler's method.

Exercise 2.10

Solve the Verhulst model

$$\frac{dp}{dt} = p(1-p), \quad p(0) = \frac{1}{2}, \quad 0 \le t \le 5,$$

with the trapezoidal rule. Derive the quadratic equation for p_{n+1} as a function of p_n. Which branch (root) of the quadratic equation is appropriate for numerical integration? Repeat the computation, this time solving the quadratic equation numerically using Newton's method.

3 Exercises for Chapter 3

Exercise 3.1

Consider the function $f(x) = A(1-x)x^2$ with $0 \le A \le 27/4$ on the interval $[0, 1]$.

(a) Show that f maps the interval $[0, 1]$ to itself.

Consider the recursion

$$x_0 \in [0, 1], \quad x_{n+1} = f(x_n), \qquad n \ge 0.$$

(b) For general A determine the fixed points of the recursion and check whether they are stable or unstable.

Exercise 3.2

Consider the recursion

$$x_{n+1} = f(x_n), \qquad f(x) = r^2 x(1-x)(1-rx+rx^2), \quad r = 1+\sqrt{5}.$$

Use the graphical analysis method to identify (graphically) the steady states of this function and establish their stability. There are two stable steady states; call them α and β. Associated to these are sets A and B of initial conditions whose iterates eventually converge to α and β, respectively. That is

$$A = \{x \in [0,1] \mid x_0 = x \Rightarrow \lim_{n\to\infty} x_n = \alpha\},$$

and similarly for B. Consider how you might construct the sets A and B. (*Hint:* $f'(\alpha) = f'(\beta) = 0$.)

Exercise 3.3

Consider a recursion $x_{n+1} = f(x_n)$ where $f : \mathbf{R} \longrightarrow \mathbf{R}$.

(a) Construct an example (d.m.v. formule/grafiek/programma/...) of a function f that has a fixed point that is attracting, but not stable in the sense of Lyapunov. *Hint: what value will the derivative of f take at the fixed point?*

(b) Does such a function exist with the property of (a) if we demand that f be continuous?

(c) Think of a continuous function f with a stable fixed point α such that for a certain sequence (x_n) of iterates that converge to α, it holds that $|x_{2n+1} - \alpha| \geq 10\,|x_{2n} - \alpha|$ $(n = 0, 1, 2, \ldots)$.

(d) Does there exist a function with the property of (c) if we demand that f be continuously differentiable?

Exercise 3.4

Consider the recursion $x_{n+1} = f(x_n)$ in which f is a continuously differentiable function. A solution of this recursion of the form $a, b, a, b, a, \ldots$ with $a \neq b$ is called a period-2 orbit. Apparently $a = f(b)$ and $b = f(a)$. The composite function $f(f(x))$ is denoted $(f \circ f)(x)$. The constant sequences $a, a, a, \ldots$ and $b, b, b, \ldots$ are hence solutions of the recursion $x_{n+1} = (f \circ f)(x_n)$.

(a) Prove using the chain rule for differentiation that $(f \circ f)'(a) = (f \circ f)'(b) = f'(a)f'(b)$.

We call a period-2 orbit $a, b, a, b, \ldots$ stable if $|f'(a)f'(b)| < 1$ and unstable if $|f'(a)f'(b)| > 1$. Choose $f(x) = A(1-x)x$ met $1 < A < 4$.

(b) Write out the equation $(f \circ f)(x) = x$ and note that two solutions are already known (which ones?).

(c) Prove that for $A = 2$ there is no period-2 orbit and for $A = 7/2$ there is one. In the latter case, find the period-2 orbit and check if it is stable.

Exercise 3.5

Consider the function $f : [0,1] \longrightarrow [0,1]$ given by

$$f(x) \quad := \quad \begin{cases} 2x & \text{als} \quad x \in [0, \frac{1}{2}), \\ 2-2x & \text{als} \quad x \in [\frac{1}{2}, 1]. \end{cases}$$

(a) Show that f possesses periodic orbits of arbitrarily high order.

(b) Check that in every neighborhood $(x-\varepsilon, x+\varepsilon)$ of a give point x there exists a point $y \in (x-\varepsilon, x+\varepsilon)$ for which the orbit $y, f(y), f^2(y), \ldots$ is periodic. Why are these orbits unstable? *Hint: how does the graph of f^n look, in particular for for large values of n?*

(c) Define a symbolic dynamics using the binary representation of numbers $x \in [0,1)$. Show that there exists a dense orbit. Reason that the dynamics generated by f exhibits sensitive dependence on initial conditions, and conclude that the dynamics is chaotic.

Exercise 3.6

Consider the iteration $x_{n+1} = 4(1-x_n)x_n$, in other words, de logistic map with $A = 4$. Choose $\phi_0 \in [0,1]$ such that $x_0 = (\sin(\frac{1}{2}\pi\phi_0))^2$. Prove that for every n it holds that $x_n = (\sin(2^n \frac{1}{2}\pi\phi_0))^2$. For which values of ϕ_0 kan we expect a periodic orbit? Are these stable? Relate this process to that of Exercise 3.5.

Exercise 3.7

Construct a bifurcation diagram for the recursion

$$x_{n+1} = F(x_n), \qquad F(x) = \exp(-8x^2) + \beta,$$

for $-1 \le \beta \le 1$, and $-1.2 \le x \le 1.2$. Choose $x_0 = 0$ for your iterations. (An example Mathematica code can be found on page 289 of Lynch.)

Exercise 3.8

Consider a bacteria population in a reservoir in which $N(t)$ the number of bacteria per unit volume. The growth factor k of this population is dependent on the available food concentration C, i.e. $k = k(C)$. Assume the function $k(C)$ satisfies $k(0) = 0$, that it is increasing, and that it approaches a limit as $C \to \infty$. We assume the nutrition consumption is $\gamma(C)N$, where $\gamma(C)$, the nutrition consumption per bacteria per unit time, is an increasing function of C that also approaches a limit as $C \to \infty$. For simplicity we assume $\gamma(C) = \alpha k(C)$ (Segel) voor a certain constant α.

(a) Derive a system of first order differential equations for N and C. Show that $C + \alpha N$ is a constant function of time (i.e. for a solution $C(t)$ and $N(t)$ of your differential equation, show that $\frac{d}{dt}[C(t) + \alpha N(t)] = 0$). Call this constant C_0. Eliminate N from the equation and derive a first order equation for C.

(b) Suppose we take $k(C) = k \cdot C$. This does not satisfy the assumptions, but show anyway that we obtain the equation

$$C' = -kC(C_0 - C)$$

(*a Verhulst equation*). Show that $C \to 0$ as $t \to \infty$ if $0 < C(0) < 1$. What does this imply for $N(t)$?

(c) Show now for general $k(C)$ that $\lim_{t\to\infty} C(t) = 0$ if $C(0) < C_0$. (Hint: check that $C(t)$ is a positive, decreasing function.)

Exercise 3.9

Identify the equilibria and classify their stability for the differential equation

$$\frac{dy}{dt} = \frac{\sin y}{y}.$$

For each stable equilibrium $y = \alpha$, identify the set of initial conditions $y(0)$ for which $\lim_{t\to\infty} y(t) = \alpha$.

Exercise 3.10

Consider a function $V(x)$ that is twice continuously differentiable. Suppose that $V(x)$ possesses a local minimum at $x = \alpha$. Let $g(x) = -V'(x)$, and prove that the local minimum is an asymptotically stable equilibrium of the differential equation

$$\frac{dx}{dt} = g(x), \qquad x(0) = x_0.$$

Exercise 3.11

Consider the function

$$V(x) = \frac{x^6}{6} + \frac{2x^5}{5} - \frac{13x^4}{4} - \frac{14x^3}{3} + 12x^2.$$

Suppose you wish to find the local minima of V using Euler's method (see the previous exercise). What is the maximum stable stepsize that allows you to find all the minima (from appropriate initial conditions).

4 Exercises for Chapter 4

Exercise 4.1

In this exercise, γ_1 and γ_2 are complex numbers and $\lambda_1 = \rho + i\sigma$, $\lambda_2 = \overline{\lambda_1} = \rho - i\sigma$ with $\rho, \sigma \in \mathbf{R}$.

(a) Show that the function $y(t) := \gamma_1 \exp(\lambda_1 t) + \gamma_2 \exp(\lambda_2 t)$ is real-valued precisely when $\gamma_1 = \overline{\gamma_2}$.

(b) Write $\gamma_1 = a + ib = r\exp(i\delta)$ for appropriate real numbers a, b, r and δ (Is this possible? What is the relation between γ_1, a, b, r and δ?). Suppose $\gamma_2 = \overline{\gamma_1}$.
Show that $y(t) = ae^{\rho t}\cos(\sigma t) - be^{\rho t}\sin(\sigma t) = re^{\rho t}\cos(\sigma t + \delta)$.

Exercise 4.2

Consider the following matrix

$$Q(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}.$$

(a) Show that given a vector $v \in \mathbf{R}^2$, the vector $Q(\theta)v$ rotates v through an angle θ.

(b) Consider the recursion $x_{n+1} = Q(\theta)x_n$. How do the iterates look in the phase plane? Under what condition on θ is the solution periodic?

(c) Compute the eigenvalues and eigenvectors of $Q(\theta)$. Is the origin asymptotically stable for the recursion? Stable in the sense of Lyapunov?

(d) Next consider the differential equation

$$\frac{dy}{dt} = Jy, \qquad y(t) \in \mathbf{R}^2, \quad J = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}.$$

Show that the solution of this differential equation at time t is given by $y(t) = Q(\omega t)y(0)$.

Exercise 4.3

The three supermarkets in a town, supermarkets 1, 2 and 3, are engaged in a fierce price war. The residents of the town do not appear to be very loyal customers: there are a lot who decide weekly where they will do their shopping.

We define, for $1 \le i \le 3$,
$k_n^{(i)}$ = the number of customers in week n (since the beginning of the price war) who shop in store i, and

$$k_n = (k_n^{(1)}, k_n^{(2)}, k_n^{(3)})^T.$$

It turns out that

$$k_{n+1} = Pk_n,$$

with

$$P = \frac{1}{10}\begin{pmatrix} 6 & 3 & 2 \\ 4 & 5 & 0 \\ 0 & 2 & 8 \end{pmatrix}.$$

(a) Does P have a dominant eigenvalue?

(b) What is, for $k = (k^{(1)}, k^{(2)}, k^{(3)})^T \in \mathbf{R}^3$ with $k_1, k_2, k_3 \geq 0$,

$$\lim_{n\to\infty} P^n k?$$

Supermarket 1 does a lot of advertising out of town, and consequently attracts weekly 10 customers from out of town who previously bought groceries in their own town. A side effect of this is that it becomes so busy in supermarket 1 that a number of customers decide to go out of town for groceries. This situation is modelled as

$$k_{n+1} = Ak_n + b, \qquad (1)$$

with

$$A = \frac{1}{10}\begin{pmatrix} 6-\alpha & 3 & 2 \\ 4 & 5 & 0 \\ 0 & 2 & 8 \end{pmatrix}, \qquad b = \begin{pmatrix} 10 \\ 0 \\ 0 \end{pmatrix},$$

met $0 < \alpha \leq 6$.

(e) Determine the equilibrium state of (1). Is this state stable for $\alpha = 4$?

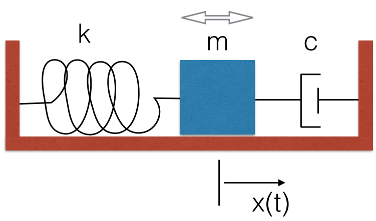
Exercise 4.4

Determine the general solution of each of the following differential equations. First, write it as a linear system of first order differential equations. Second, solve the system by eigenvalue decomposition. Discuss the stability of the point at the origin in phase space.

1. $x'' - 3x' - 4x = 0$
2. $x'' - 2x' + x = 0$
3. $x'' + 4x' + 5x = 0$

Exercise 4.5

The spring-mass-damper system



is modelled by a second order differential equation

$$m\,x'' \;+\; c\,x' \;+\; k\,x \;\;=\;\; 0,$$

where the mass $m = \frac{1}{5}$ and spring constant $k = 25\pi$ are fixed, whereas the damping c depends on the viscosity of the oil used in the damper. We have 3 standard types of oil available, leading to three values $6, 8$ and 10 for c.

The mass is released from a stretched position $x(0) = 1$. We would like the system to return to steady state as quickly as possible. That is, we want $|x(t)| \leq 10^{-3}$ for all $t \geq \tau$ with $\tau > 0$ as small as possible. The initial velocity at release is $x'(0) = 0$.

(a) Which kind of oil gives the smallest value of τ?

(b) The oil manufacturer can produce special oils on demand. For what value of c is τ be minimal?

(c) Suppose you can give the mass an impulse upon release, i.e. you may choose the initial speed $\dot{x}(0)$ yourself. With which standard oil can you make τ smallest in this case?

Exercise 4.6

Consider the nonlinear recursion

$$\begin{aligned} x_{n+1} &= (a - x_n - y_n)x_n \\ y_{n+1} &= b x_n y_n \end{aligned}$$

which describes, for certain values of $a > 0$ and $b > 0$, a (naive) model of a caterpillar-scorpion wasp population.

(a) Determine the equilibria for arbitrary a, b.

(b) Suppose $b = 1$. Determine for which values of a the equilibria are stable.

(c) According to the above calculation, in the case $a = 2.9$, $b = 1$ the point $(1, 0.9)$ is a stable equilibrium. Check its stability by taking nearby points as initial conditions. What can you say about the eigenvalues near this equilibrium?

(d) Investigate what happens as we gradually increase a by steps of, say, $1/10$. Is the equilibrium mentioned above still stable? Consider also values of a that are smaller than 2.9.

(e) Repeat the above investigation, now for $b = 1.2$.

Exercise 4.7

The Nicholson-Bailey recursion

$$\begin{aligned} G_{n+1} &= \lambda G_n e^{-aP_n} \\ P_{n+1} &= \mu G_n (1 - e^{-aP_n}) \end{aligned}$$

is a classical host-parasite model.

(a) Show that by rescaling P_n and G_n (i.e. let $x_n = \nu G_n$ and $y_n = \mu P_n$ for appropriate constant scaling factors ν and μ) this model can be simplified to

$$\begin{aligned} x_{n+1} &= \lambda x_n e^{-y_n} \\ y_{n+1} &= x_n(1 - e^{-y_n}) \end{aligned}$$

(b) Suppose $\lambda > 1$. Besides $(0,0)$ there is another equilibrium. Determine this and establish also the nature of $(0,0)$. Carry out a stability analysis on the non-trivial equilibrium and show that this yields the eigenvalue equation

$$X^2 - (1 + \frac{\log \lambda}{\lambda - 1})X + \frac{\lambda \log \lambda}{\lambda - 1} = 0.$$

Show that the constant term is strictly greater than 1 if $\lambda > 1$. What does this imply about the nature of the equilibrium?

The instability of the Nicholson-Bailey model is the reason that it is not considered to be realistic. An improved version can be obtained by replacing the factor λx_n by the logistic term $x_n \exp(r(1 - x_n/k))$. We obtain

$$\begin{aligned} x_{n+1} &= x_n \exp(r(1 - x_n/k) - y_n) \\ y_{n+1} &= x_n(1 - e^{-y_n}). \end{aligned}$$

(c) Finding an explicit expression for the equilibrium has now become problematic. Instead we turn to the computer. Investigate the behavior of the system for self-chosen values $0 < r < 5$ and $0 < k < 5$. Some suggestions: $r = 0.5$ and $k = 3$, $r = 2$ and $k = 4$, $r = 2.5$ and $k = 5$. As a rule-of-thumb we take for the plots: $0 < x < 1.5k$ and $0 < y < 1.5r$.

Exercise 4.8

(Borrowed from I. Hoveijn, Symplectic reversible maps, tiles and chaos, preprint 665, 1991, RUU Utrecht.) Consider the nonlinear recursion

$$\begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} = \begin{pmatrix} \cos a & -\sin a \\ \sin a & \cos a \end{pmatrix} \begin{pmatrix} u_n \\ v_n \end{pmatrix} + \sin u_n \begin{pmatrix} -\sin a \\ \cos a \end{pmatrix}.$$

(a) Show that the origin $(0,0)$ is an equilibrium.

(b) Write the linearization of this recursion around the origin. Determine the eigenvalues of the associated matrix.

(c) Take $a = 3\pi/10$. Try to gain a good impression of the orbits in the plane by computing orbits from a representative set of intitial conditions.

(d) Do the same for $a = 7\pi/10$.

(e) Do the same for $a = 9\pi/10$. What has happened to the point $(0,0)$?

(f) Investigate the recursion for $a = \pi/2$. Take among others $(3,0)$ as initial condition. Can you find a period-4 orbit?

Exercise 4.9

A projectile is shot up from the earth with a speed of V_0. We neglect air resistance, nbut do take into account the dependence on gravity as a function of distance from the projectile to the earth. The attracting force of the earth is mgR^2/r^2, where m is the mass of the projectile, g acceleration due to gravity, R the radius of the earth and r the distance from the projectile to the center of the earth. We would like too determine the minimal V_0 for whcich the projectile does not fall back to earth. In other words we want to know the "vertical escape speed of earth's gravity."

(a) Derive the equation of motion:

$$mr'' = -mg\frac{R^2}{r^2}.$$

(b) Multiply both sides of this equation with r' and determine the antiderivative. (We get an integration constant E and the resulting relation for r' and r expresses the conservation of energy).

(c) Now make use of $(r')^2 \geq 0$ to determine an upper bound for r that depends on E. For which E can r grow unbounded?

(d) Next compute the excape velocity as a function of g and R. Finally, let $g = 10m/s^2$ and $R = 6400km$. Compute an explicit value for the escape velocity.

There is also an escape velocity with a smaller value, if we shoot the projectile off in a horizontal direction.

Exercise 4.10

Determine the equilibria. For every equilibrium, find the eigenvalues of the linearized system of equations.

(a) $x' = y - xy$
$y' = -x + xy$

(b) $x' = x^2 - 3x + 2$
$y' = xy^2 - x$

(c) $x' = \cos x + \sin y$
$y' = \sin x + \cos y$

Exercise 4.11

Consider:

$$x' = 5\left(1 - kx - \frac{2y}{1+x}\right)x$$
$$y' = \left(-1 + \frac{2x}{1+x}\right)y$$

(a) Determine the equilibria.

(b) Determine for each equilibrium the values of k for which the equilibrium is stable.

Exercise 4.12

The following system models two species that compete with each other for the same food source:

$$\begin{aligned} N_1' &= (a_1 - d_1(bN_1 + cN_2))N_1 \\ N_2' &= (a_2 - d_2(bN_1 + cN_2))N_2. \end{aligned}$$

All constants are positive. Determine the equilibria for this system. Show that if $a_1 d_2 > a_2 d_1$ population N_2 dies out and N_1 converges upon an equilibrium (Volterra's exclusion principle).

Exercise 4.13

There is a lake that has a reputation to be a great location for fishing and consequently attracts a large number of anglers (= fishermen). To study the fluctuations in the fish population we make a simple model of the fish-angler interaction.

Assume for the fish:

- Fish populations grow exponentially in the absence of anglers.
- The presence of anglers slows the population growth by an amount proportional to the number of anglers and the number of fish.

Assume for the anglers:

- Anglers are attracted to the lake at a rate proportional to the number of fish.
- Anglers are discouraged from coming to the lake at a rate proportional to the number of anglers present at the lake.

Derive a model for this interaction and analyse it. Suppose that a sportfishing club decides to release fish into the lake. How does the model change then? Analyse this situation too. What is the effect of releasing fish on the fish population?

Exercise 4.14

The chemostat. For experiments on the growth of bacteria colonies it is necessary to continuously maintain a supply of available bacteria. An appropriate culture is that of a *chemostat*. This consists of a fluid reservoir of volume V in which the bacteria are kept. Let $N(t)$ be the number of bacteria per unit volume at time t and let $C(t)$ be the concentration of nutrient. Fluid is pumped into the bacteria reservoir at a constant rate, and the same amount of fluid is removed. The inflow contains a fixed concentration C_0 of nutrient. Some of the bacteria and nutrient is removed via the outflow. Assume the inflow rate is F per time unit. In the following we assume the bacteria growth obeys the same rules as before. In particular the function $k(C)$ is that of Exercise 4.8. The question is how the in- and outflow of fluid into the chemostat influences the population. In particular we must avoid that F becomes too large such that all bacteria are flushed from the system. What is the critical value of F?

(a) Note that $N(t)V$ is the total number of bacteria at time t and $C(t)V$ the total amont of nutrient in the reservoir. Consider now which terms influence $d(NV)/dt$ and $d(CV)/dt$. Derive the following system of differential equations:

$$\begin{aligned} N' &= k(C)N - \frac{F}{V}N \\ C' &= -\alpha k(C)N + \frac{F}{V}(C_0 - C). \end{aligned}$$

(b) Besides the point $(0, C_0)$ (=no bacteria, only nutrient) there is precisely one other equilibrium $(\bar{N}, \bar{C})$ given by $k(\bar{C}) = F/V$ and $\bar{C} + \alpha\bar{N} = C_0$. Show this.

(c) Prove, under the assumption that k is strictly monotonically increasing, that $\bar{N} > 0$ if and only if $k(C_0) > F/V$.

(d) Show that the eigenvalues of the Jacobian matrix near $(0, C_0)$ are $-F/V$, $k(C_0) - F/V$ and near $(\bar{N}, \bar{C})$ are $-F/V$, $-\alpha k'(\bar{C})\bar{N}$. Investigate the nature of both equilibria. Recall that $\bar{N}$ and $k(C_0) - F/V$ have the same sign.

(e) Show that the from the differential equations it follows that $(C + \alpha N)' = (-F/V)(C + \alpha N) + FC_0/V$. Solve this equation for $C + \alpha N$ and conclude that $\lim_{t\to\infty}(C(t) + \alpha N(t)) = C_0$.

(f) What restriction is needed on F and C_0 to prevent all bacteria from being washed away?

Exercise 4.15

The following method, called the θ-method, is popular in engineering calculations. The parameter θ may be chosen in the range $\theta \in [0, 1]$.

$$y_{n+1} = y_n + \tau\left[(1-\theta)f(y_n) + \theta f(y_{n+1})\right].$$

Identify the methods corresponding to $\theta = 0$, $\theta = 1/2$, and $\theta = 1$. Plot the stability regions for $\theta = \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$. Do the same for the *explicit θ-method*

$$y_{n+1} = y_n + \tau\left[(1-\theta)f(y_n) + \theta f(y_n + \tau f(y_n))\right].$$

Exercise 4.16

Consider the spring-mass-damper model

$$\frac{d^2x}{dt^2} + 2\rho\frac{dx}{dt} + \nu^2 x = 0,$$

with small damping $\rho \ll \nu$. Determine the maximum stepsize for which the equilibrium $(x \equiv dx/dt \equiv 0)$ is stable when this model is solved using Euler's method. Write the maximum stepsize as a function of ρ and ν. Hint: write the model as a first order system to apply Euler's method. How does the stepsize scale with decreasing ρ? What is the maximum stepsize when implicit Euler is used?

Exercise 4.17

For each of the differential equations given in Lynch, Chapter 2, Exercise 4b,c,d: (i) determine the equilibria and classify the type (stable/unstable, saddle/node/spiral, etc.); (ii) sketch the phase plot using nullclines and eigenvectors; (iii) check your answers using Mathematica.

Exercise 4.18

Work Exercise 8 in Chapter 2 of Lynch.

5 Exercises for Chapter 5

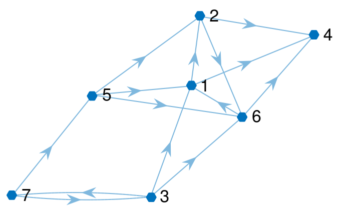
(No exercises.)

6 Exercises for Chapter 6

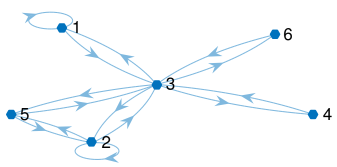
Exercise 6.1

Consider the following graphs:

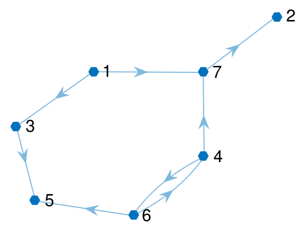
a)



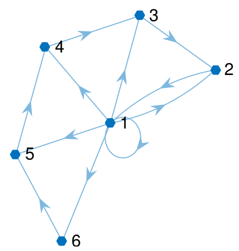
b)



c)



d)



For each graph:

(a) Construct the associated incidence matrix.

(b) Compute the number of paths of length 3 from node 2 to node 6.

(c) Compute the number of paths of length 4 from node 3 to node 2.

(d) Investigate the periodicity and irreducibility of the graph.

Exercise 6.2

An animal population is divided in three age groups with associated Leslie Matrix

$$L = \begin{bmatrix} 0 & 6 & 12 \\ 1/2 & 0 & 0 \\ 0 & 1/3 & 0 \end{bmatrix}$$

At $t = 0$ the population distribution is $x_0 = (4, 3, 1)^T$.

(a) Determine the population distribution at times $t = 1$, 2, 3.

(b) Determine the eigenvalues of the matrix L.

(c) Determine the relative population distribution in the limit $t \to \infty$.

Exercise 6.3

The survival rate $s > 0$ of newborns in an animal population is dependent on the severity of the winter. We study a model that describes the first three years. We divide the population up into 1-,2- and 3-year olds, and count the population every year. The number of k-year old animals in time period n is denoted $N_n(k)$. Let L be the associated Leslie matrix

$$L = \begin{bmatrix} 0 & 2 & 4 \\ s & 0 & 0 \\ 0 & 1/2 & 0 \end{bmatrix}.$$

Our population distribution is described by the recursion

$$N_{n+1} = LN_n, \qquad N_n = \begin{pmatrix} N_n(1) \\ N_n(2) \\ N_n(3) \end{pmatrix},$$

where the population distribution N_0 is given.

(a) Draw the graph associated with L and determine its irreducibility and periodicity.

(b) Prove the L possesses a positive dominant eigenvalue.

(c) Suppose $\lambda \neq 0$ is an eigenvalue of L. Determine an eigenvector associated to λ in terms of s and λ.

(d) Derive the characteristic polynomial of L.

(e) Determine the smallest survival rate s_0 such that the population can survive indefinitely. Is there a stable equilibrium for this s_0? If so, compute this equilibrium.

Exercise 6.4

A system resides in one of three states and behaves according to the Markov chain with transition matrix

$$P = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix}.$$

(a) Draw the graph of P.

(b) Determine the stationary probability vector.

(c) We are given that the system is in state 3. What is the chance that state 1 is reached in two steps? And the chance that after 2 steps the system is in state 3?

(d) We are given that the system is in state 1. How many steps on average does it take to reach state 3?

Exercise 6.5

Two dogs are begin terrorized by four fleas. Between time n and $n+1$, a randomly selected flea jumps from one dog to the other.

(a) Show that this process can be modelled as a Markov chain with the number of fleas on one of the dogs as state vector and the following transition matrix

$$P = \begin{bmatrix} 0 & 1/4 & 0 & 0 & 0 \\ 1 & 0 & 1/2 & 0 & 0 \\ 0 & 3/4 & 0 & 3/4 & 0 \\ 0 & 0 & 1/2 & 0 & 1 \\ 0 & 0 & 0 & 1/4 & 0 \end{bmatrix}.$$

(b) Show that the matrix is irreducible.

(c) Determine the period of the matrix.

(d) Show that there exists precisely one stationary state and determine this.

(e) Provide an example of an initial distribution p such that $P^n p$ has no limit as $n \to \infty$.

(f) Determine P^n as $n \to \infty$.

(g) Determine $\lim_{n\to\infty} P^n p$ for an arbitrary initial distribution p. Is this limit independent of p?

(This model is a variant of the urn model of Ehrenfest.)

Exercise 6.6

A mouse has a simple house: three rooms next to each other with a passage between 1 and 2 and between 2 and 3. If the mouse is in room 1 or 3, it moves to 2 in the following time step. If it is in room 2, is flips a coin and moves with probability p to room 1 and probability $1-p$ to room 3. The mouse never stays in the same place.

(a) Model this situation as a Markov chain with transition matrix P.

(b) Show that the chain is irreducible.

(c) Determine the periodicity.

(d) Compute P^n for arbitrary n.

(e) Sho that there is precisely one stationary distribution.

Exercise 6.7

An instructor tells a student in confidence that there will be a problem about Markov chains on the final exam. The student is unable to contain him/herself and tells this further on student 2, who tells it further to student 3 and so forth. Due to the noise in the lunchroom, is there a chance p that the message is correctly passed on and a chance $q = 1 - p$ that it is incorrectly communicated.

(a) Model this situation with a Markov chain with 2×2 transition matrix P.

(b) Compute P^n for arbitrary n.

(c) Show that the probability distribution for the chance that the message is correctly communicated is in the end independent of what the instructor actually said.

7 Exercises for Chapter 7

Exercise 7.1

Find the general solution of the following differential equations:

(a) $x'' + x = \sin 2t$.

(b) $x'' + 3x' + 2x = t$.

(c) $x'' + 3x' + 2x = e^{\alpha t}$ for $\alpha = -1$ and $\alpha = 1$.

Exercise 7.2

A well-known oil company has the following problem. When drilling for oil, the drill shaft tends to vibrate. Since the drill bit is wider than the drill shaft, this is initially not a problem, but when the amplitude of the vibration become big enough, the shaft begins to knock agains the side of the well. Consequently the shaft can be damaged or broken. We consider the following model:

$$x'' + 2\mu x' + \omega^2 x = a \sin \nu t,$$

where x is the horizontal displacement of the drill shaft, μ is a damping constant, $\omega = 1/\sqrt{L}$ is the eigenfrequency of the drill staff of length L, a and ν are respectively the amplitude and frequency of the forcing, determined by the rotation speed of the drill shaft.

The oil company provides us with the following parameter values

$$\mu = 1/4, \quad a = 1, \quad \nu^2 = 8/9.$$

The length L (in kilometers) of the drill shaft is equal to the depth of the drill bit. The drill shaft knocks agains the well wall when its horizontal displacement $x(t)$ (in inches) reaches 2 inches. We assume that the damped part of the displacement $x(t)$ can be neglected. At what depth do you expect the first problem.