WISB134

# Modellen & Simulatie

***Lecture 7 - Lineaire modellen in meerdere dimensies***

Universiteit Utrecht

# Overzicht van ModSim

Meeste aandacht (t/m 1 apr.)

- Basisbegrippen dynamische modellen
  - Definities recursies, DVs, numerieke methoden
  - Oplossingen DVs
  - Convergentie numerieke methoden
- Dynamica
  - Scalaire dynamica
  - ➡ Dynamica op $\mathbf{R}^d$
  - Lineaire dynamica op $\mathbf{R}^2$
- Bijzondere gevallen
  - Lineaire kansmodellen (Markovketens)
  - Niet-autonome systemen (Resonantie)
  - Hogere orde numerieke methoden

# Dynamica op $\boldsymbol{R}^d$

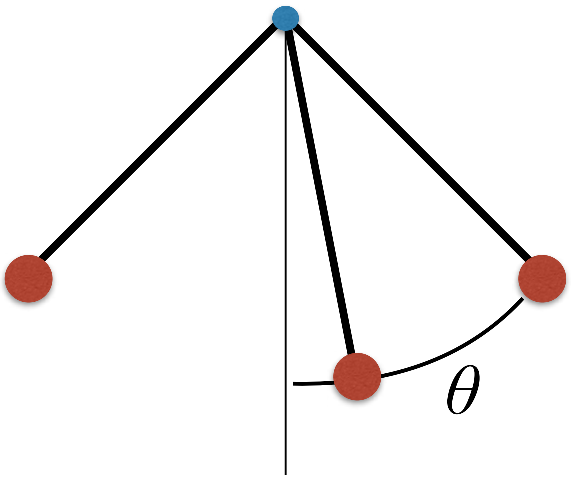
vandaag

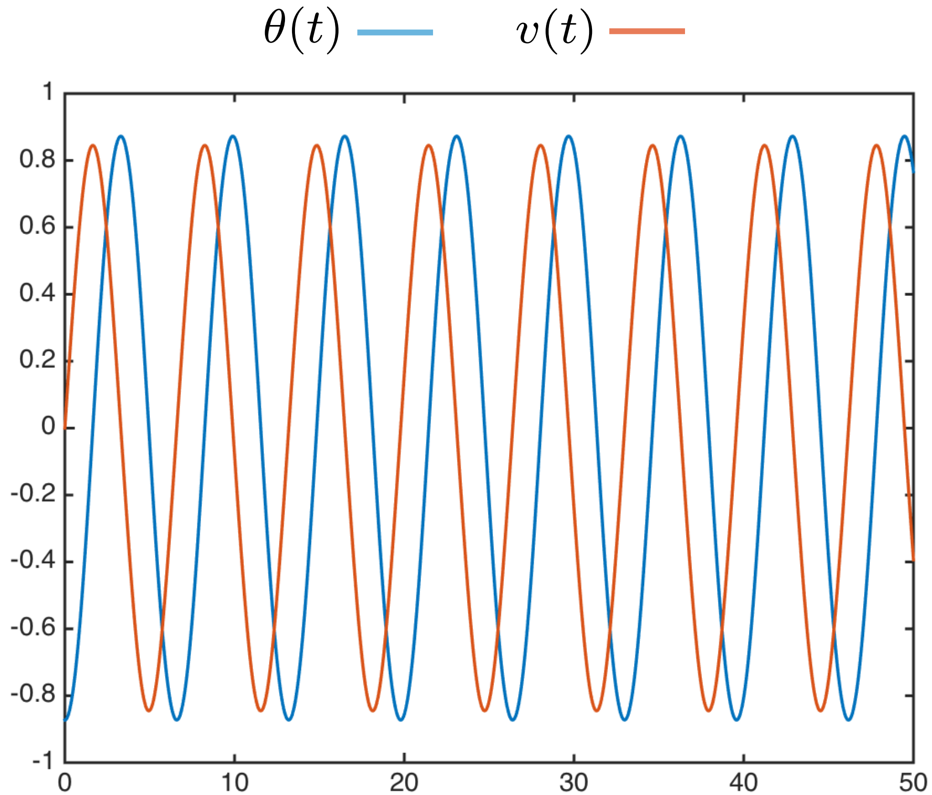
- Tijdsreeks en fase ruimte
- Lineaire dynamica:
    - Lineaire recursies
    - Lineaire DVs
- Niet-lineaire recursies
- Niet-lineaire DVs
- Stabiliteit van numerieke methoden

# Slinger

$$\dot{\theta} = v$$

$$\dot{v} = -\sin\theta$$



# Fase ruimte

**Dynamica op $R^d$**

- Tijdsreeks en fase ruimte
- Lineaire dynamica:
  - Lineaire recursies
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- Stabiliteit van numerieke methoden

- De toestand van ons model op tijd $t$ wordt weergegeven door een vector $y(t) \in \mathbf{R}^d$
- De componenten van de vector kunnen ook beschouwd worden als de coordinaten van een punt in $\mathbf{R}^d$
- Het punt 'beweegt' in de tijd, en tekent daarbij een kromme.
- De ruimte $\mathbf{R}^d$ met coordinaten y, noemen we de *fase ruimte.*

## Slinger

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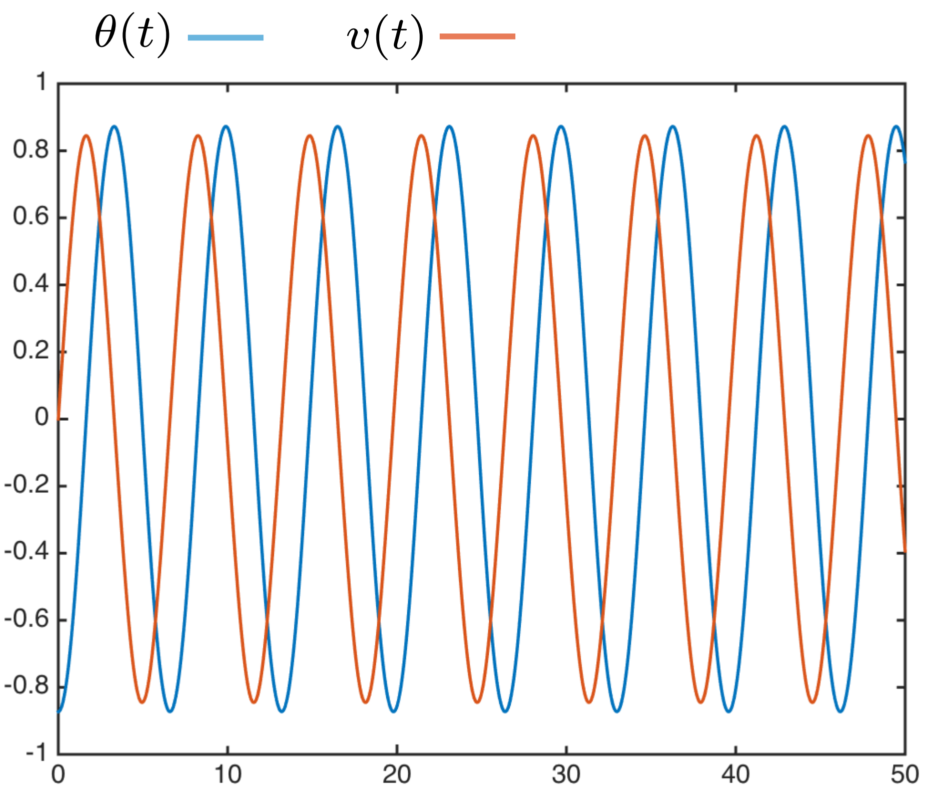


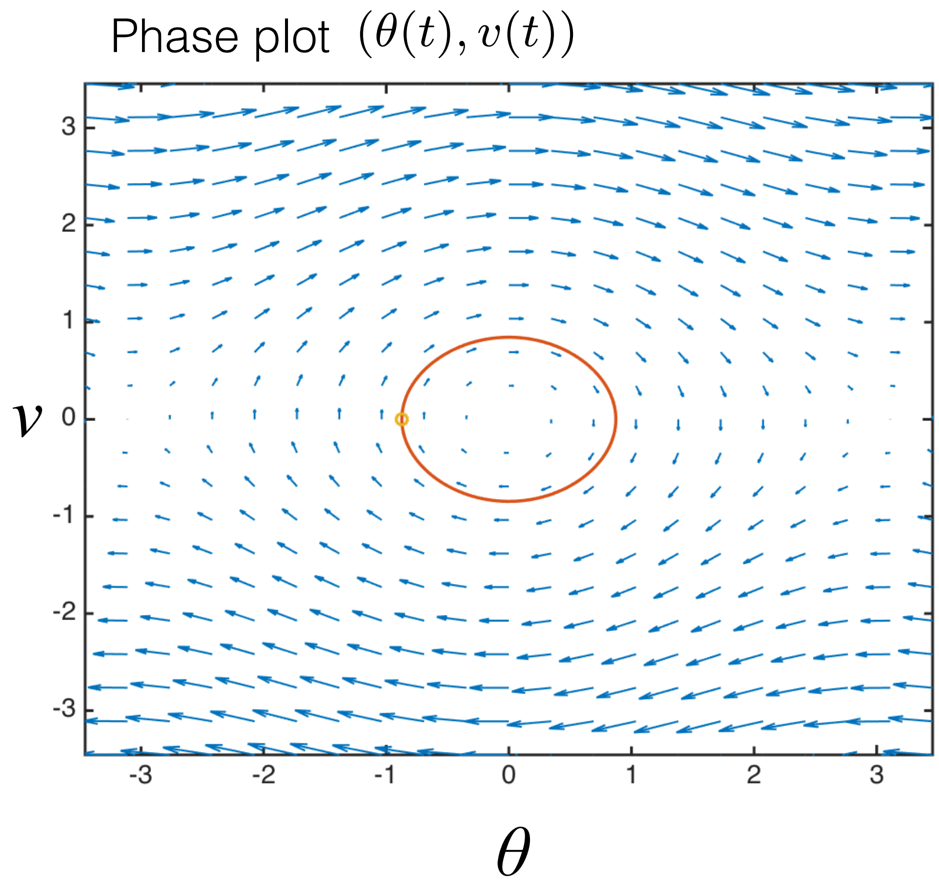

# Fase ruimte

**Dynamica op $R^d$**

- Tijdsreeks en fase ruimte
- Lineaire dynamica:
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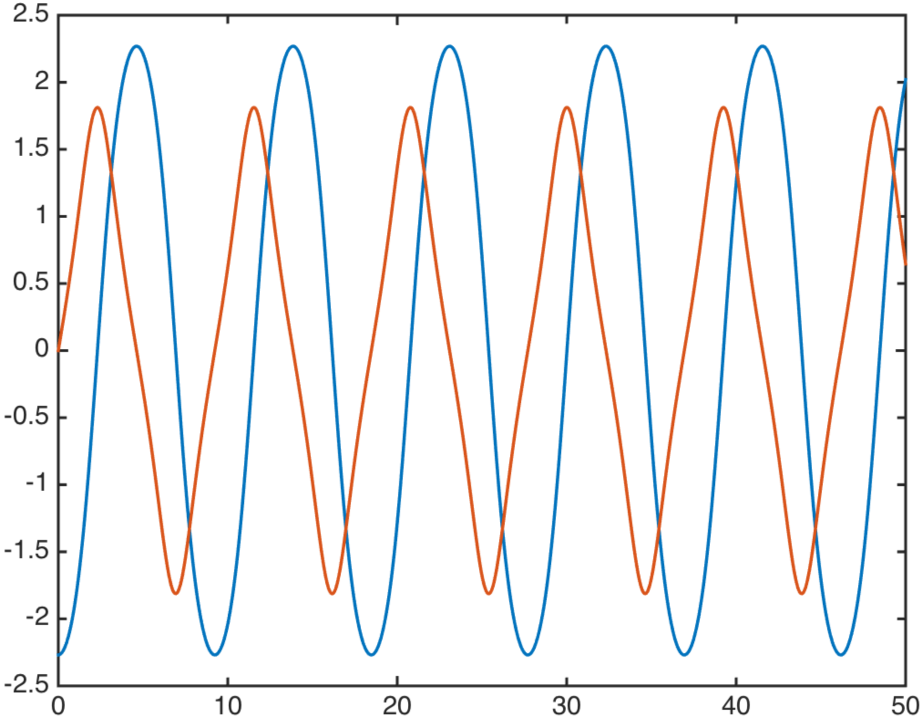
- Stabiliteit van numerieke methoden

- Op elk punt $y \in \mathbf{R}^d$ wordt geassocieerd een vector $f(y) : \mathbf{R}^d \to \mathbf{R}^d$
- Ook wel een *vectorveld* genoemd.
- Aangeeft de richting en grootte van de verandering van de oplossing $\left( \frac{dy}{dt} = f(y) \right)$
- De oplossings krommen zijn tangent aan de vectoren $f(y)$.

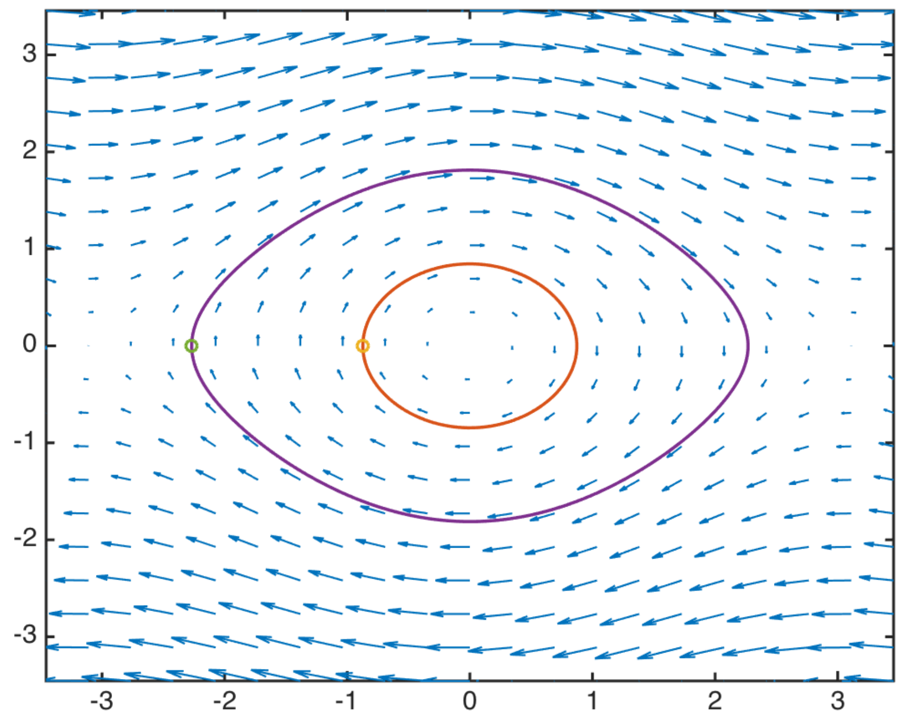
Slinger $\dot{\theta} = v$

$$\dot{v} = -\sin\theta$$

$\theta(t)$ $v(t)$




Phase plot $(\theta(t), v(t))$



Slinger

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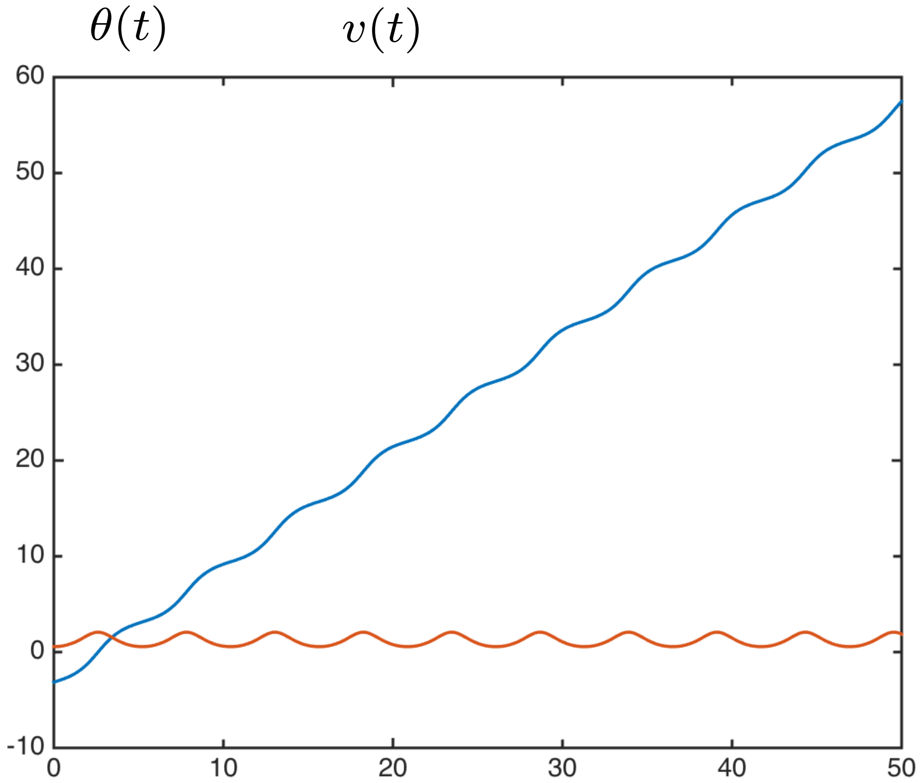


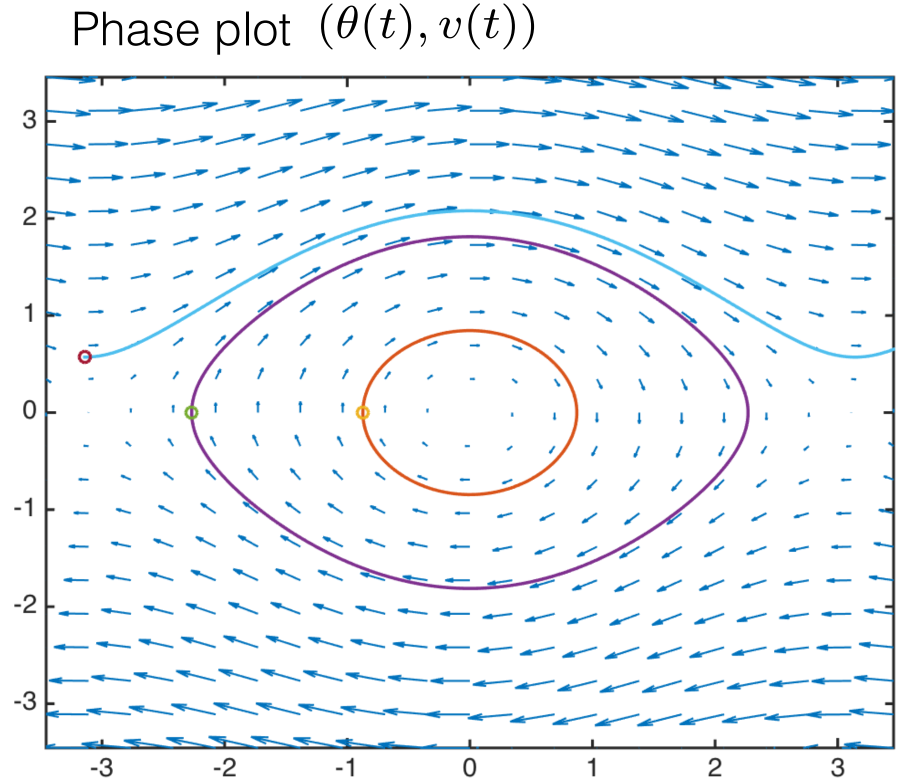

# Fase ruimte

**Dynamica op $R^d$**

- Tijdsreeks en fase ruimte
- Lineaire dynamica:
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- Stabiliteit van numerieke methoden

- Door willekeurig $y \in \mathbf{R}^d$ kan een oplossings kromme getekend worden.
- Als we dit doen voor een aantal begincondities krijgen we een beeld: de faseportret.
- Deze krommen elkaar niet kruisen, want dat zou betekenen dat er twee oplossingen zijn voor een beginconditie (het snijpunt), en dat gaat tegen de uniciteit van oplossingen in.

## Slinger

$$\dot{\theta} = v$$

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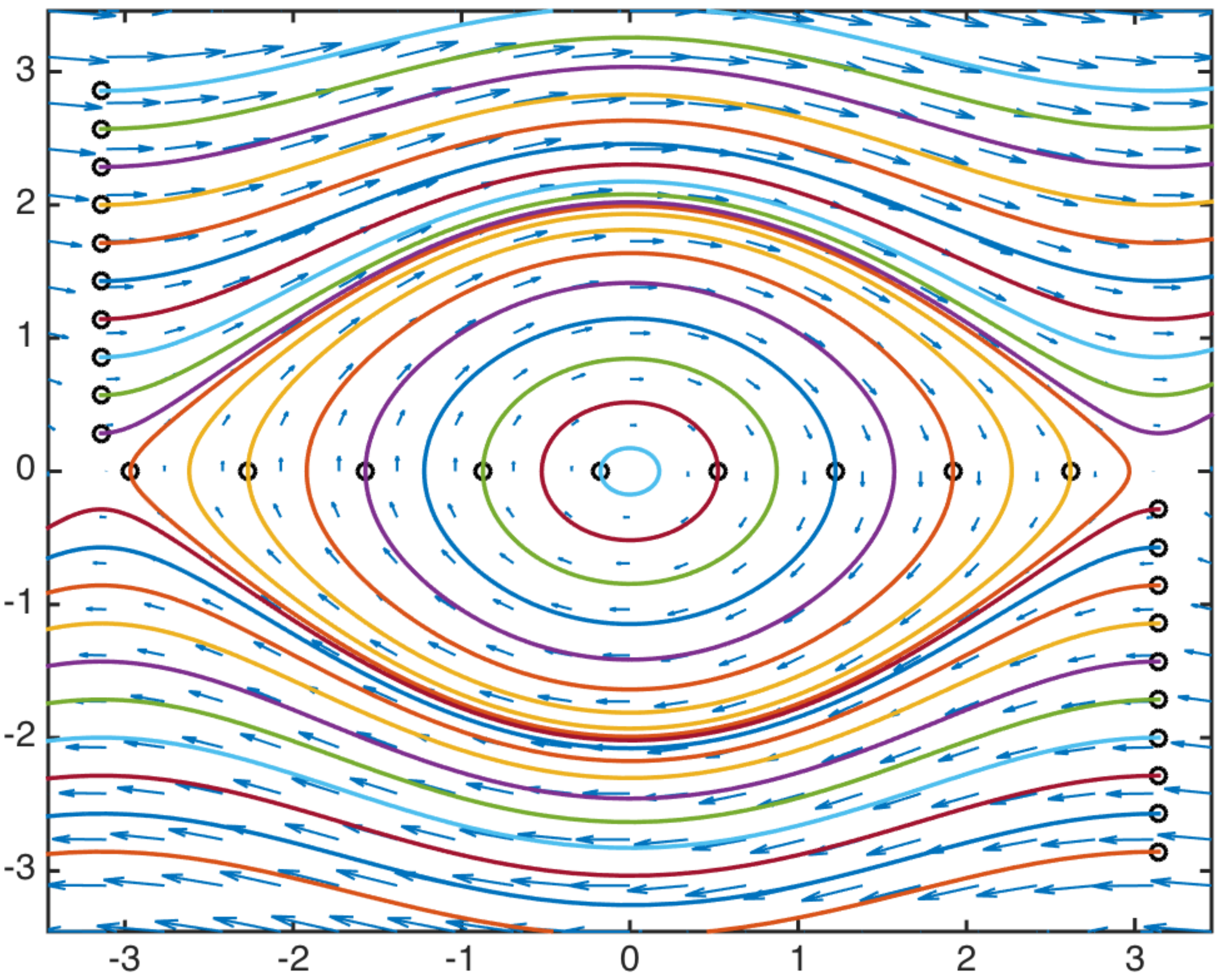
Phase plot $(\theta(t), v(t))$



# Twee jarige planten

In jaar $n$ $N_1(n)$: aantal planten in jaar $n$ ontkiemt
$N_2(n)$: aantal planten in jaar $n-1$ ontkiemt

**Aanname**:
- 50% van de 0-jarige geeft zaad
- gemiddeld twee zaden van 0-jarige ontkiemen
- 75% planten overleeft de winter
- gemiddeld vier zaden van de 1-jarige ontkiemen

**Model**: $N_1(n+1) = 2 \cdot \frac{1}{2} \cdot N_1(n) + 4\, N_2(n)$
en $N_2(n+1) = \frac{3}{4} N_1(n)$.

In "matrix taal":

$$\begin{bmatrix} N_1(n+1) \\ N_2(n+1) \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ \frac{3}{4} & 0 \end{bmatrix} \begin{bmatrix} N_1(n) \\ N_2(n) \end{bmatrix}$$

**Evenwicht** als

voor alle $n$: $N_1(n+1) = N_1(n) = \alpha_1$ en

$N_2(n+1) = N_2(n) = \alpha_2$

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ \frac{3}{4} & 0 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$$

$$\Leftrightarrow \quad 1 \text{ eigenwaarde } \mathbf{A} = \begin{bmatrix} 1 & 4 \\ \frac{3}{4} & 0 \end{bmatrix}$$

**Evenwichtige opbouw in leeftijd** als

voor alle $n$: $N_1(n+1)/N_2(n+1) = N_1(n)/N_2(n)$

$$\lambda \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ \frac{3}{4} & 0 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}$$

$$\Leftrightarrow \quad \lambda \text{ eigenwaarde } \mathbf{A}$$

*Wanneer is evenwicht stabiel?*
*Wanneer is evenwichtige leeftijdsopbouw stabiel?*

Beschouw de **iteratie** in $\mathbb{R}^2$

$$\begin{bmatrix} x_1(n+1) \\ x_2(n+1) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1(n) \\ x_2(n) \end{bmatrix}$$

$$\mathbf{x}_{n+1} = \mathbf{A}\,\mathbf{x}_n$$

Dan $\qquad \mathbf{x}_n = \mathbf{A}\,\mathbf{x}_{n-1} = \ldots = \mathbf{A}^n\,\mathbf{x}_0$

**Eigenwaarden** en **eigenvektoren** van **A**:

$$\mathbf{A}\mathbf{v}_1 = \lambda_1\,\mathbf{v}_1 \quad \text{en} \quad \mathbf{A}\mathbf{v}_2 = \lambda_2\,\mathbf{v}_2$$

$$\mathbf{x}_0 = \gamma_1\mathbf{v}_1 + \gamma_2\mathbf{v}_2 \quad \Rightarrow$$

$$\begin{aligned} \mathbf{x}_n &= \mathbf{A}^n\,\mathbf{x}_0 = \mathbf{A}^n\,(\gamma_1\mathbf{v}_1 + \gamma_2\mathbf{v}_2) \\ &= \gamma_1\,\mathbf{A}^n\,\mathbf{v}_1 + \gamma_2\,\mathbf{A}^n\,\mathbf{v}_2 \\ &= \gamma_1\,\lambda_1^n\,\mathbf{v}_1 + \gamma_2\,\lambda_2^n\,\mathbf{v}_2 \end{aligned}$$

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$$\mathbf{x}_0 = \gamma_1\mathbf{v}_1 + \gamma_2\mathbf{v}_2 \quad \Rightarrow \quad \mathbf{x}_n = \gamma_1\,\lambda_1^n\,\mathbf{v}_1 + \gamma_2\,\lambda_2^n\,\mathbf{v}_2$$

$$\mathbf{x}_n = \gamma_1\,\lambda_1^n \left( \mathbf{v}_1 + \frac{\gamma_2}{\gamma_1} \left( \frac{\lambda_2}{\lambda_1} \right)^n \mathbf{v}_2 \right)$$

Als $|\lambda_2| < |\lambda_1|$ en $\gamma_1 \neq 0$, dan $\mathbf{x}_n \approx \gamma_1\,\lambda_1^n\,\mathbf{v}_1$ voor grote $n$.

Als $|\lambda_2| < |\lambda_1|$, dan geldt, voor grote $n$,

$$\mathbf{x}_n = \gamma_1\,\lambda_1^n \left[\mathbf{v}_1 + \frac{\gamma_2}{\gamma_1}\left(\frac{\lambda_2}{\lambda_1}\right)^n \mathbf{v}_2\right] \approx \gamma_1\,\lambda_1^n\,\mathbf{v}_1$$

$\lambda_1$ is een **dominante eigenwaarde** als $|\lambda_2| < |\lambda_1|$

$\mathbf{v}_1$ is dan een **dominante eigenvector**

**Stelling**. **0** is een stabiel evenwicht als $1 > |\lambda_i|$, $i = 1, 2$.

**Stelling**. $\gamma\,\mathbf{v}_1$ is een stabiel evenwicht als $\lambda_1 = 1 > |\lambda_2|$.

Er is een stabiel evenwicht als 1 dominante eigenwaarde **A**.

**Stelling.** $\gamma\,\mathbf{v}_1$ is een stabiele opbouw als $\lambda_1$ dominant.

Er is een stabiele leeftijdsopbouw als

**A** een dominante eigenwaarde heeft.

**Voorbeeld**. [Twee jarige planten]

*Bereken de eigenwaarden* $\mathbf{A} = \begin{bmatrix} 1 & 4 \\ \frac{3}{4} & 0 \end{bmatrix}$.

$$\begin{aligned} \det(\mathbf{A} - \lambda \mathbf{I}) &= (1-\lambda)\cdot(-\lambda) - 4 \cdot \frac{3}{4} \\ &= \lambda^2 - \lambda - 3 \end{aligned}$$

$\lambda$ eigenwaarde $\quad \Leftrightarrow \quad \lambda^2 - \lambda - 3 = 0 \quad \Leftrightarrow$

$$\lambda = \lambda_1 = \frac{1+\sqrt{13}}{2} \quad \text{of} \quad \lambda = \lambda_2 = \frac{1-\sqrt{13}}{2}$$

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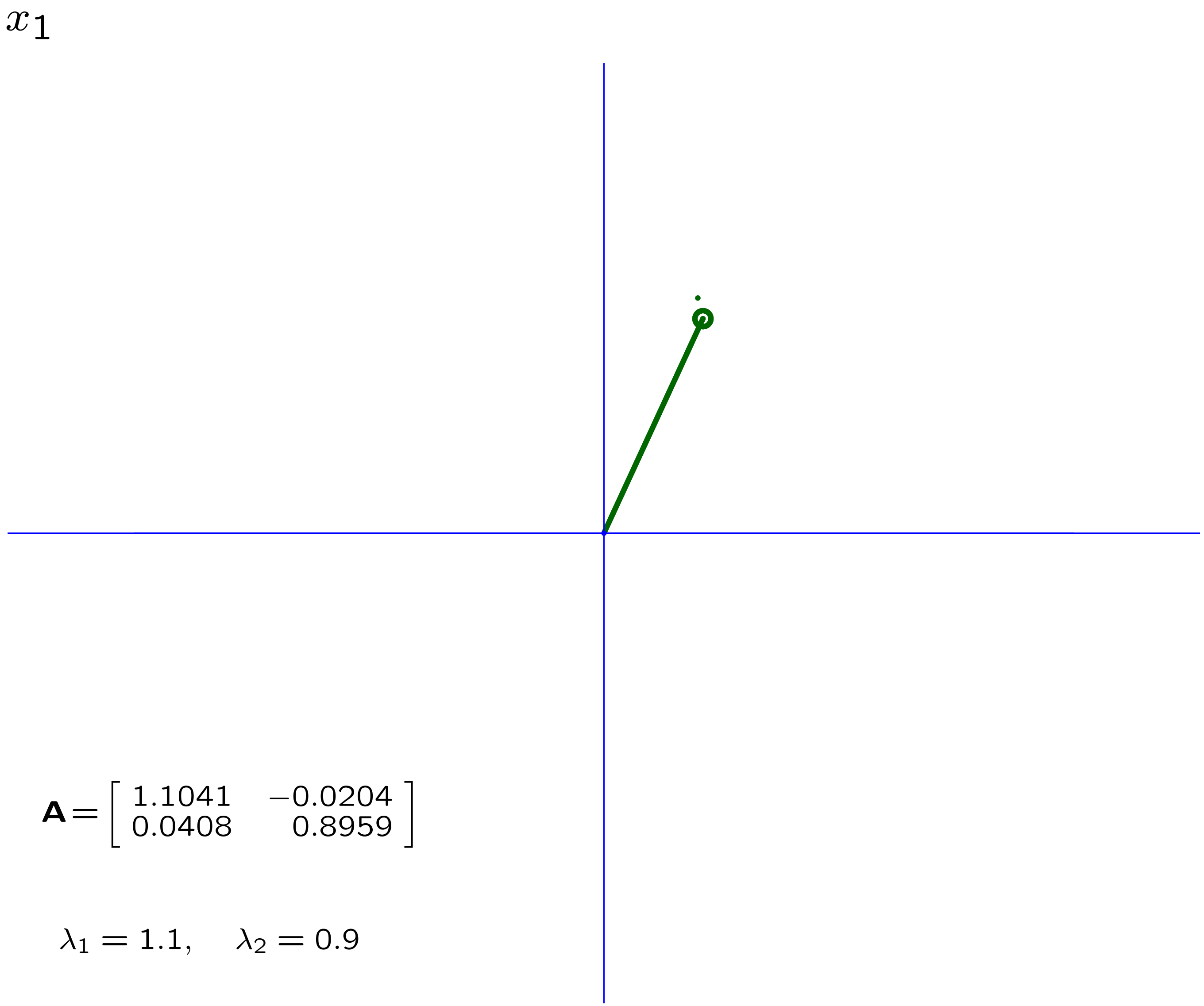
1 geen eigenwaarde: dus geen evenwicht $\neq \mathbf{0}$.

$\lambda_1 > 1$: dus $\mathbf{0}$ geen stabiel evenwicht.

$\lambda_1 > |\lambda_2|$: $\lambda_1$ dominante eigenwaarde met eigenv. $\mathbf{v}_1 = \begin{bmatrix} \lambda_1 \\ \frac{3}{4} \end{bmatrix}$.

Dus $\gamma \begin{bmatrix} \lambda_1 \\ \frac{3}{4} \end{bmatrix}$ stabiele evenwichtige leeftijdsopbouw.

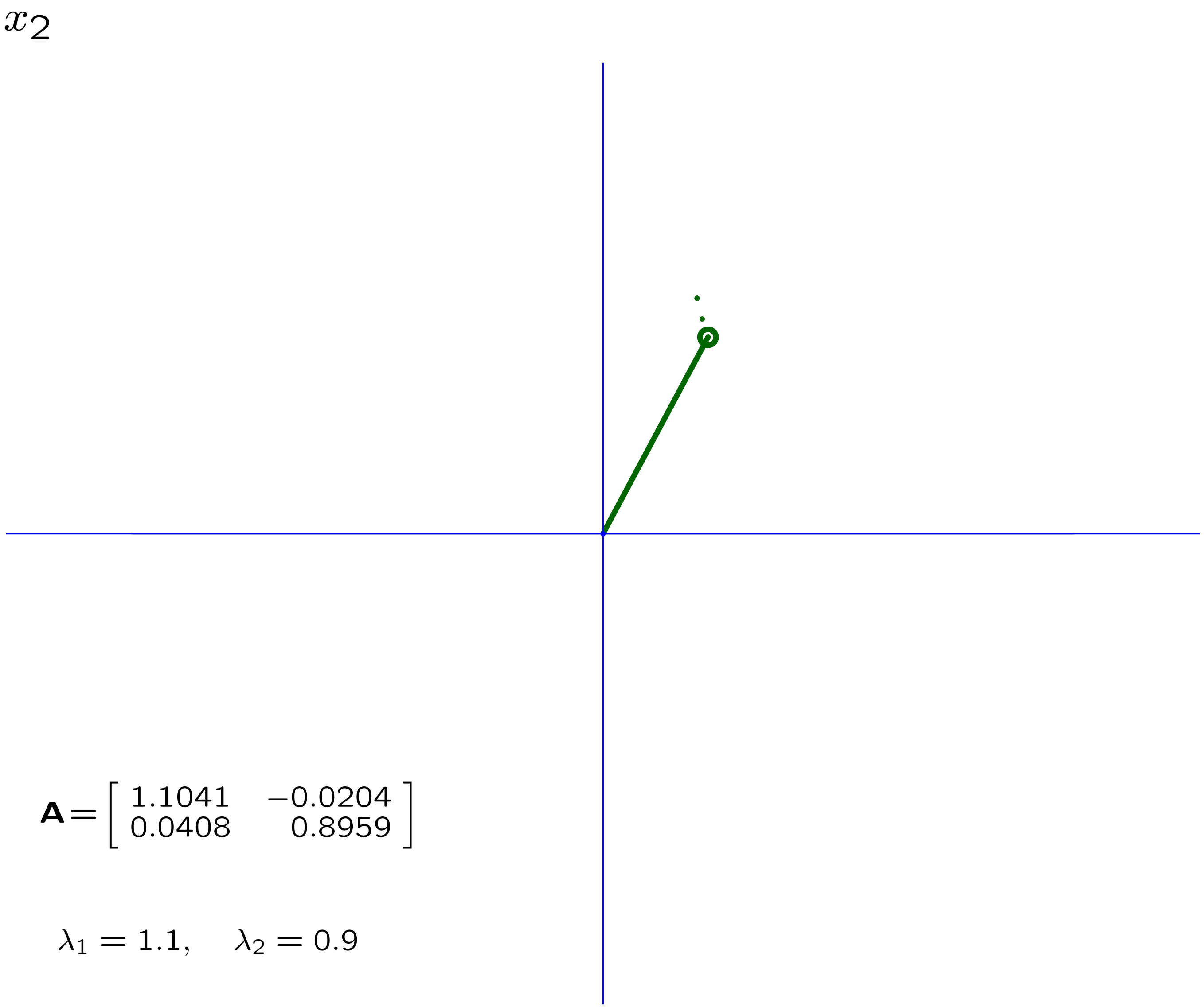
# Iteratie in beeld




$$\mathbf{A} = \begin{bmatrix} 1.1041 & -0.0204 \\ 0.0408 & 0.8959 \end{bmatrix}$$

$\lambda_1 = 1.1, \quad \lambda_2 = 0.9$

# Iteratie in beeld

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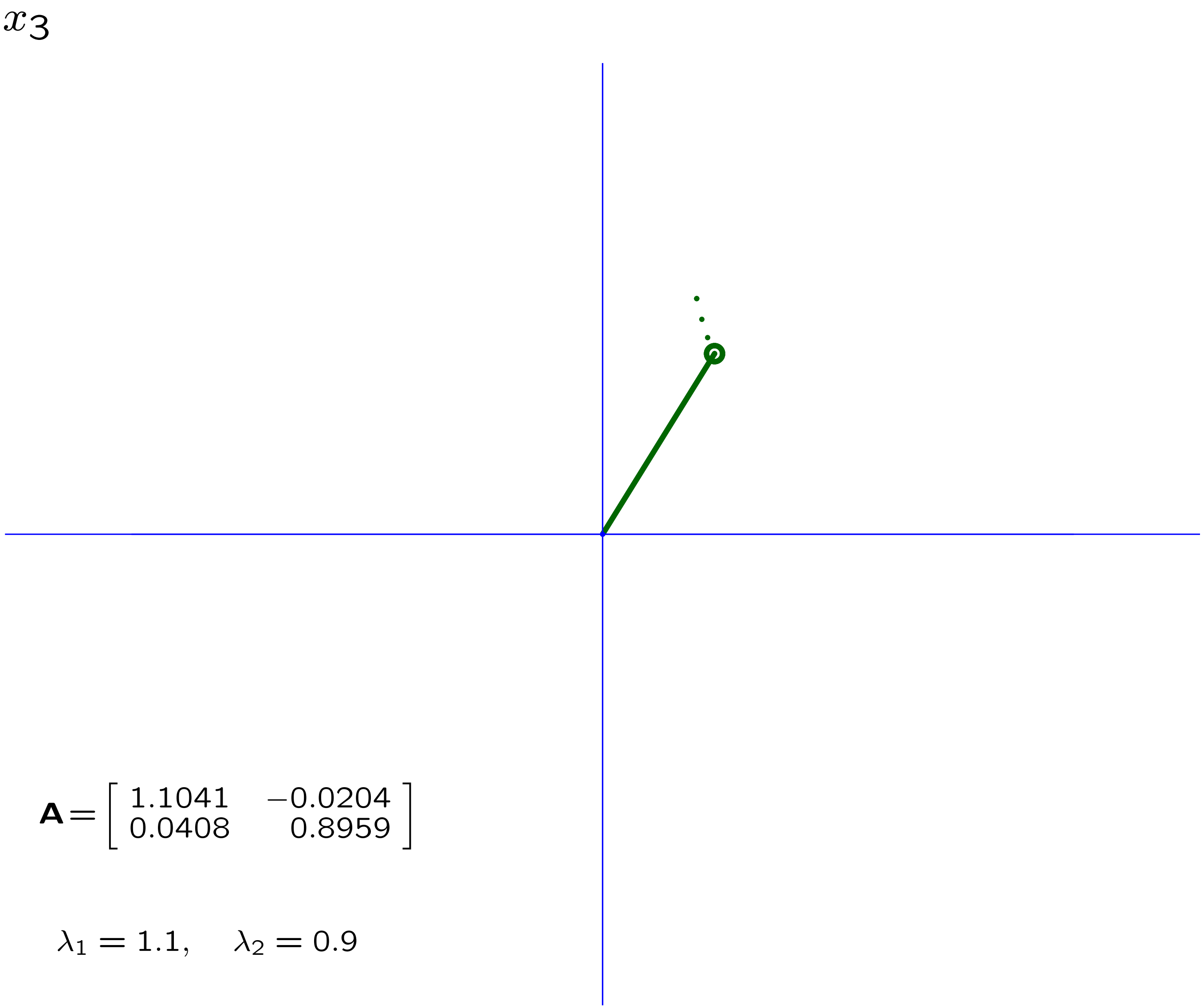


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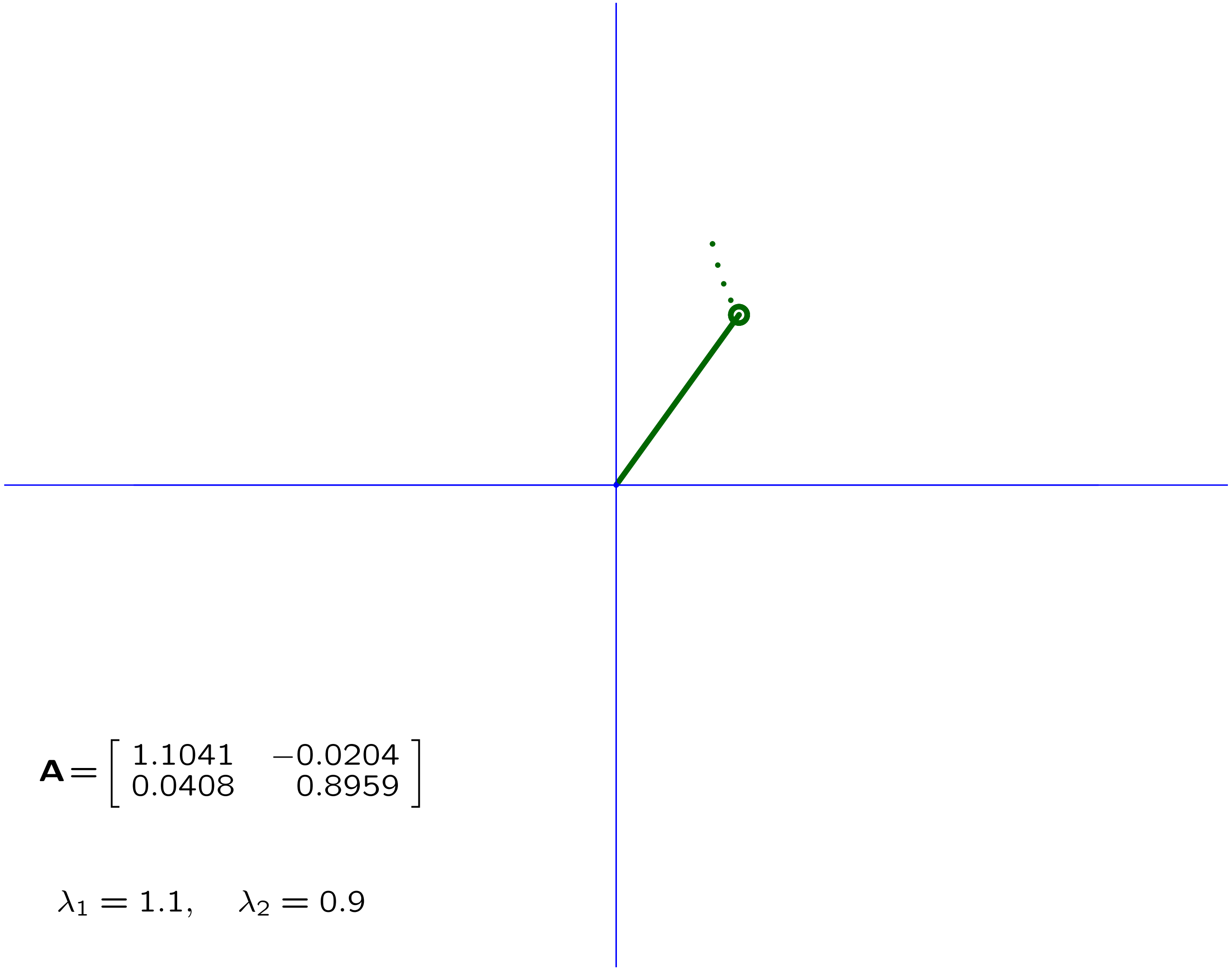
# Iteratie in beeld

$x_4$



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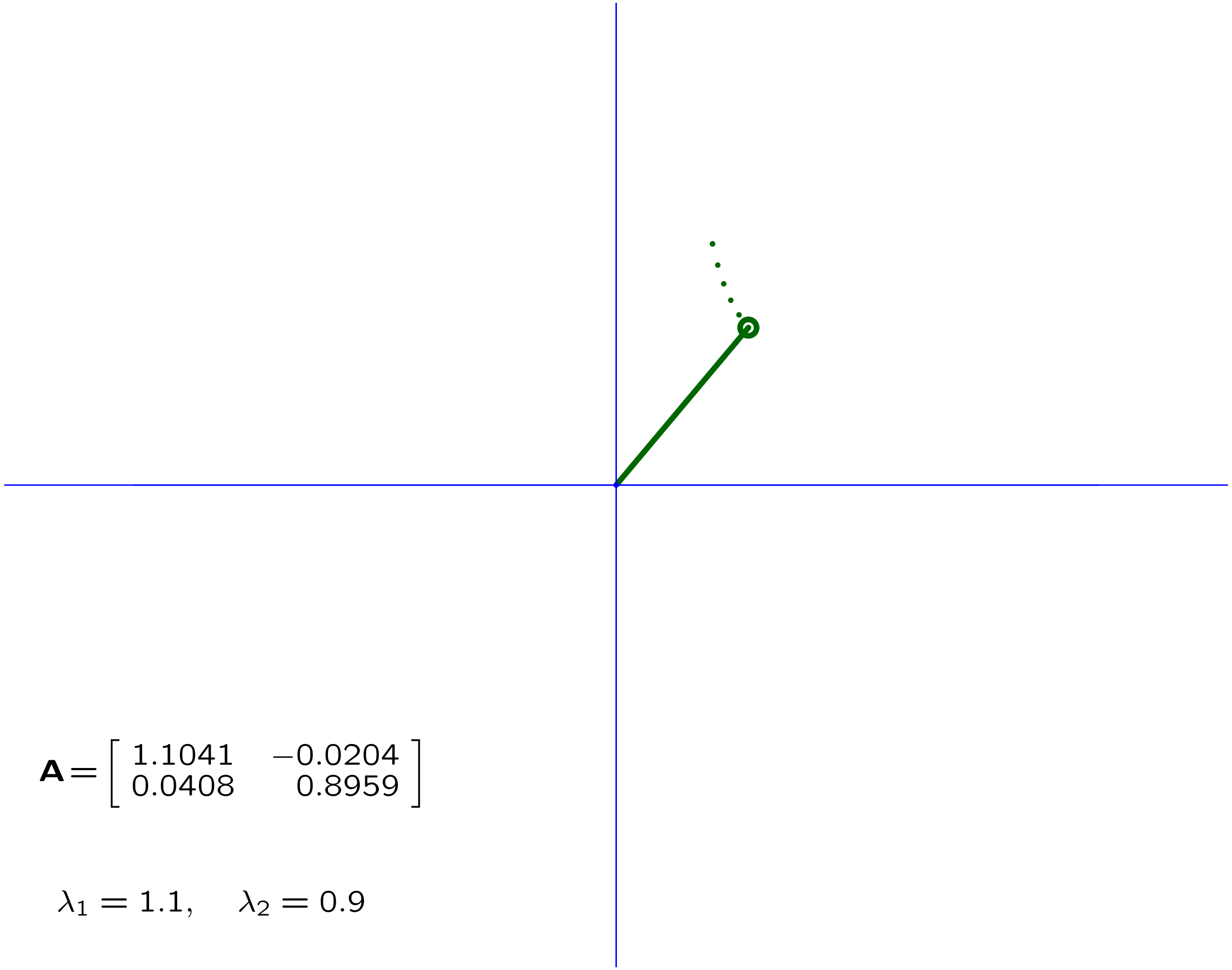
# Iteratie in beeld

$x_5$



$$\mathbf{A} = \begin{bmatrix} 1.1041 & -0.0204 \\ 0.0408 & 0.8959 \end{bmatrix}$$

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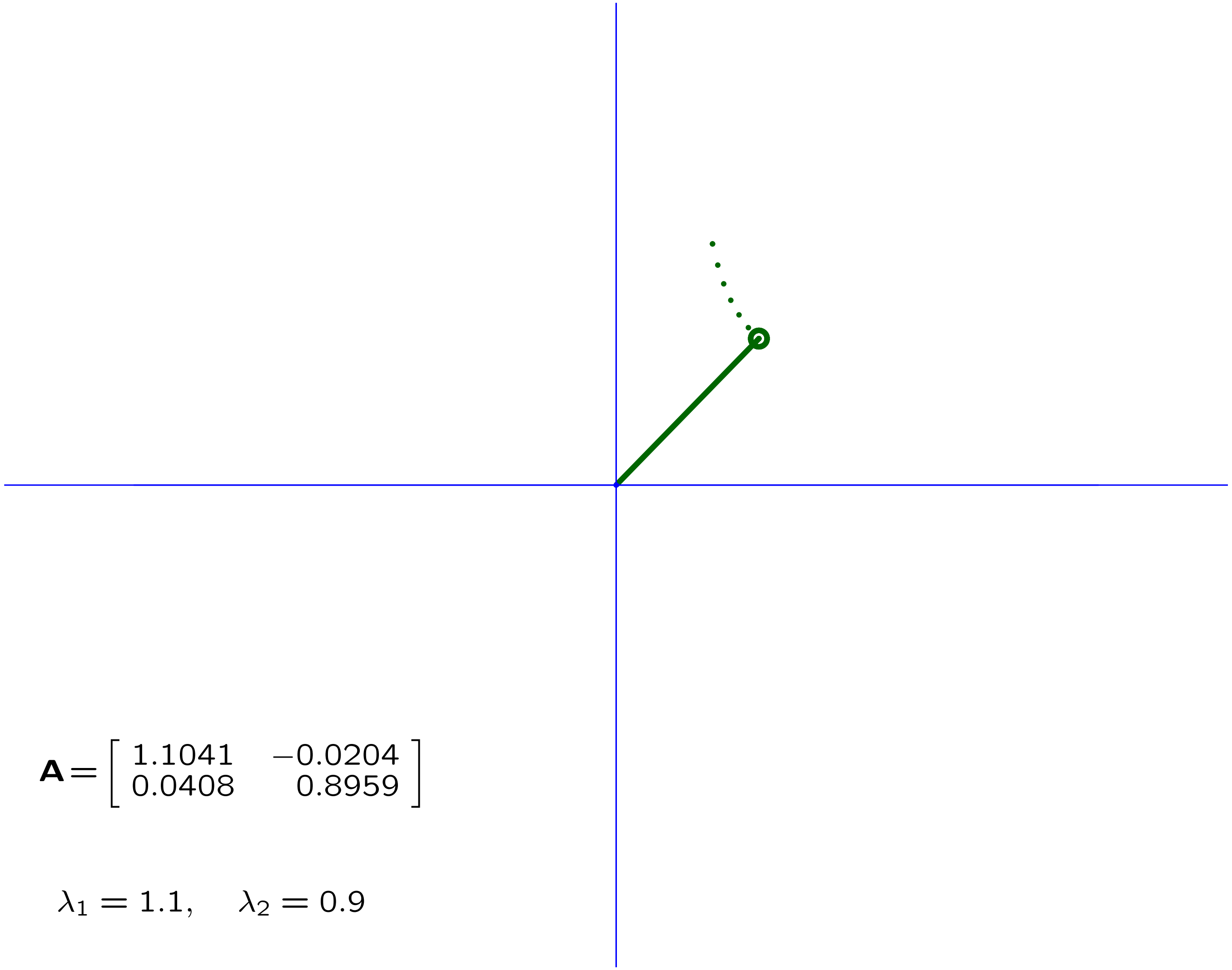
# Iteratie in beeld

$x_6$



$$\mathbf{A} = \begin{bmatrix} 1.1041 & -0.0204 \\ 0.0408 & 0.8959 \end{bmatrix}$$

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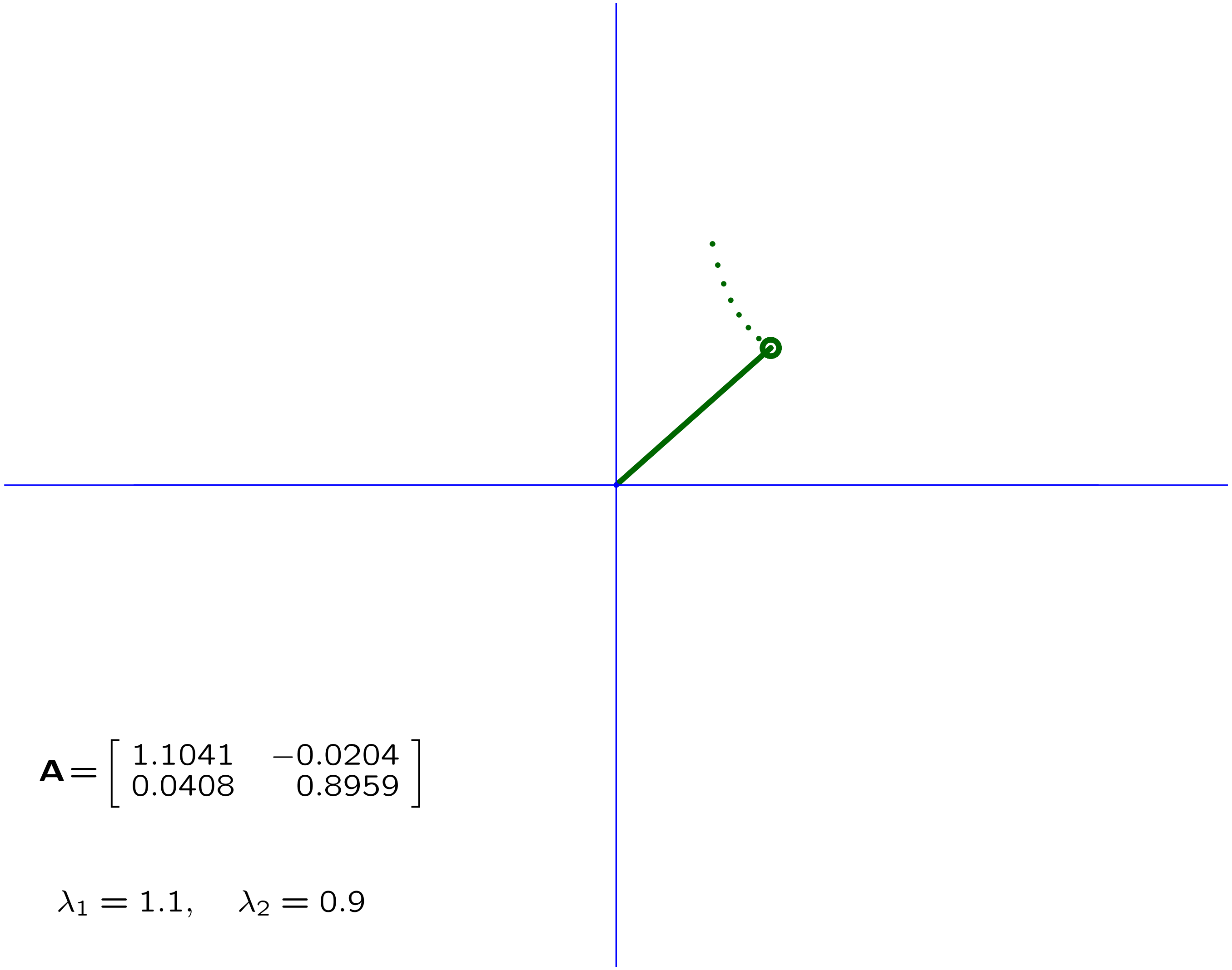
# Iteratie in beeld

$x_7$



$$\mathbf{A} = \begin{bmatrix} 1.1041 & -0.0204 \\ 0.0408 & 0.8959 \end{bmatrix}$$

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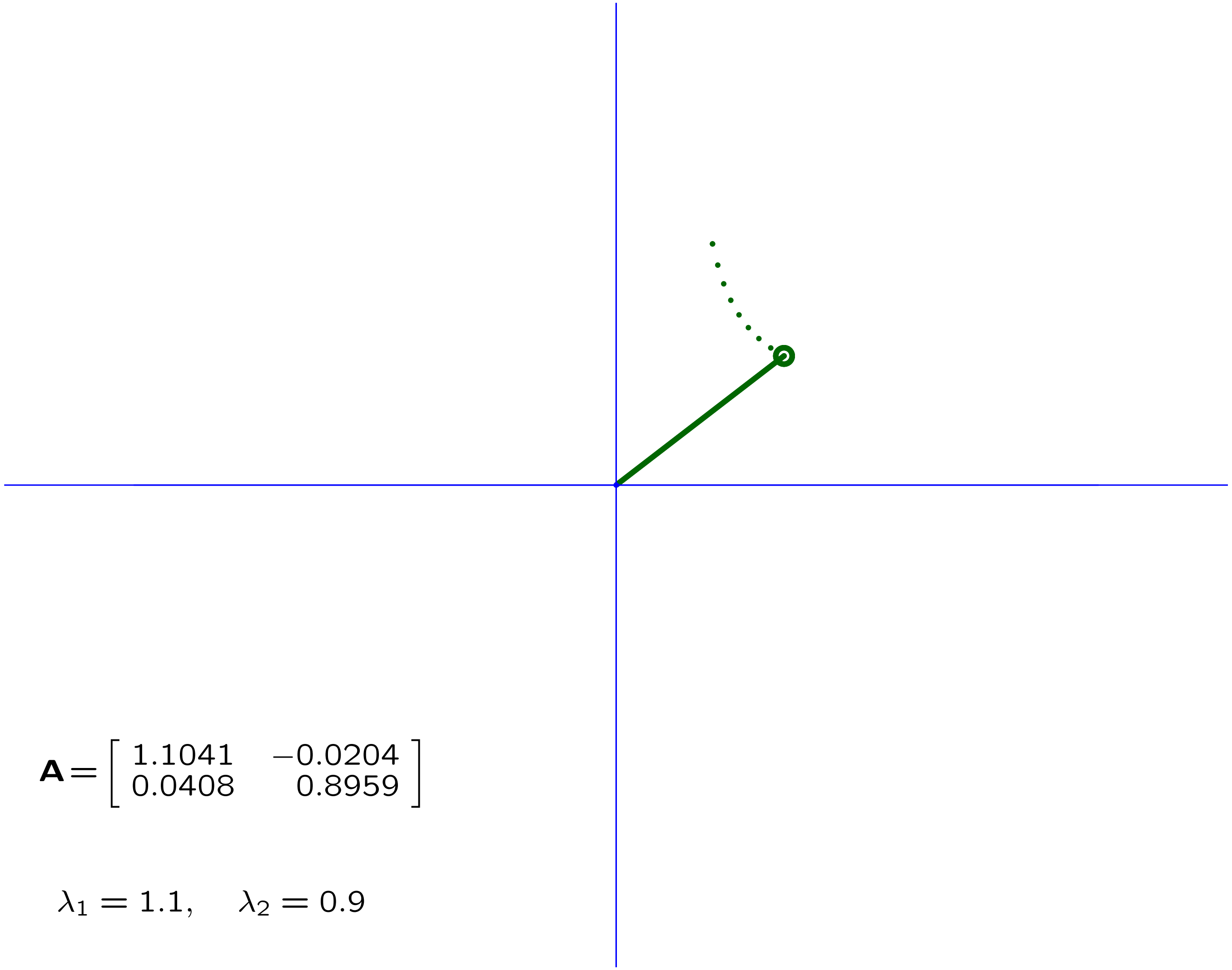
# Iteratie in beeld

$x_8$



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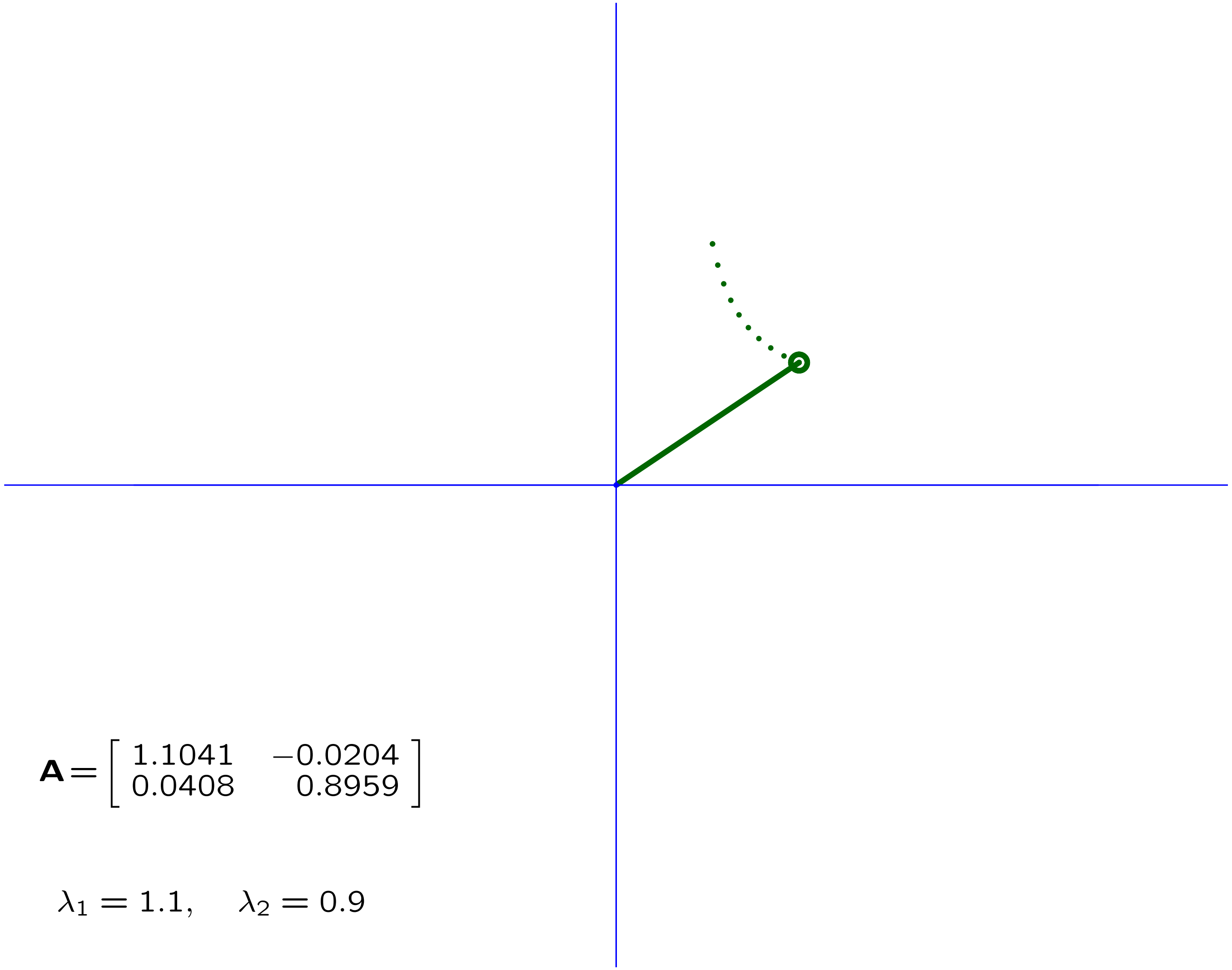
# Iteratie in beeld

$x_9$



$$\mathbf{A} = \begin{bmatrix} 1.1041 & -0.0204 \\ 0.0408 & 0.8959 \end{bmatrix}$$

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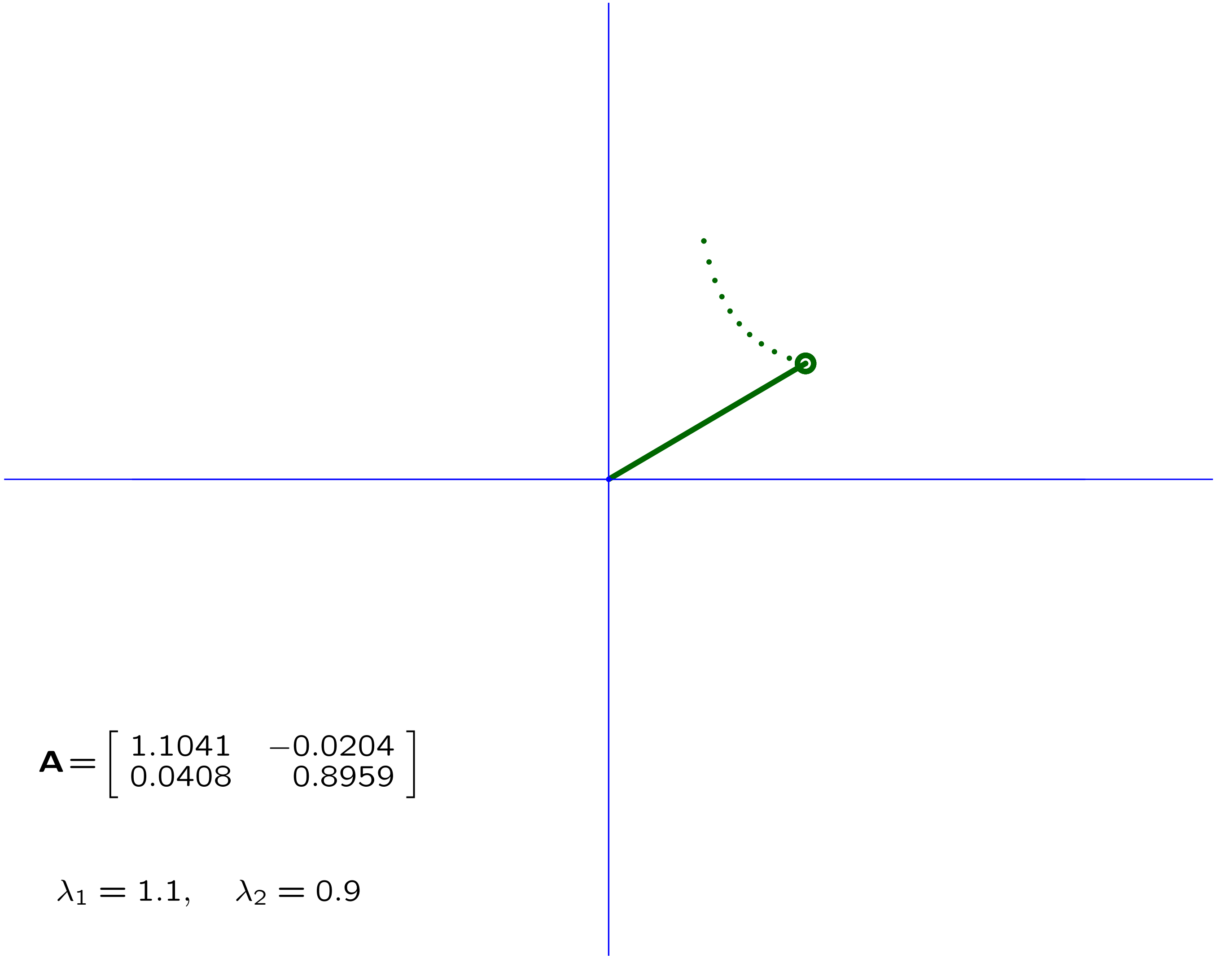
# Iteratie in beeld

$x_{10}$



$$\mathbf{A} = \begin{bmatrix} 1.1041 & -0.0204 \\ 0.0408 & 0.8959 \end{bmatrix}$$

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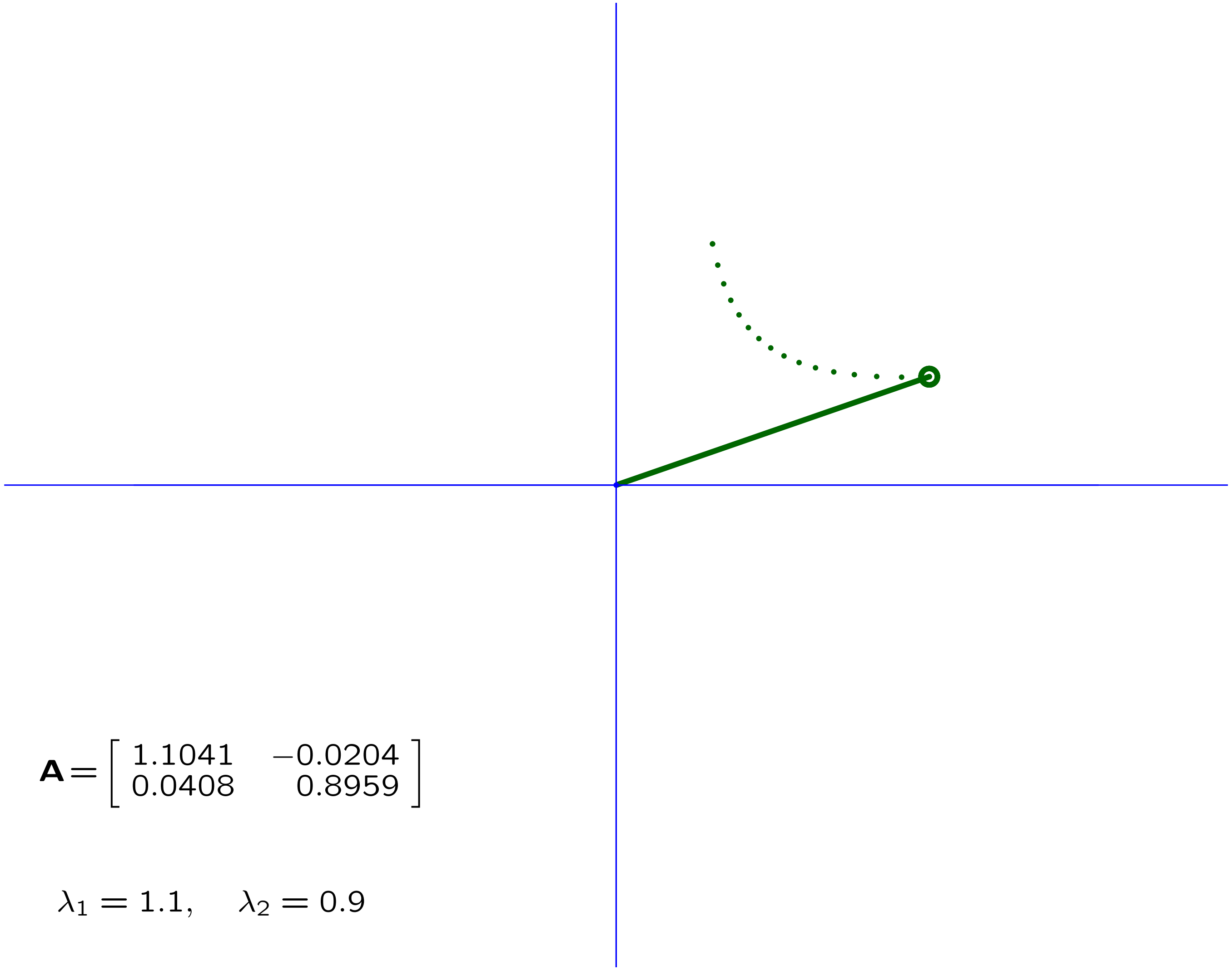
# Iteratie in beeld

$x_{15}$



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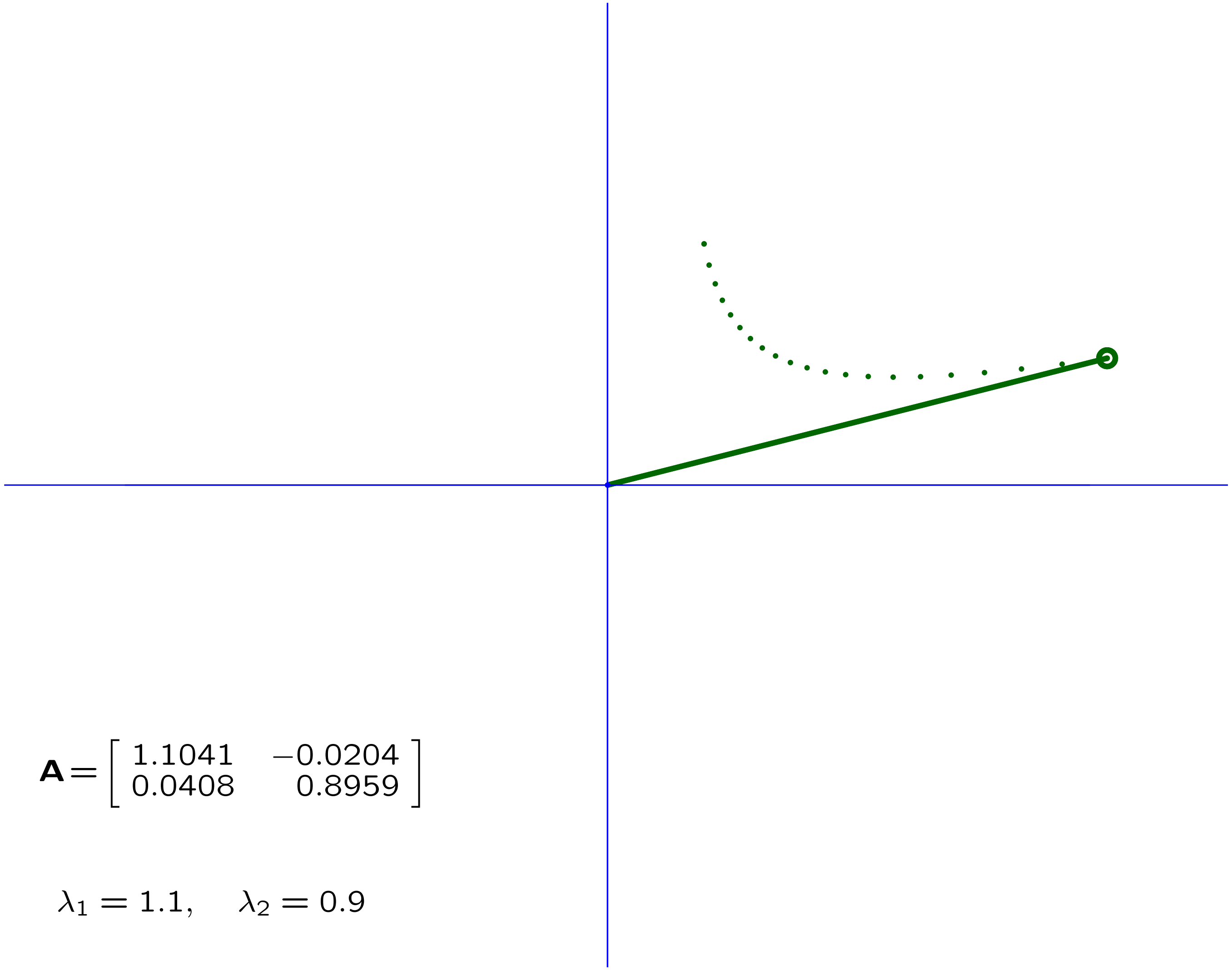
# Iteratie in beeld

$x_{20}$



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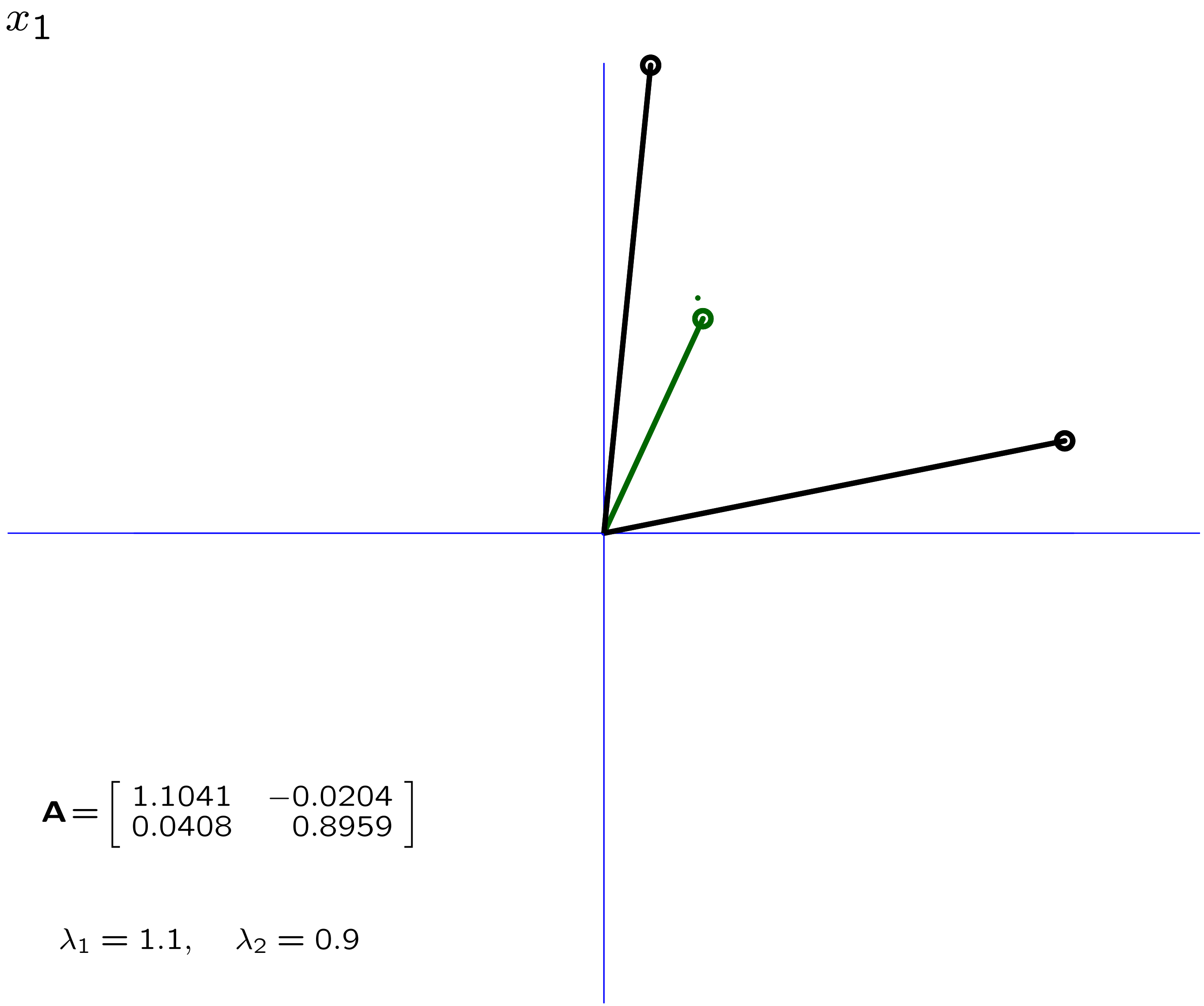
# Iteratie in beeld

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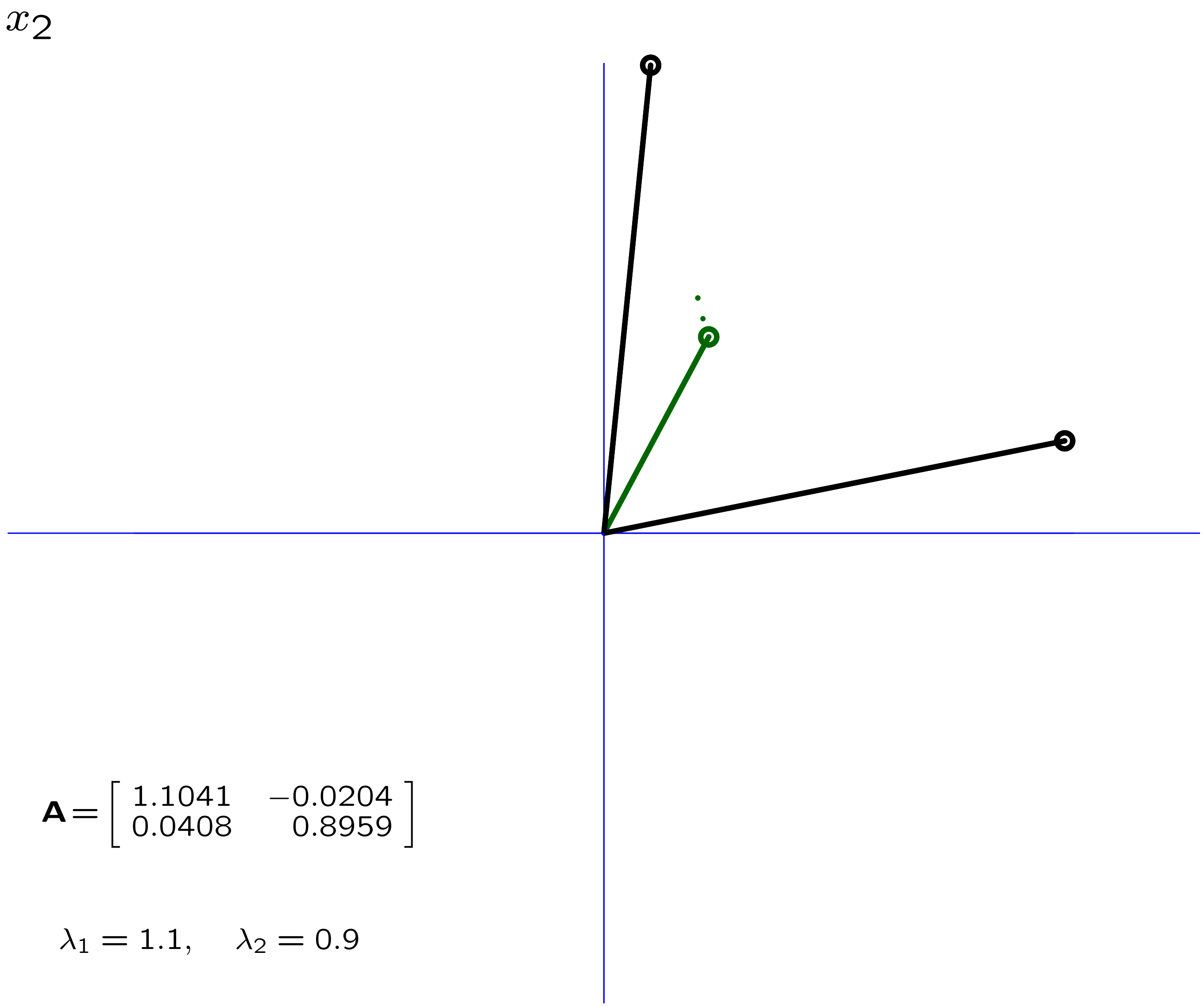


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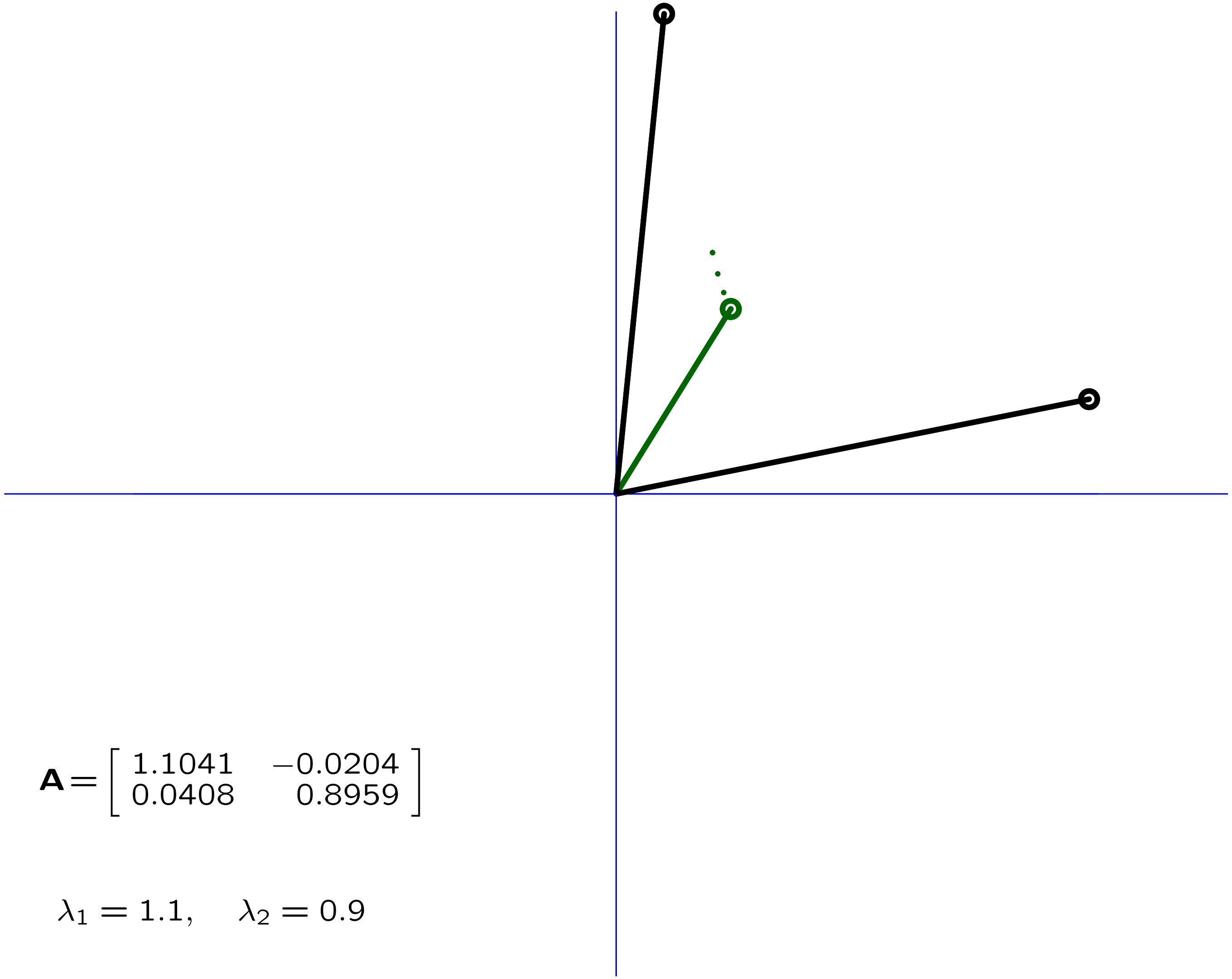
# Iteratie in beeld

$x_3$



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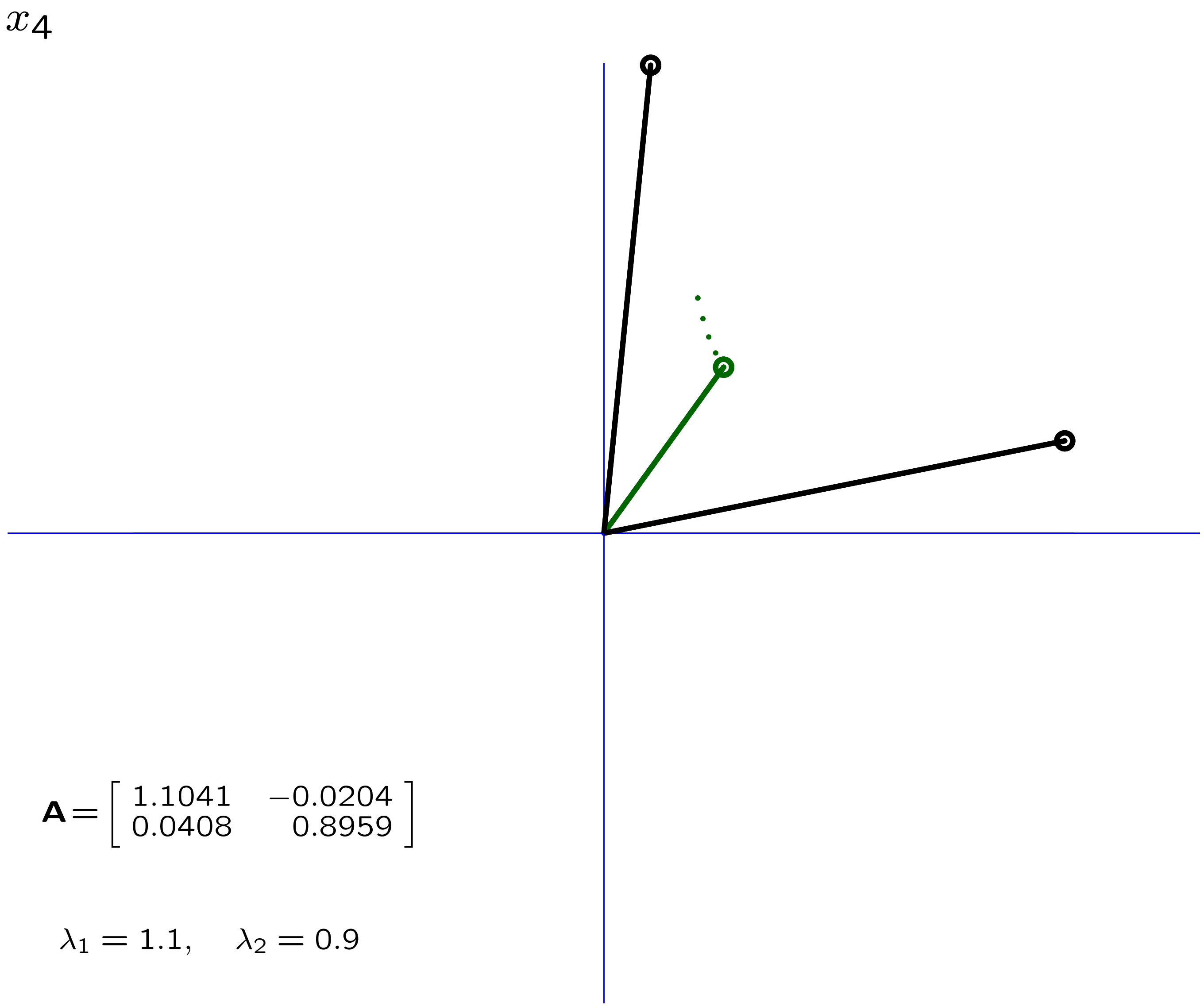


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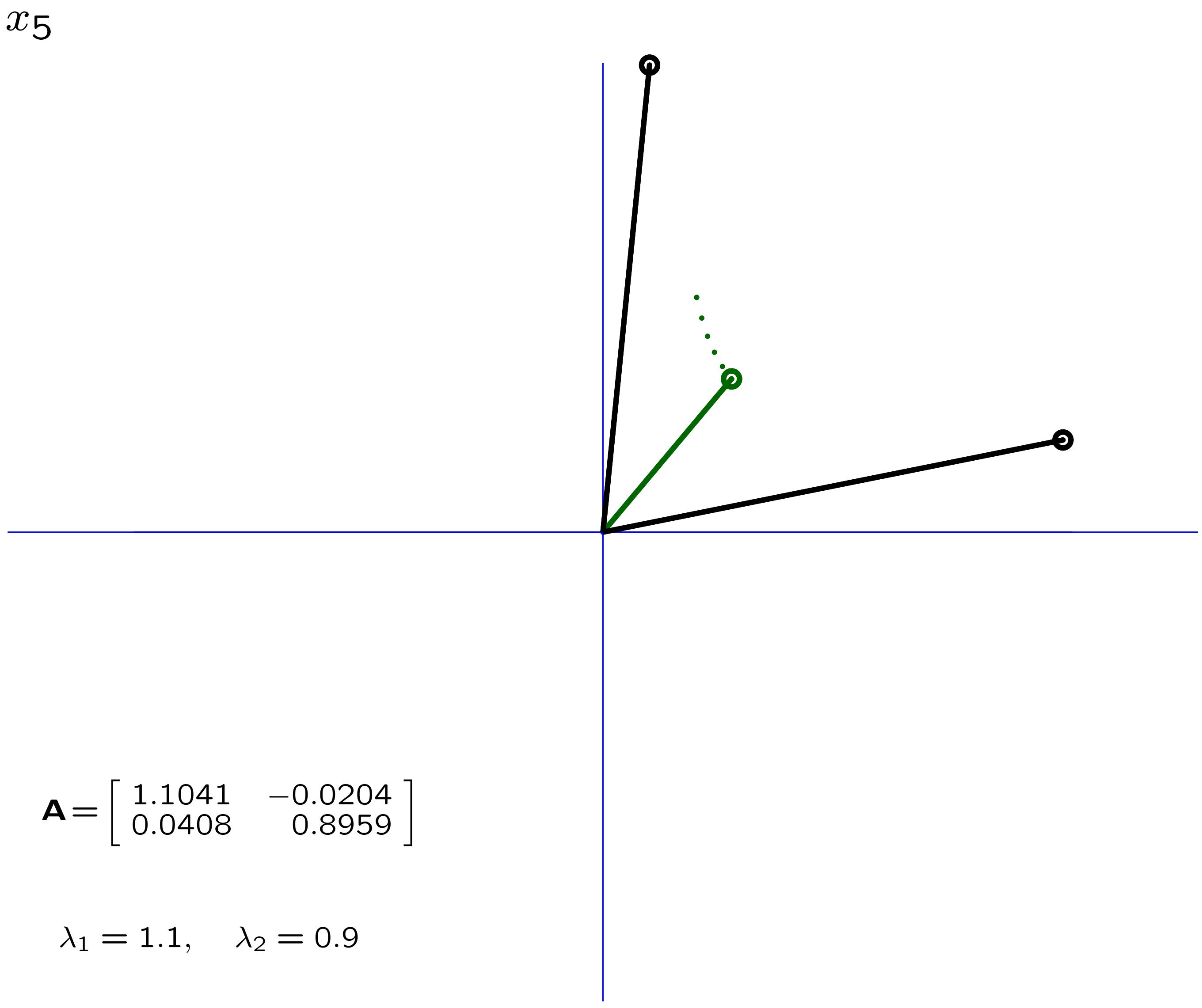


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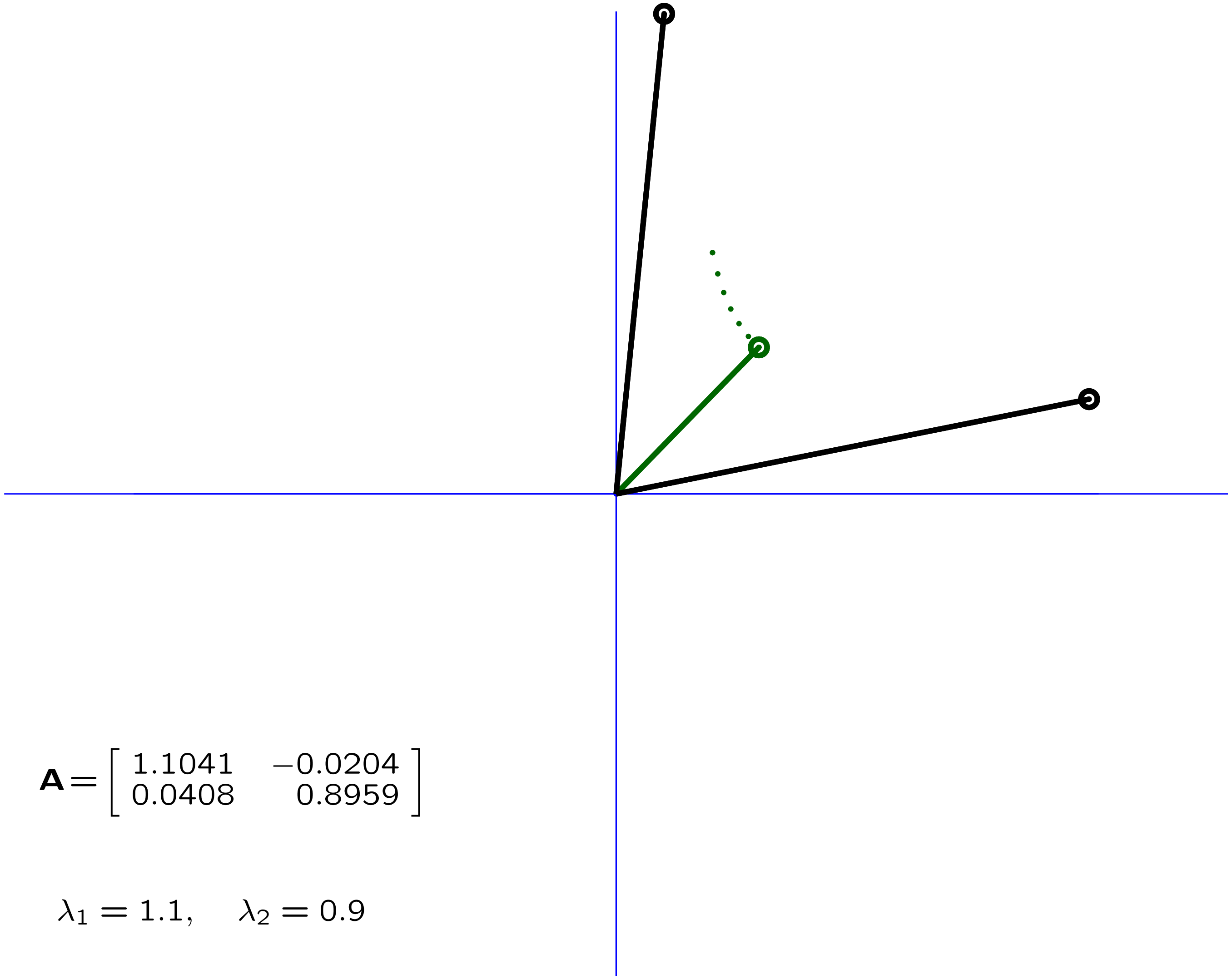
# Iteratie in beeld

$x_6$



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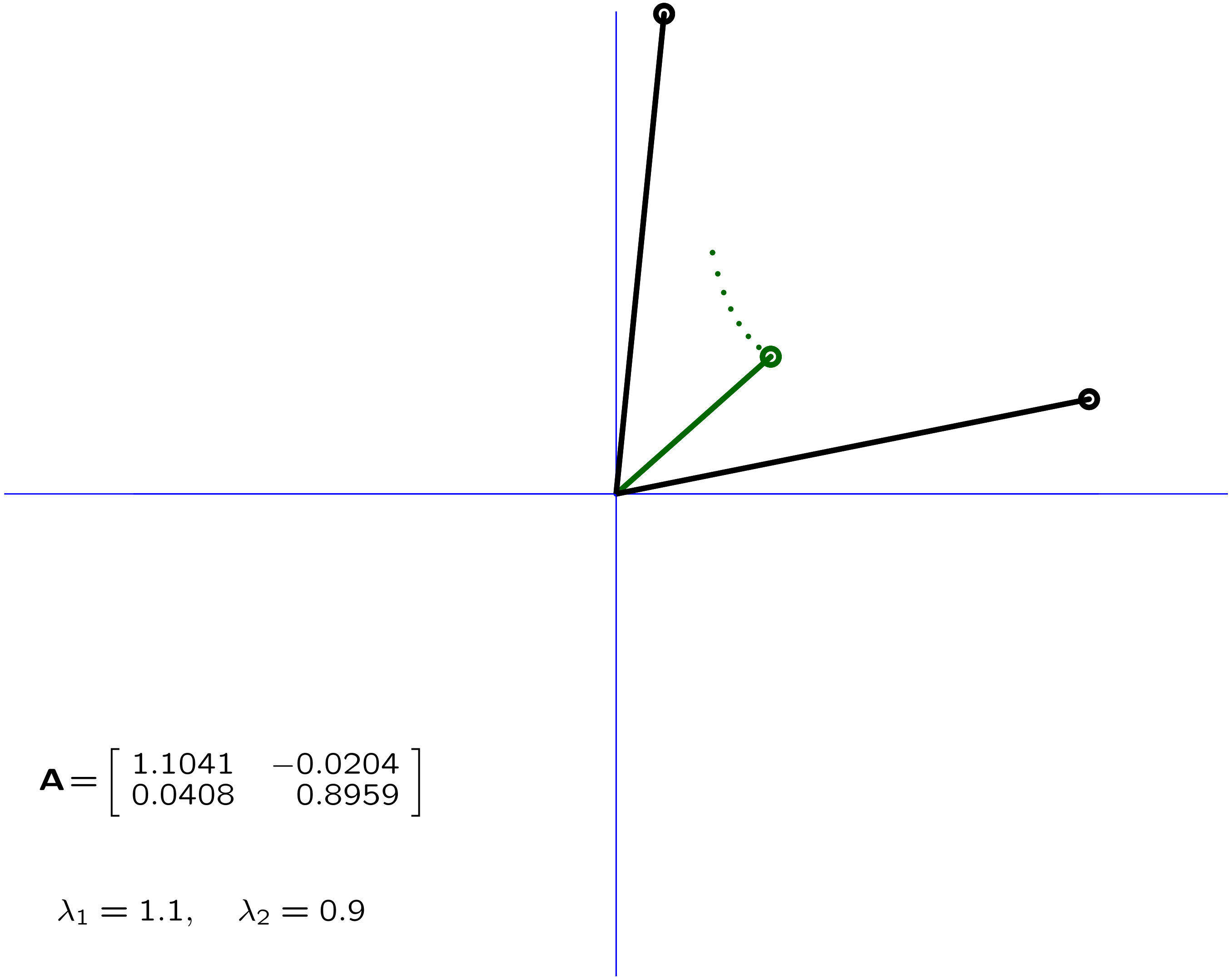
# Iteratie in beeld

$x_7$



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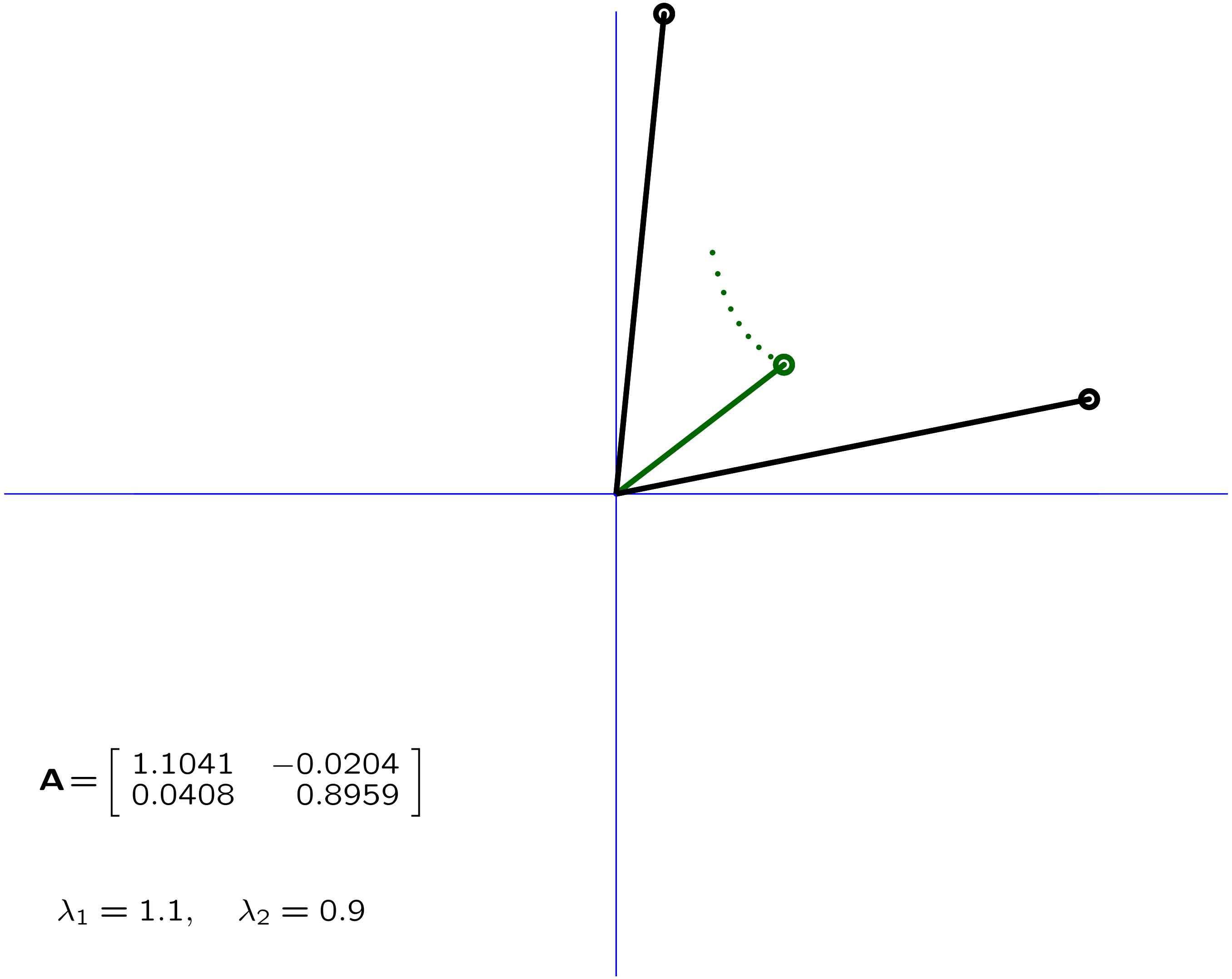

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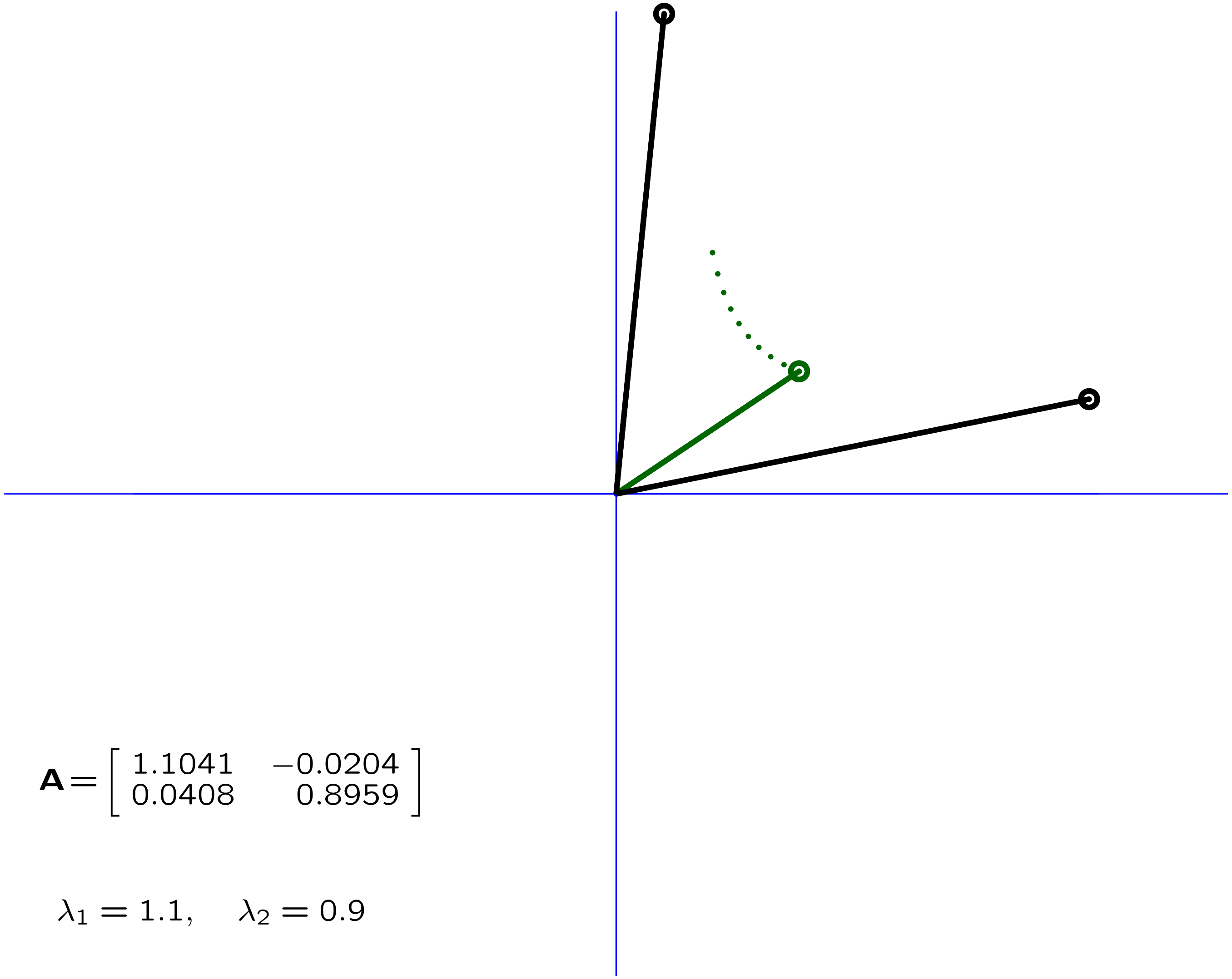
# Iteratie in beeld

$x_9$



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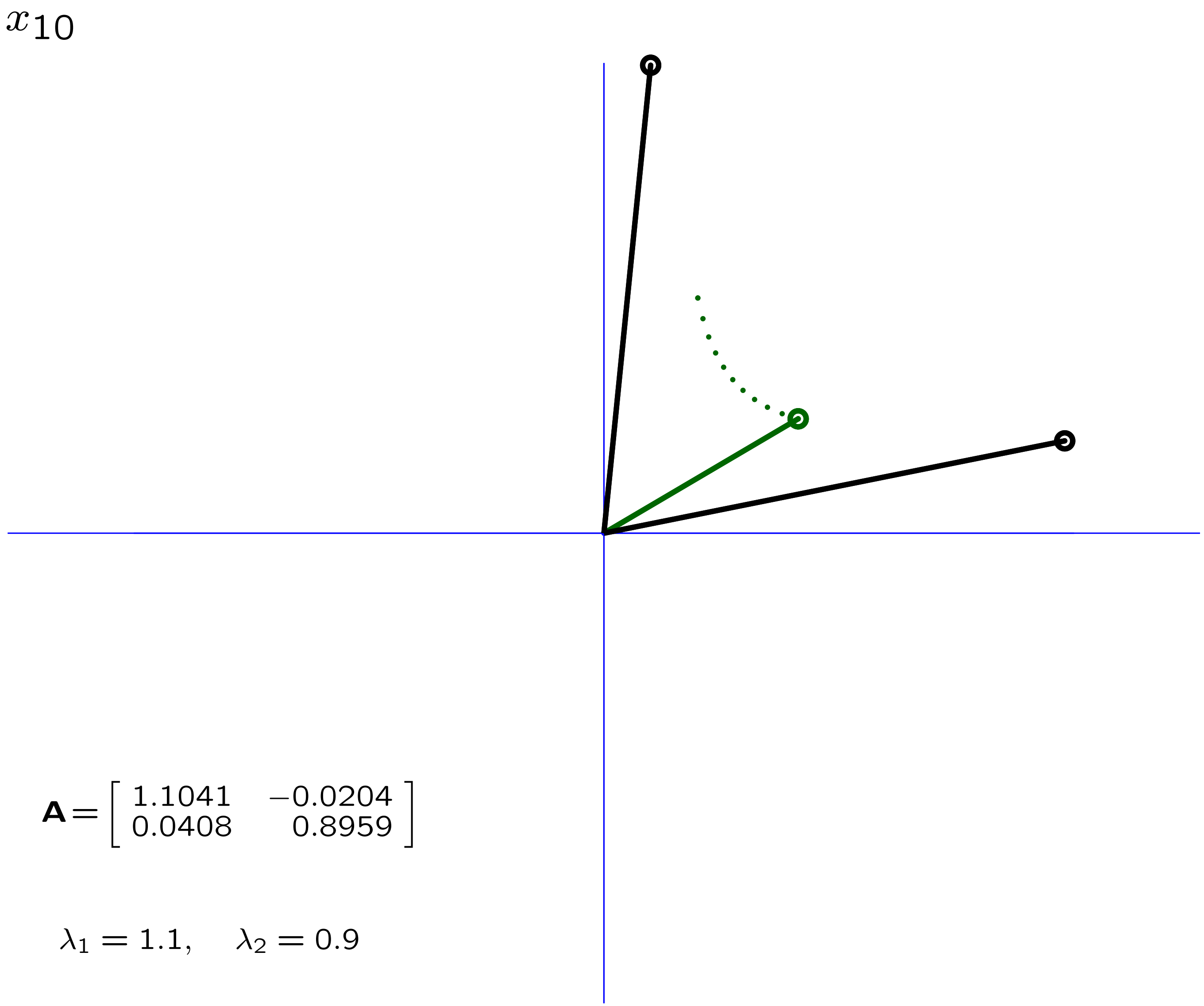
# Iteratie in beeld




$$\mathbf{A} = \begin{bmatrix} 1.1041 & -0.0204 \\ 0.0408 & 0.8959 \end{bmatrix}$$

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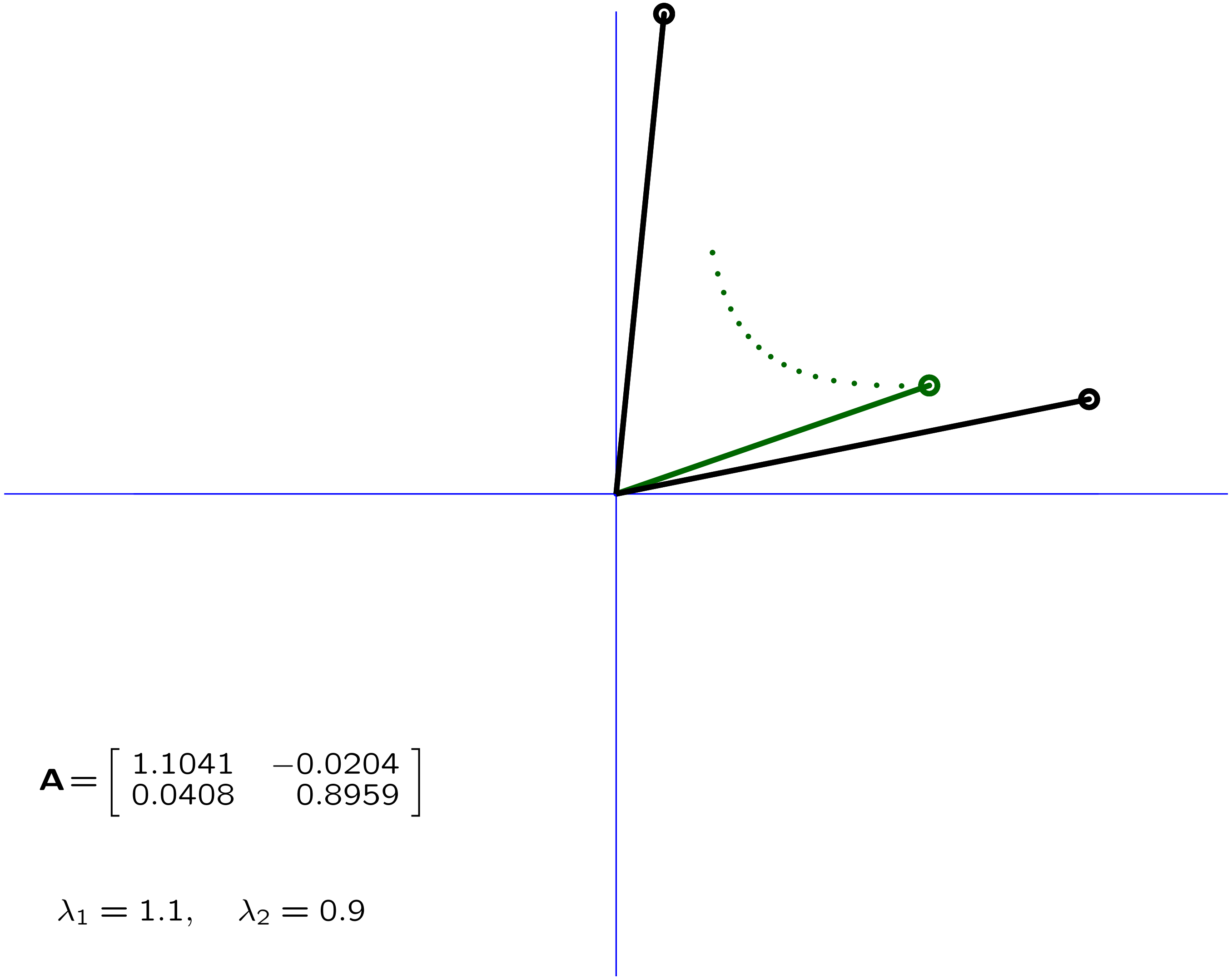
# Iteratie in beeld

$x_{15}$

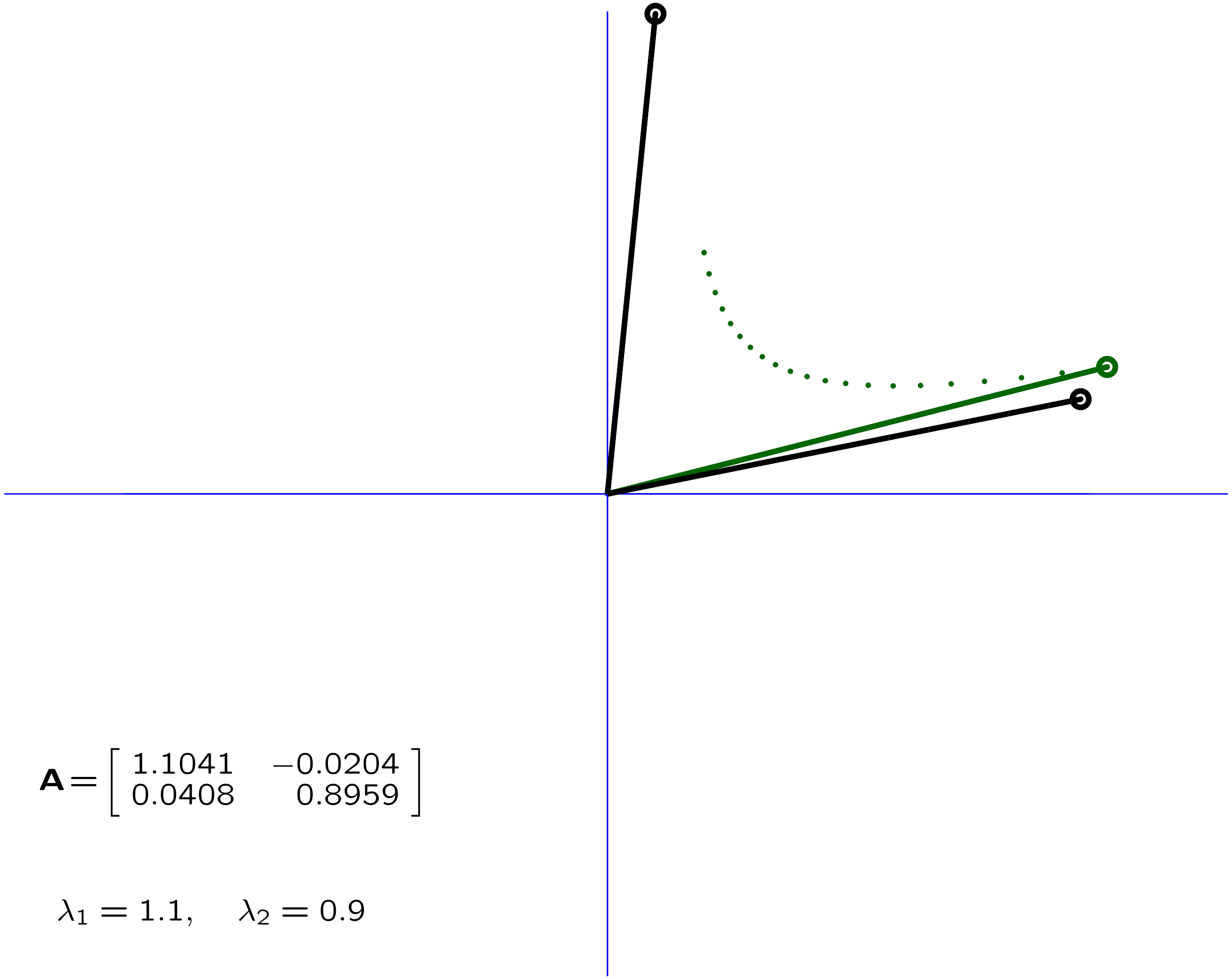


$$\mathbf{A} = \begin{bmatrix} 1.1041 & -0.0204 \\ 0.0408 & 0.8959 \end{bmatrix}$$

$$\lambda_1 = 1.1, \quad \lambda_2 = 0.9$$

# Iteratie in beeld

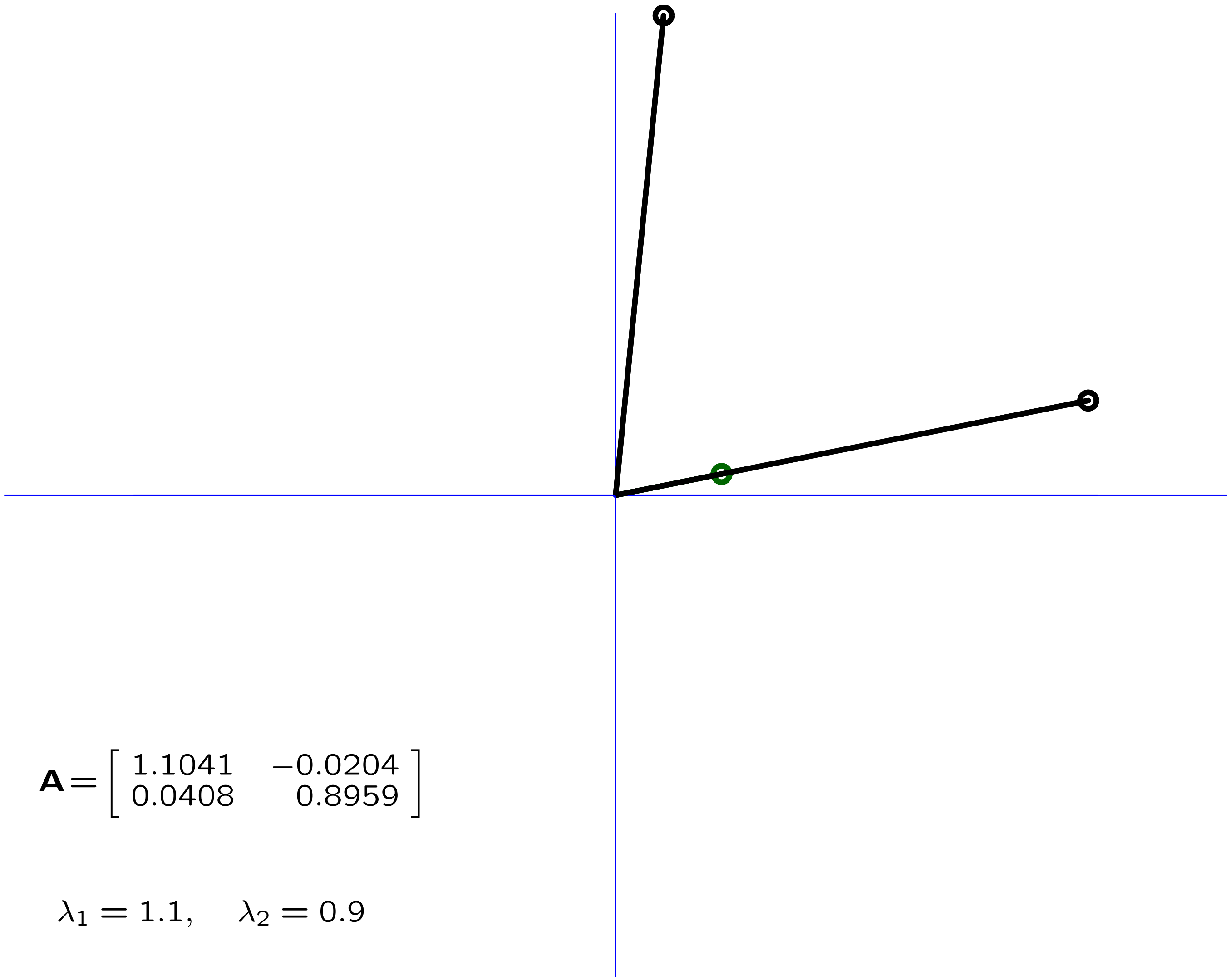
$x_{20}$



$$\mathbf{A} = \begin{bmatrix} 1.1041 & -0.0204 \\ 0.0408 & 0.8959 \end{bmatrix}$$

$$\lambda_1 = 1.1, \quad \lambda_2 = 0.9$$

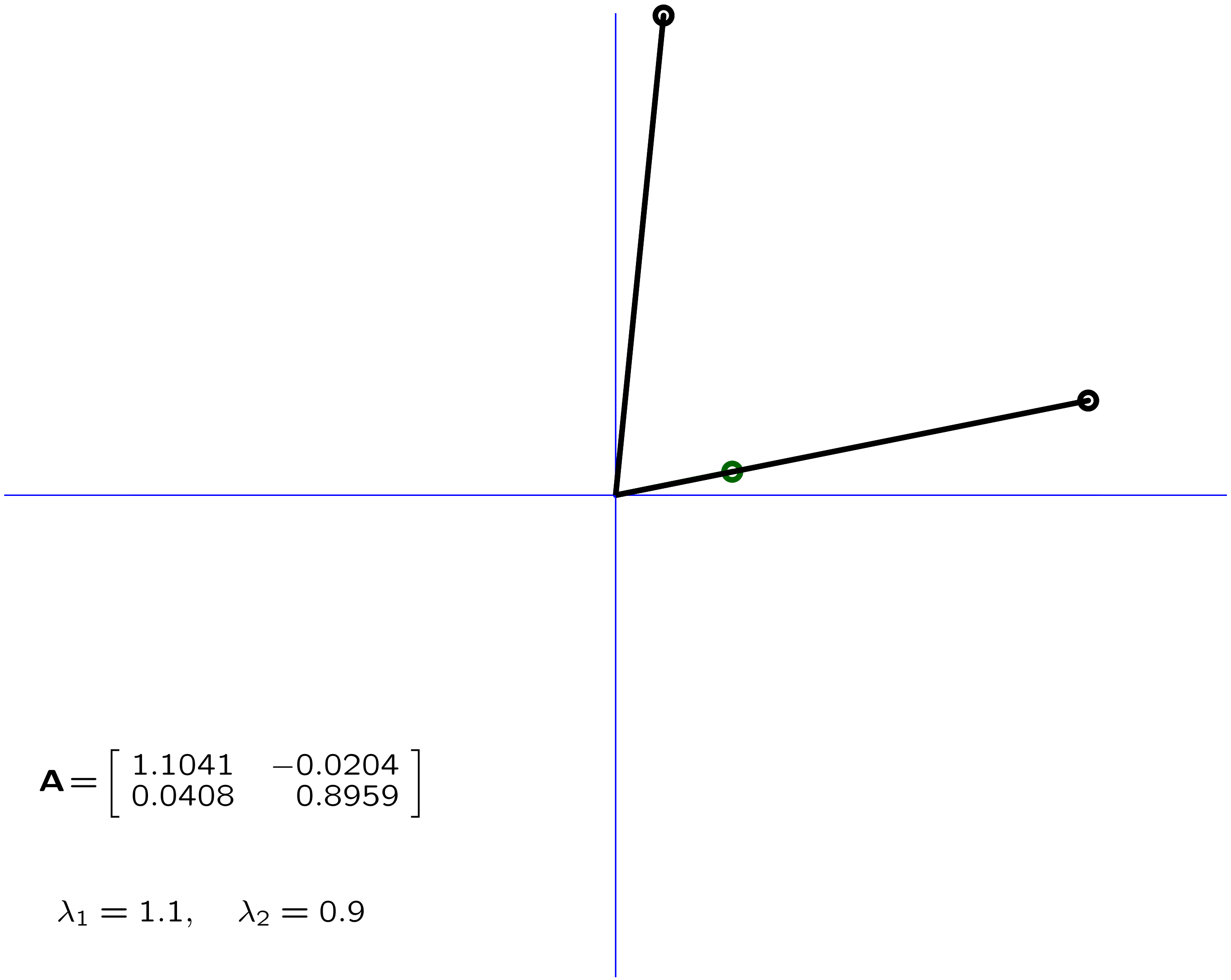
# Iteratie in beeld



$$\mathbf{A} = \begin{bmatrix} 1.1041 & -0.0204 \\ 0.0408 & 0.8959 \end{bmatrix}$$

$$\lambda_1 = 1.1, \quad \lambda_2 = 0.9$$

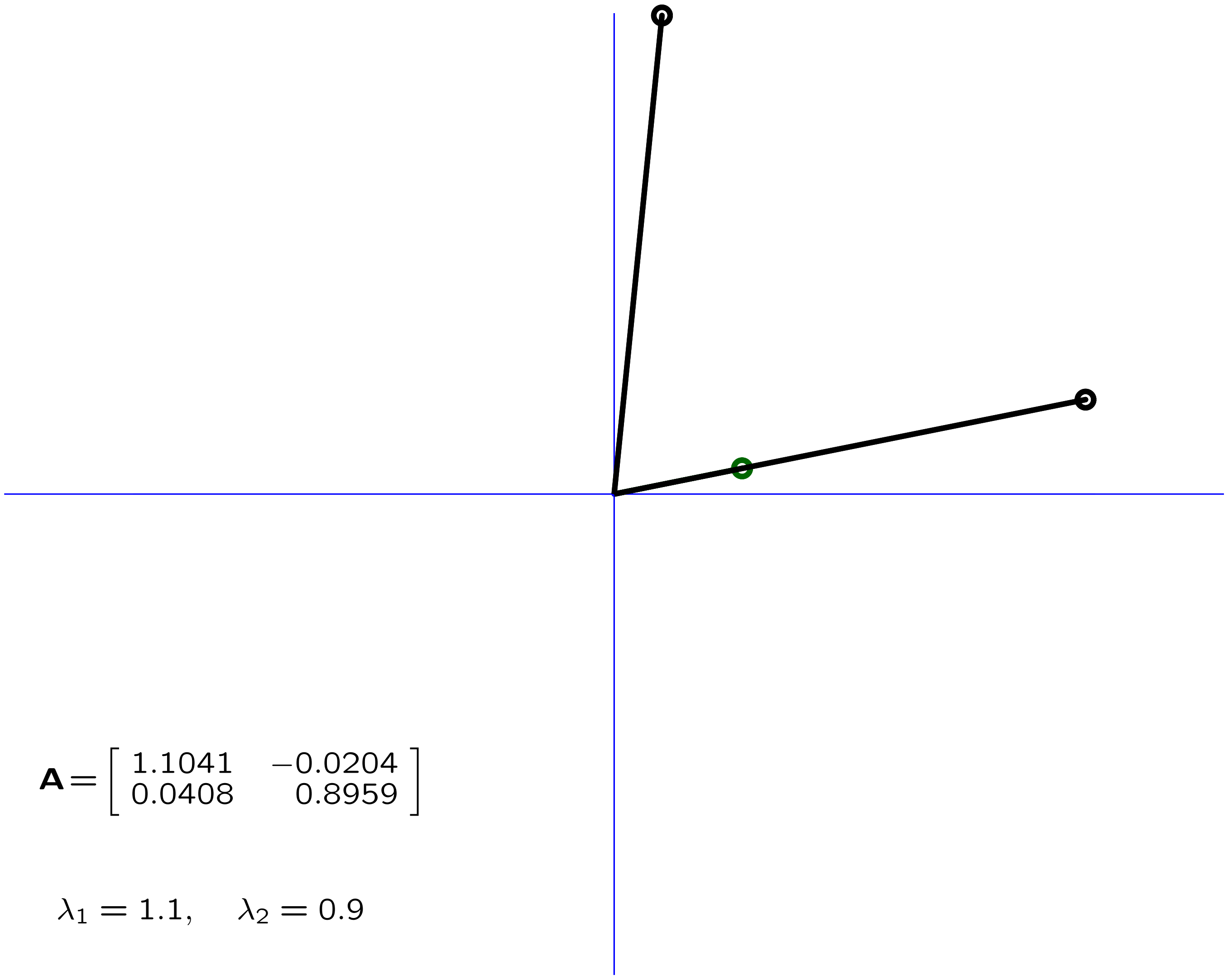
# Iteratie in beeld



$$\mathbf{A} = \begin{bmatrix} 1.1041 & -0.0204 \\ 0.0408 & 0.8959 \end{bmatrix}$$

$$\lambda_1 = 1.1, \quad \lambda_2 = 0.9$$

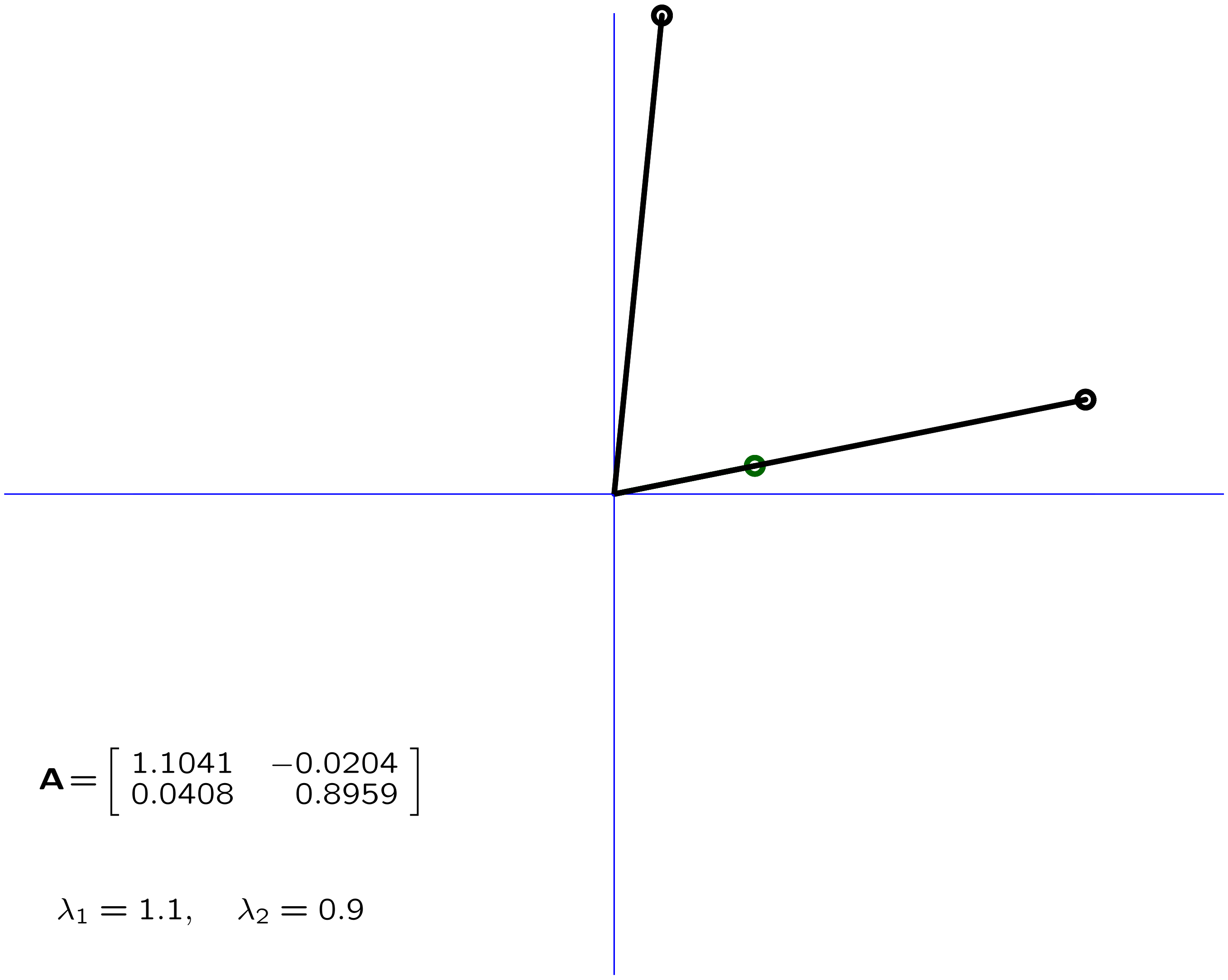
# Iteratie in beeld



$$\mathbf{A} = \begin{bmatrix} 1.1041 & -0.0204 \\ 0.0408 & 0.8959 \end{bmatrix}$$

$$\lambda_1 = 1.1, \quad \lambda_2 = 0.9$$

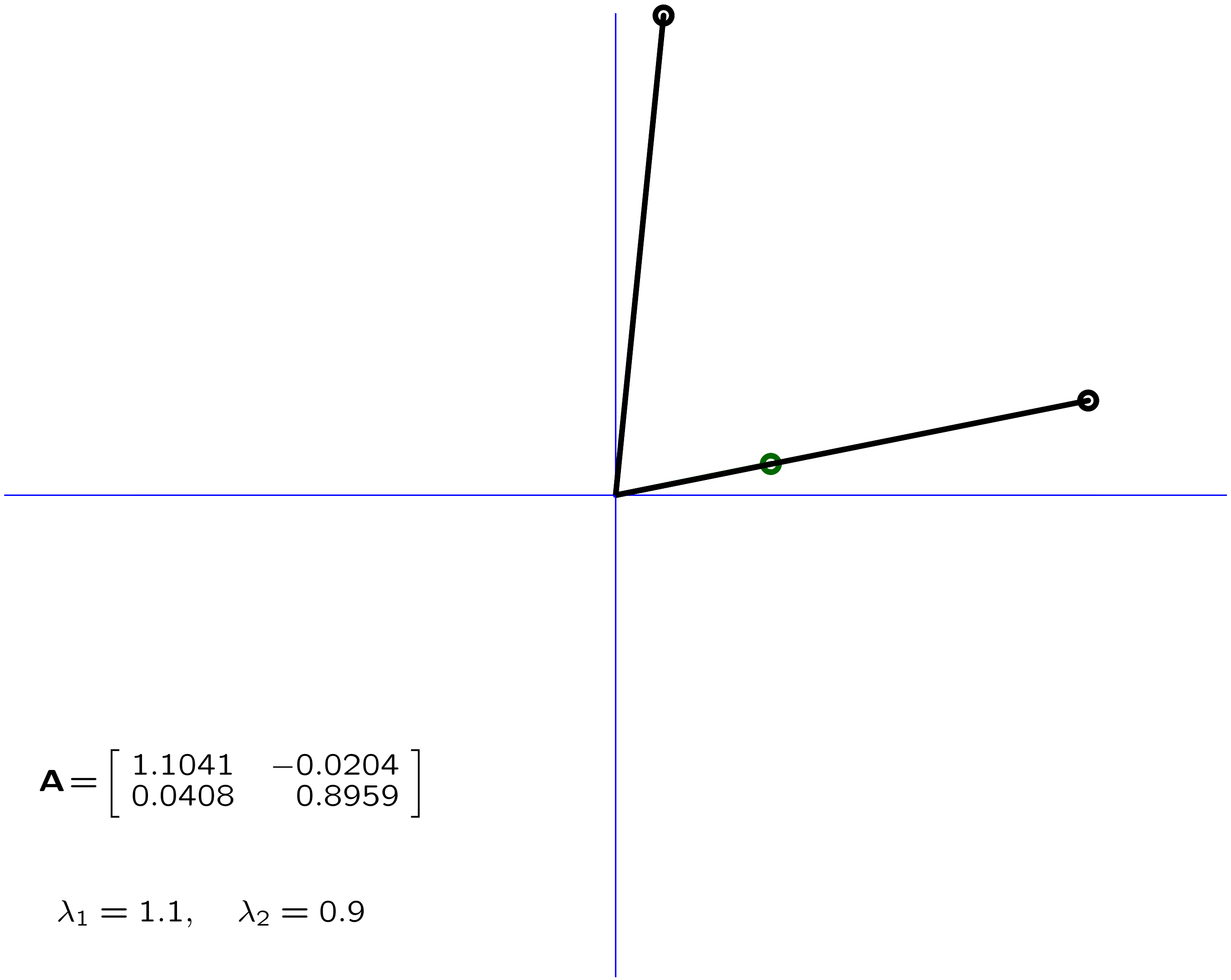
# Iteratie in beeld



$$\mathbf{A} = \begin{bmatrix} 1.1041 & -0.0204 \\ 0.0408 & 0.8959 \end{bmatrix}$$

$$\lambda_1 = 1.1, \quad \lambda_2 = 0.9$$

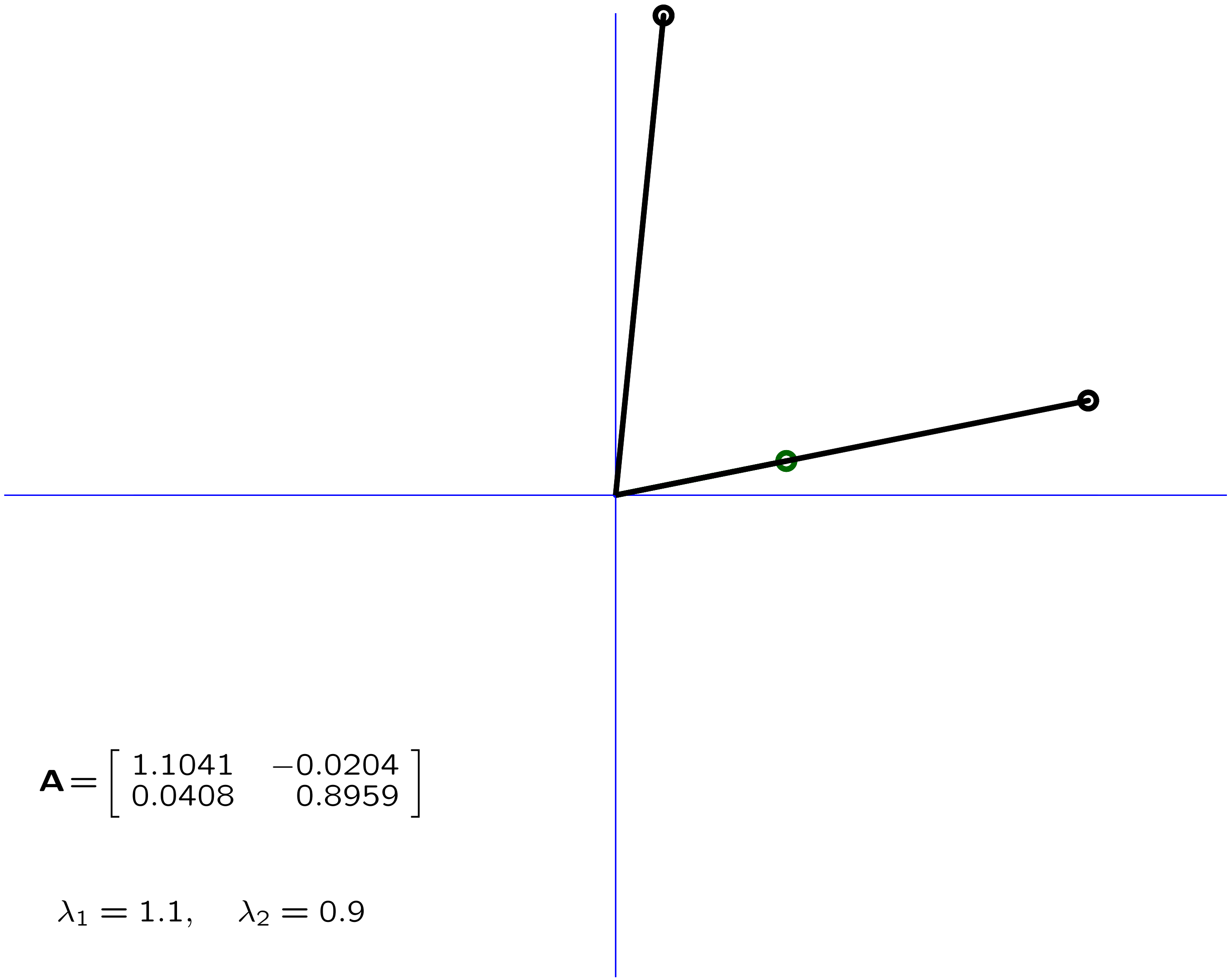
# Iteratie in beeld



$$\mathbf{A} = \begin{bmatrix} 1.1041 & -0.0204 \\ 0.0408 & 0.8959 \end{bmatrix}$$

$$\lambda_1 = 1.1, \quad \lambda_2 = 0.9$$

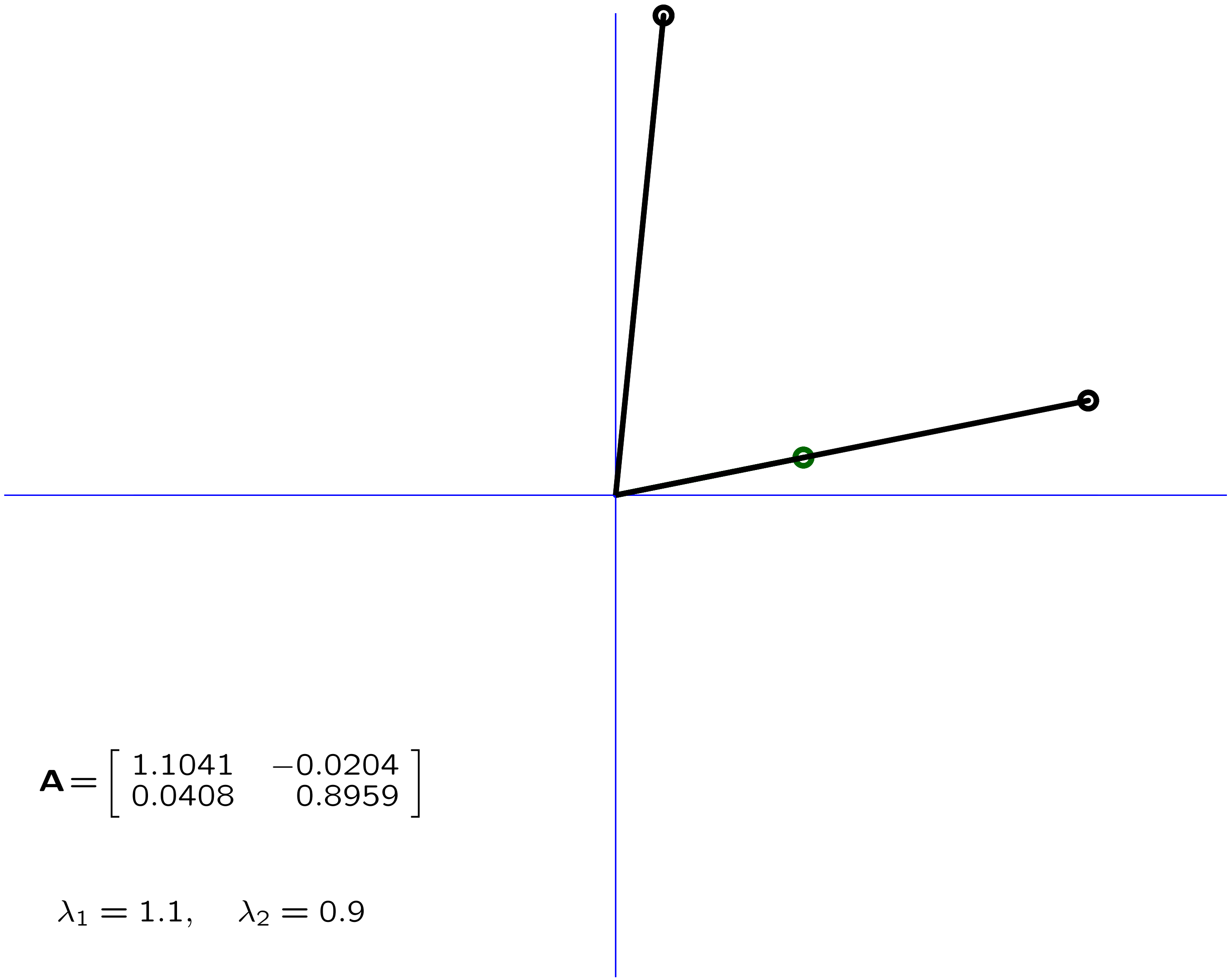
# Iteratie in beeld



$$\mathbf{A} = \begin{bmatrix} 1.1041 & -0.0204 \\ 0.0408 & 0.8959 \end{bmatrix}$$

$$\lambda_1 = 1.1, \quad \lambda_2 = 0.9$$

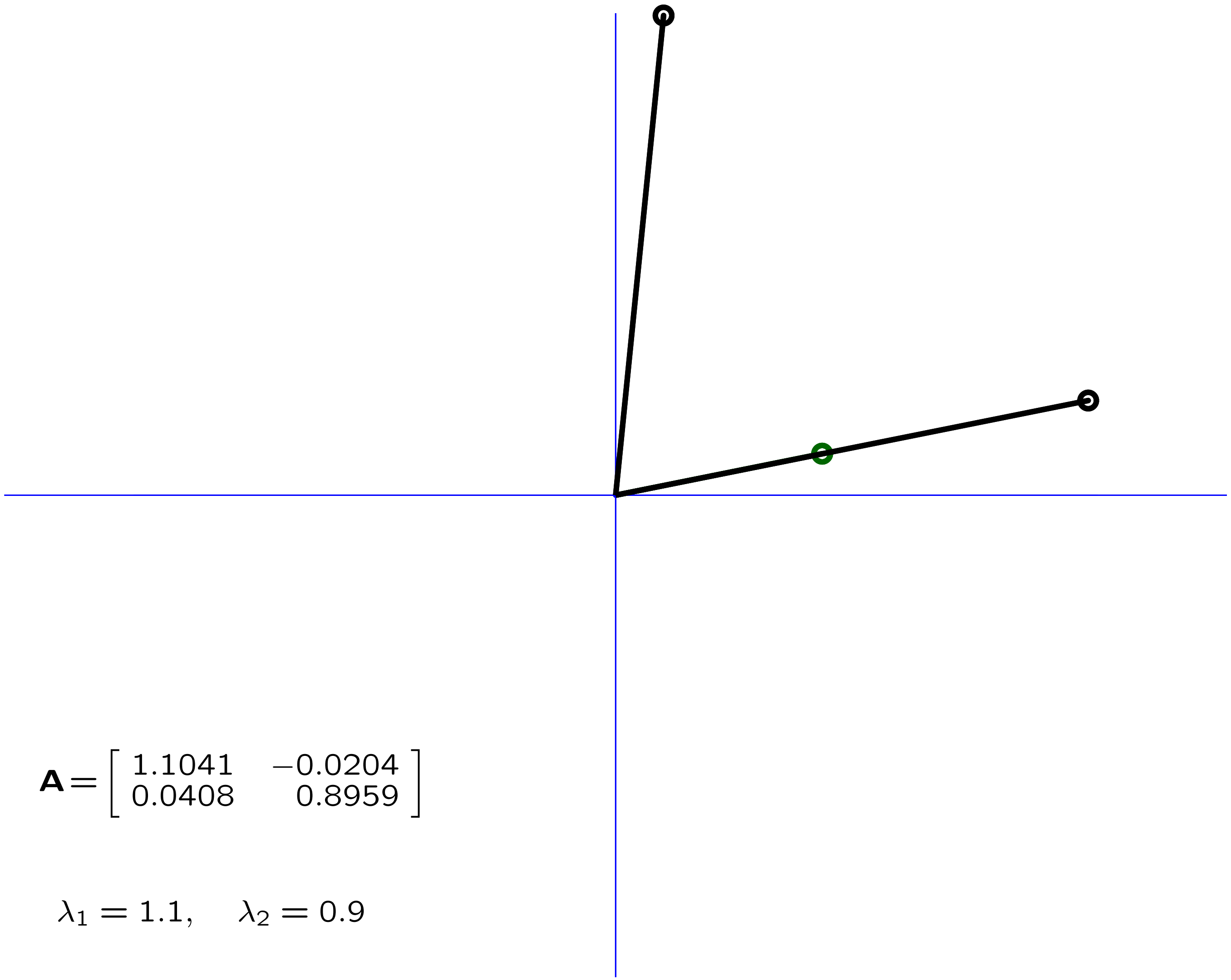
# Iteratie in beeld



$$\mathbf{A} = \begin{bmatrix} 1.1041 & -0.0204 \\ 0.0408 & 0.8959 \end{bmatrix}$$

$$\lambda_1 = 1.1, \quad \lambda_2 = 0.9$$

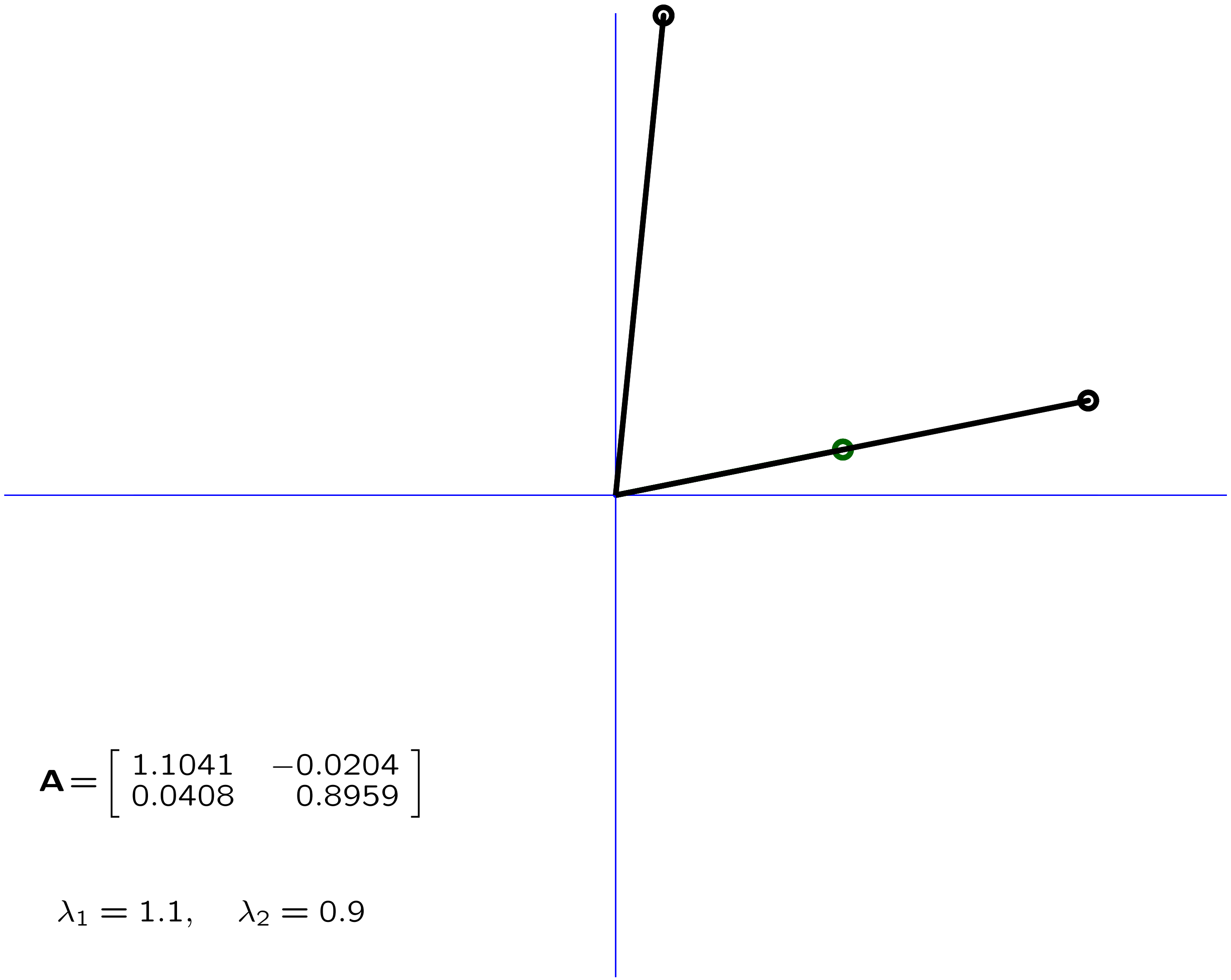
# Iteratie in beeld



$$\mathbf{A} = \begin{bmatrix} 1.1041 & -0.0204 \\ 0.0408 & 0.8959 \end{bmatrix}$$

$$\lambda_1 = 1.1, \quad \lambda_2 = 0.9$$

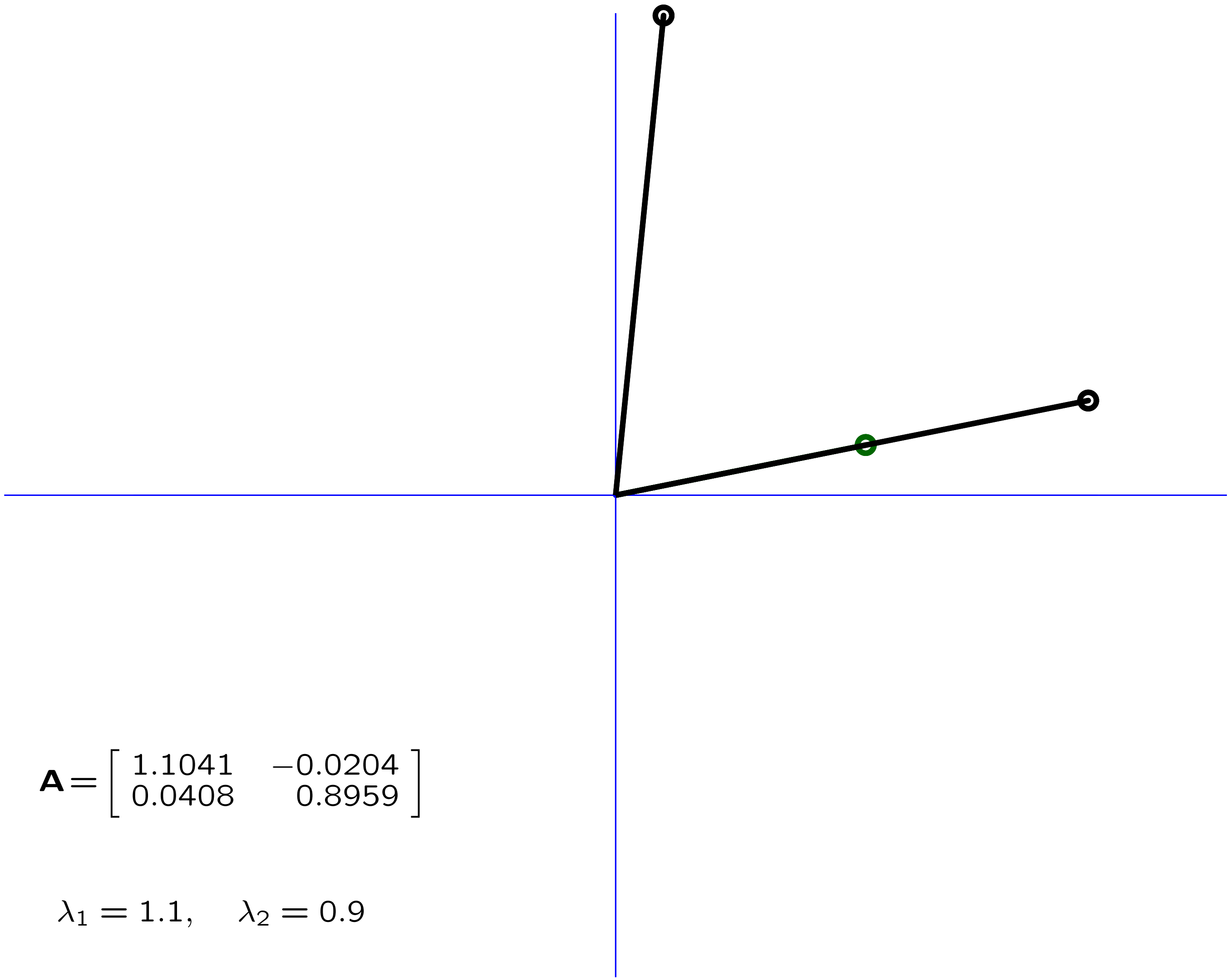
# Iteratie in beeld



$$\mathbf{A} = \begin{bmatrix} 1.1041 & -0.0204 \\ 0.0408 & 0.8959 \end{bmatrix}$$

$$\lambda_1 = 1.1, \quad \lambda_2 = 0.9$$

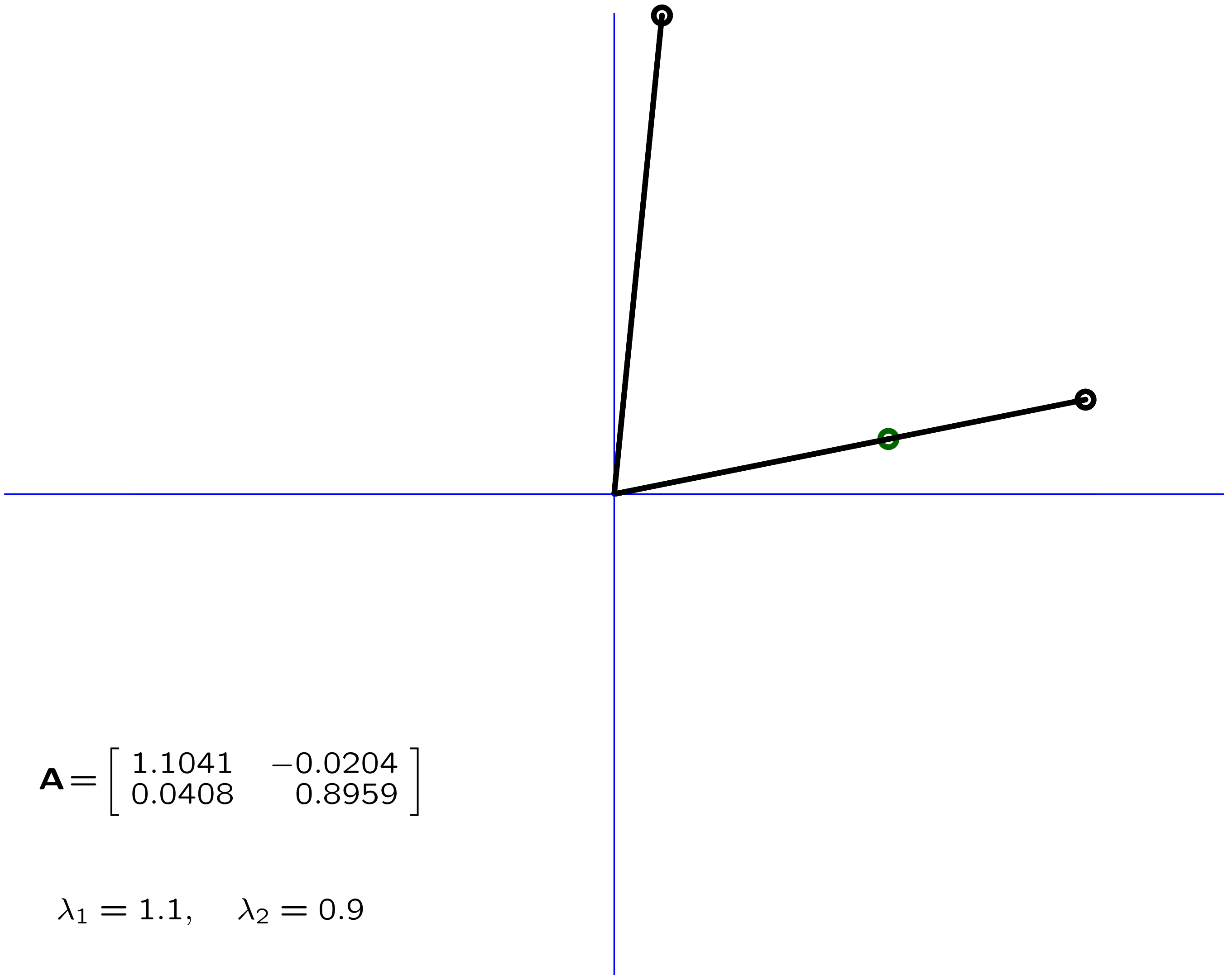
# Iteratie in beeld



$$\mathbf{A} = \begin{bmatrix} 1.1041 & -0.0204 \\ 0.0408 & 0.8959 \end{bmatrix}$$

$$\lambda_1 = 1.1, \quad \lambda_2 = 0.9$$

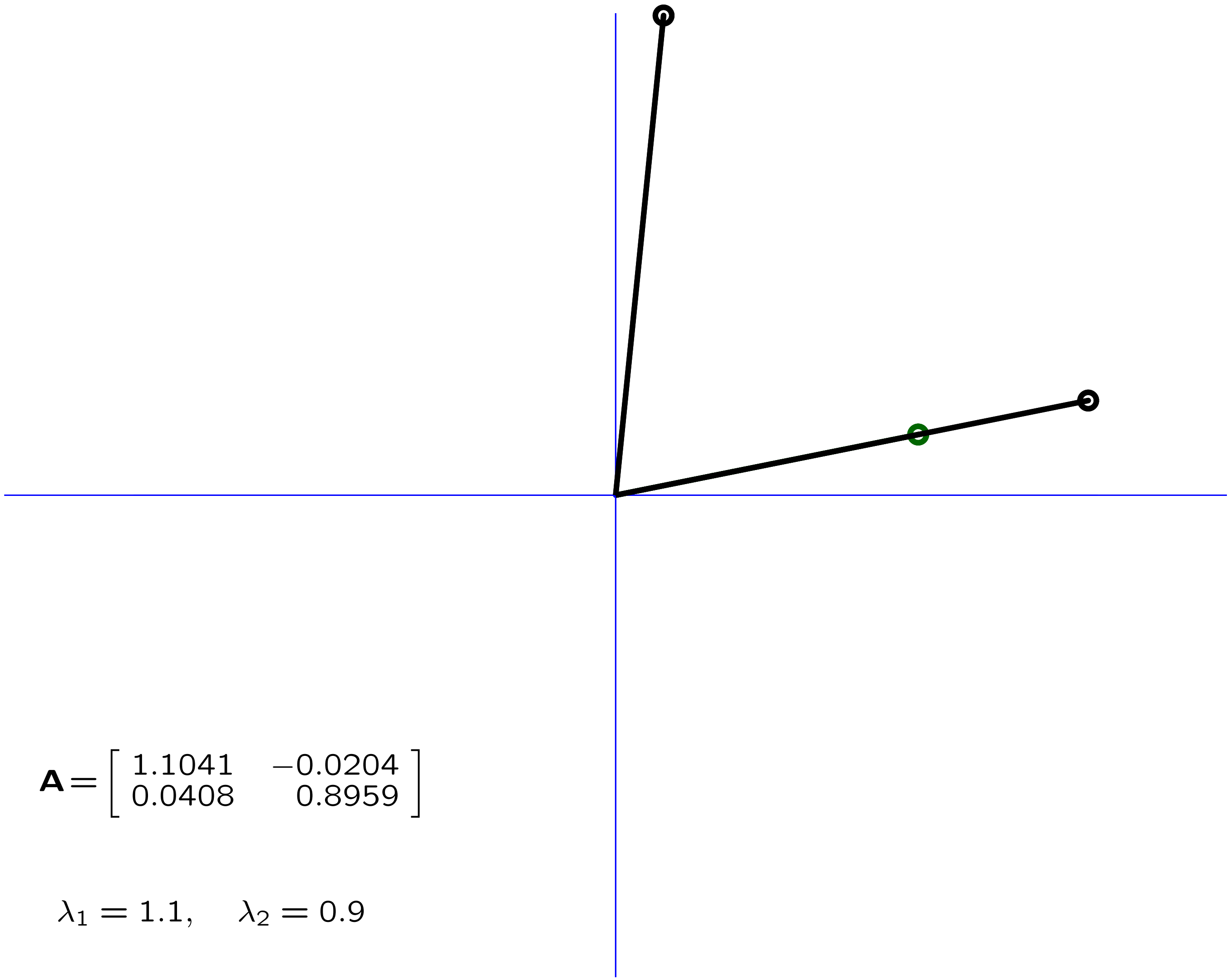
# Iteratie in beeld



$$\mathbf{A} = \begin{bmatrix} 1.1041 & -0.0204 \\ 0.0408 & 0.8959 \end{bmatrix}$$

$$\lambda_1 = 1.1, \quad \lambda_2 = 0.9$$

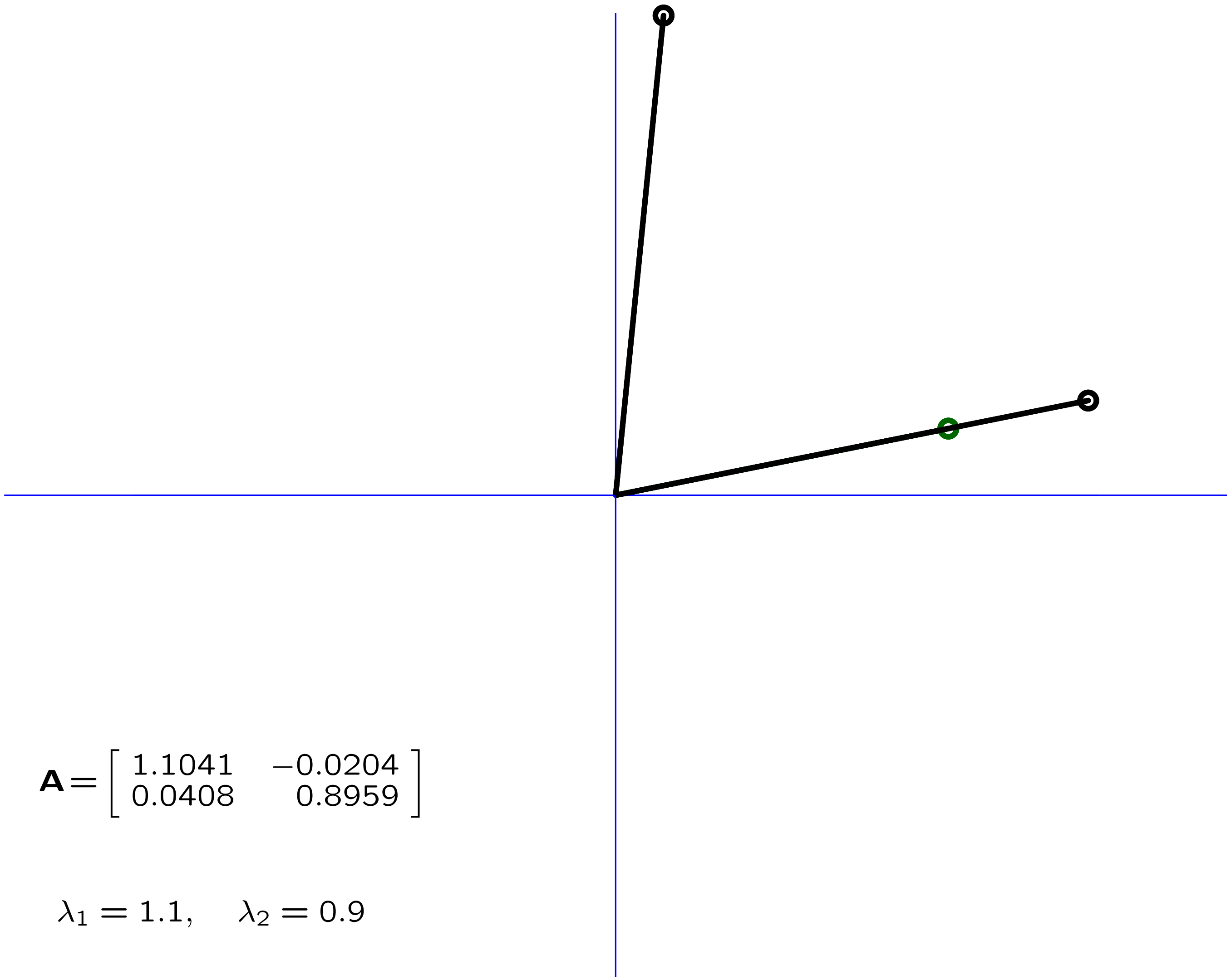
# Iteratie in beeld



$$\mathbf{A} = \begin{bmatrix} 1.1041 & -0.0204 \\ 0.0408 & 0.8959 \end{bmatrix}$$

$$\lambda_1 = 1.1, \quad \lambda_2 = 0.9$$

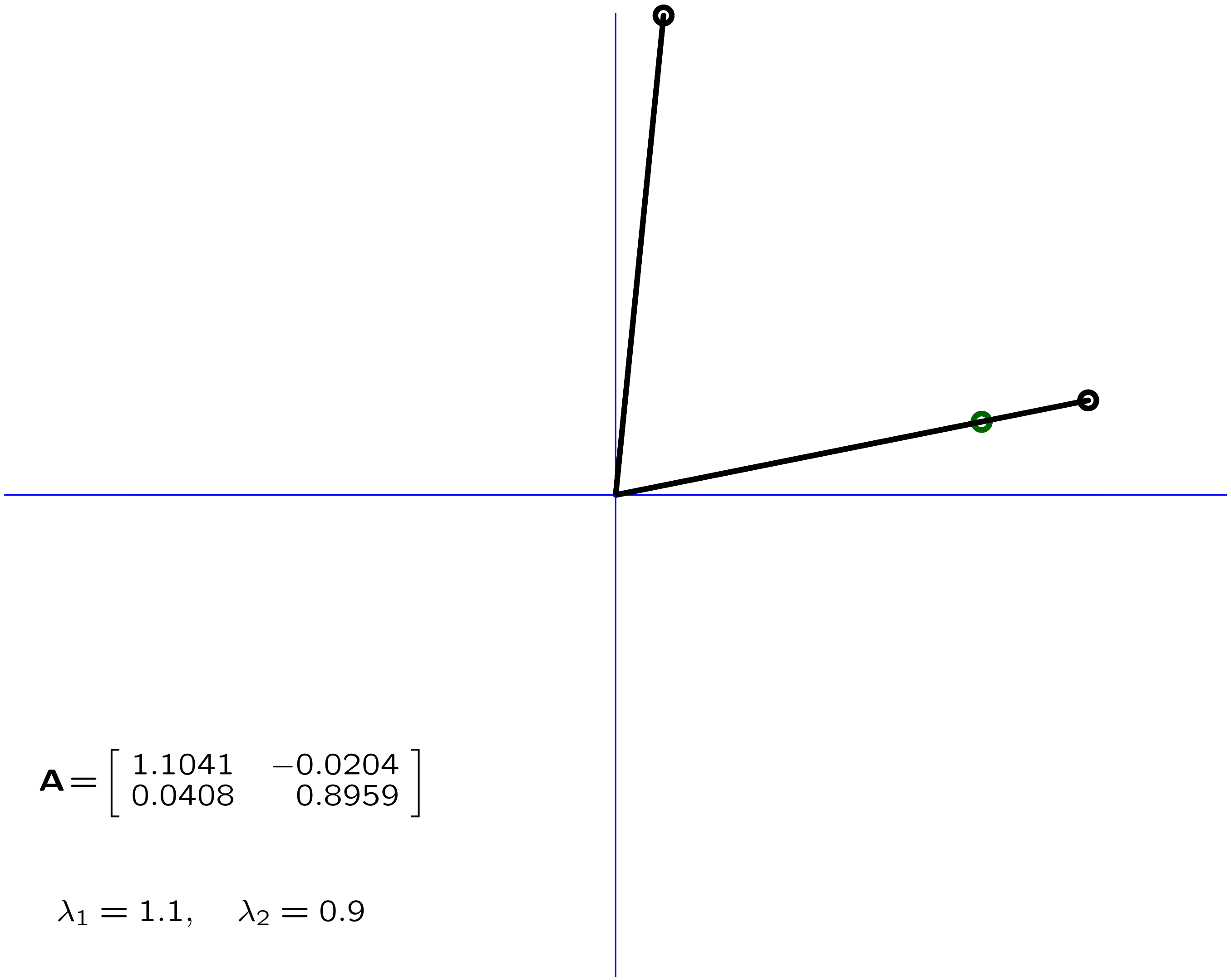
# Iteratie in beeld



$$\mathbf{A} = \begin{bmatrix} 1.1041 & -0.0204 \\ 0.0408 & 0.8959 \end{bmatrix}$$

$$\lambda_1 = 1.1, \quad \lambda_2 = 0.9$$

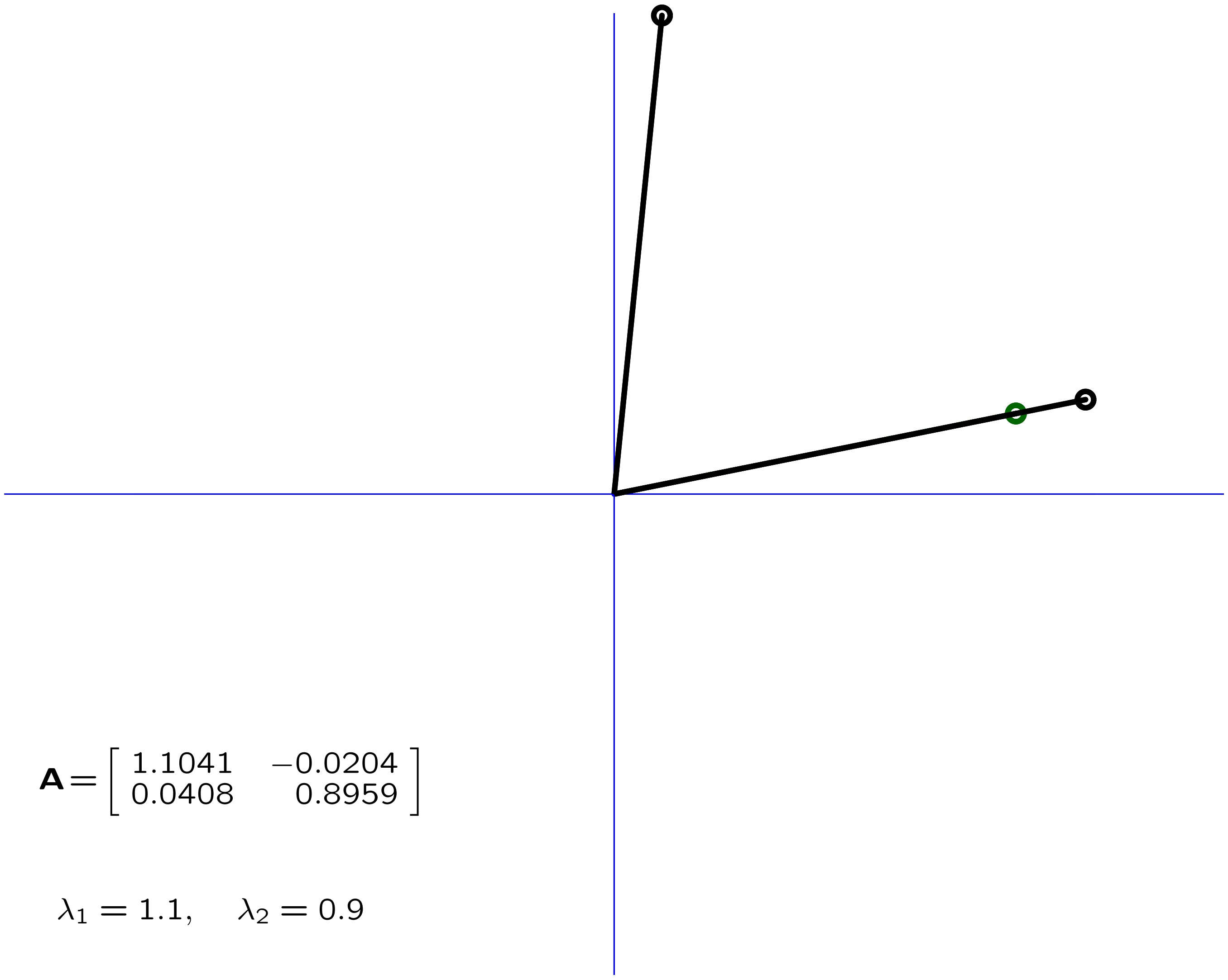
# Iteratie in beeld



$$\mathbf{A} = \begin{bmatrix} 1.1041 & -0.0204 \\ 0.0408 & 0.8959 \end{bmatrix}$$

$$\lambda_1 = 1.1, \quad \lambda_2 = 0.9$$

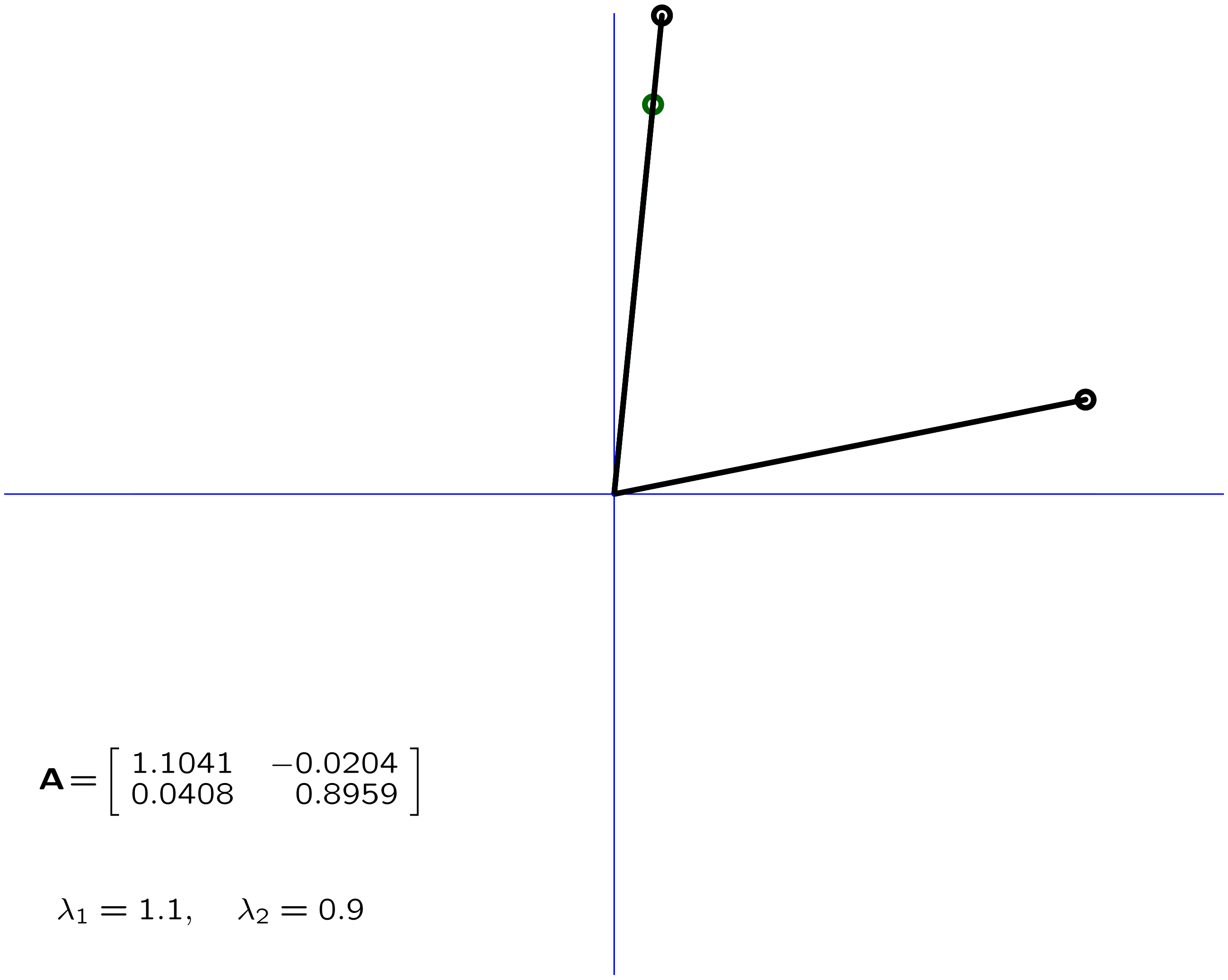
# Iteratie in beeld



$$\mathbf{A} = \begin{bmatrix} 1.1041 & -0.0204 \\ 0.0408 & 0.8959 \end{bmatrix}$$

$$\lambda_1 = 1.1, \quad \lambda_2 = 0.9$$

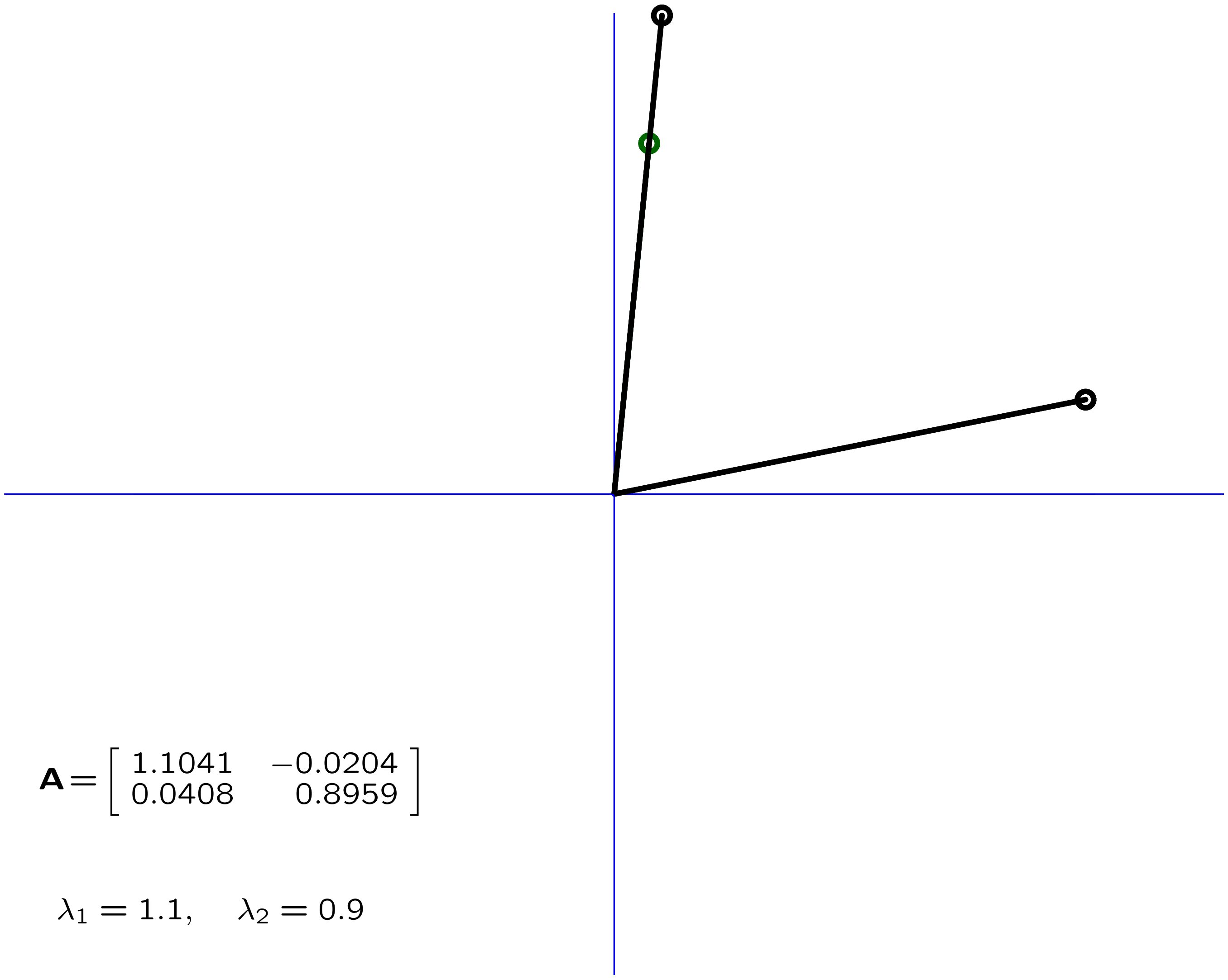
# Iteratie in beeld



$$\mathbf{A} = \begin{bmatrix} 1.1041 & -0.0204 \\ 0.0408 & 0.8959 \end{bmatrix}$$

$$\lambda_1 = 1.1, \quad \lambda_2 = 0.9$$

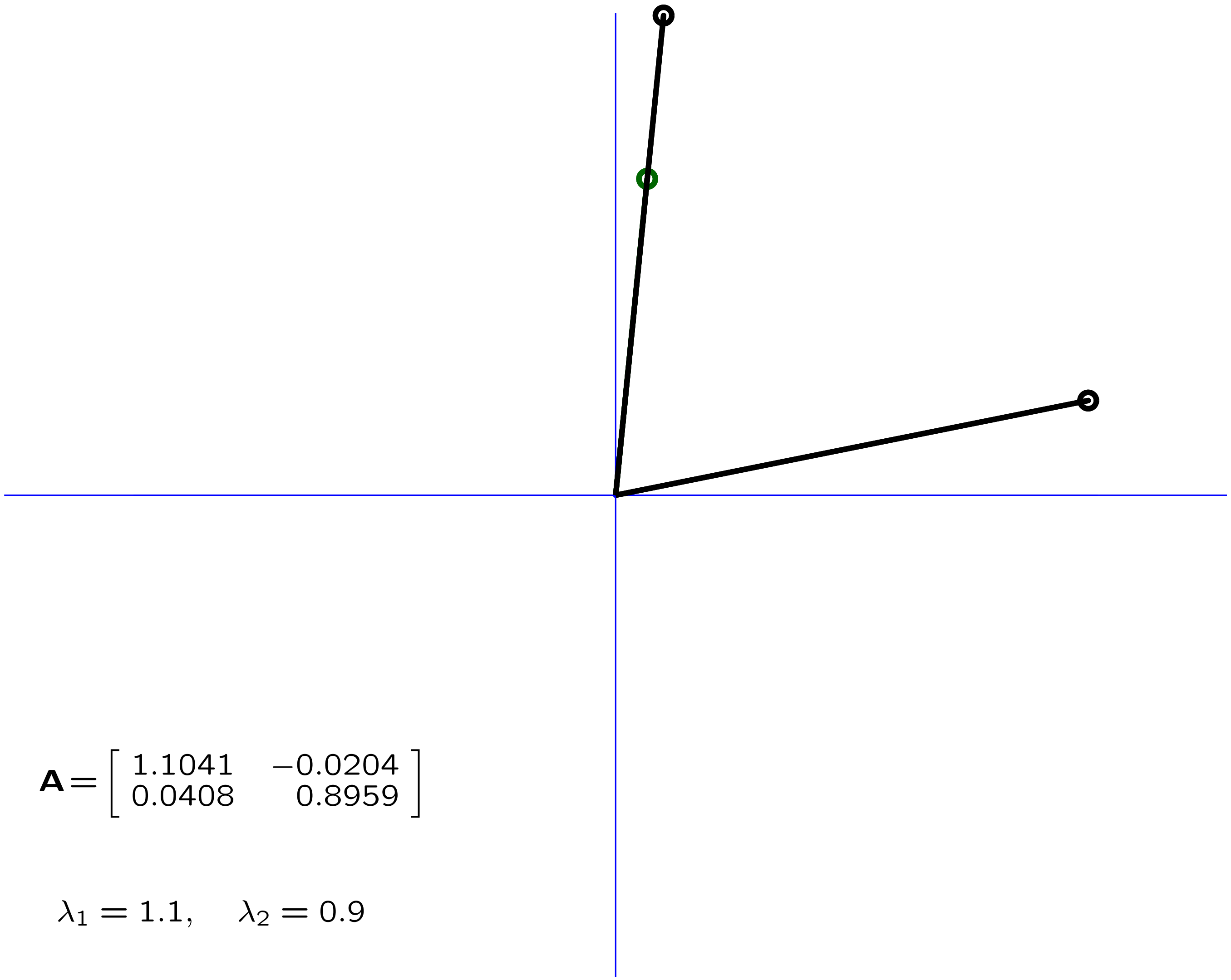
# Iteratie in beeld



$$\mathbf{A} = \begin{bmatrix} 1.1041 & -0.0204 \\ 0.0408 & 0.8959 \end{bmatrix}$$

$$\lambda_1 = 1.1, \quad \lambda_2 = 0.9$$

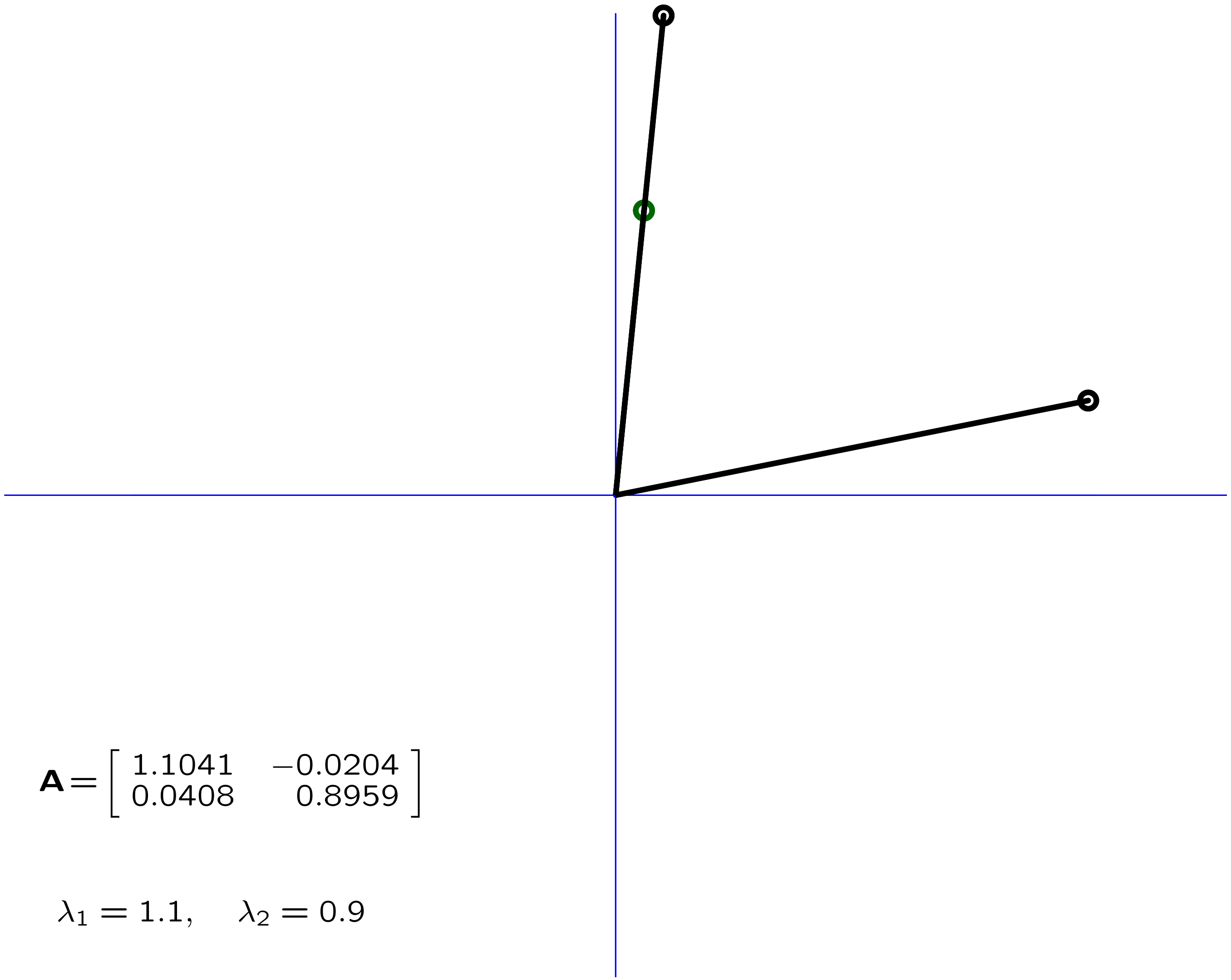
# Iteratie in beeld



$$\mathbf{A} = \begin{bmatrix} 1.1041 & -0.0204 \\ 0.0408 & 0.8959 \end{bmatrix}$$

$$\lambda_1 = 1.1, \quad \lambda_2 = 0.9$$

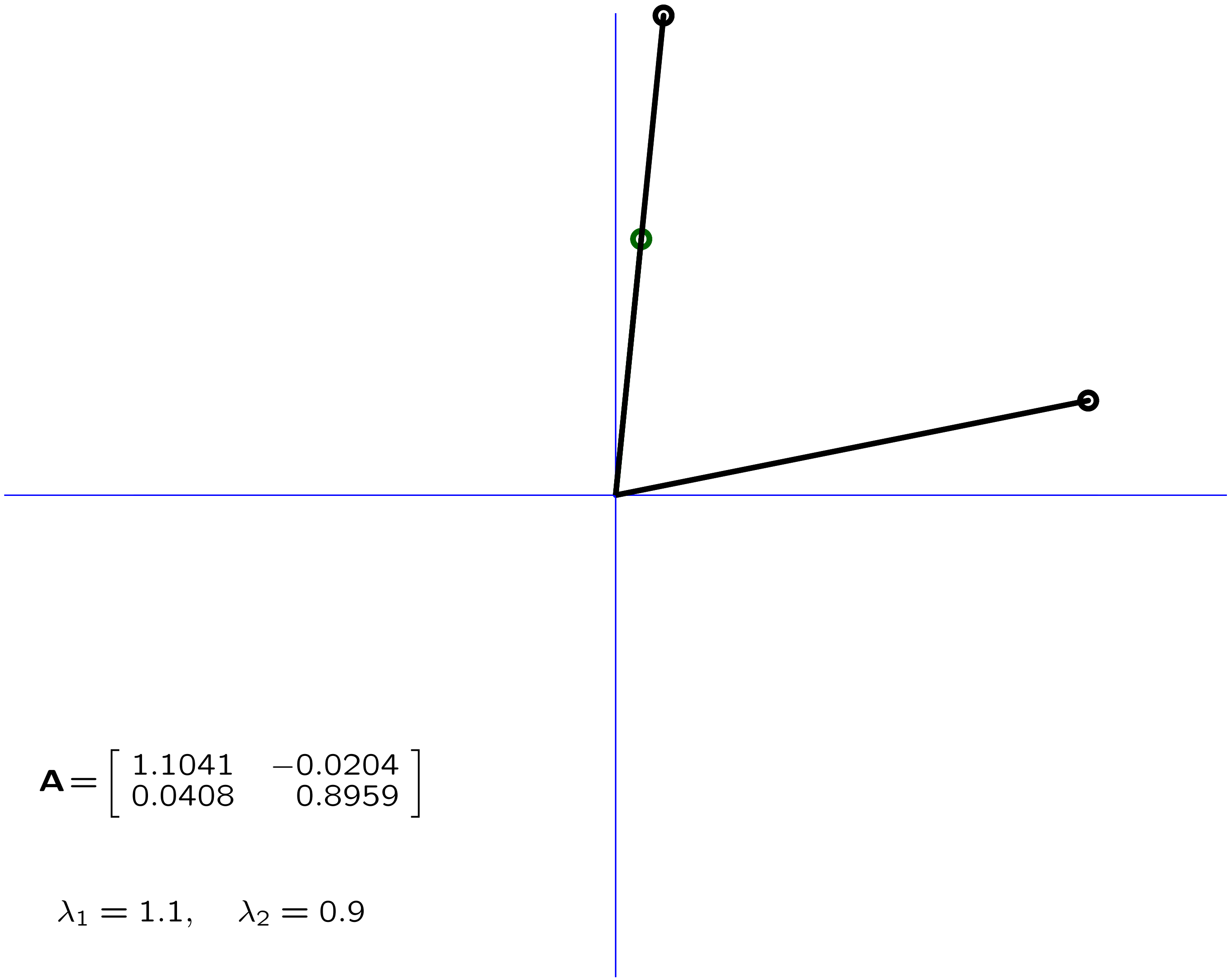
# Iteratie in beeld



$$\mathbf{A} = \begin{bmatrix} 1.1041 & -0.0204 \\ 0.0408 & 0.8959 \end{bmatrix}$$

$$\lambda_1 = 1.1, \quad \lambda_2 = 0.9$$

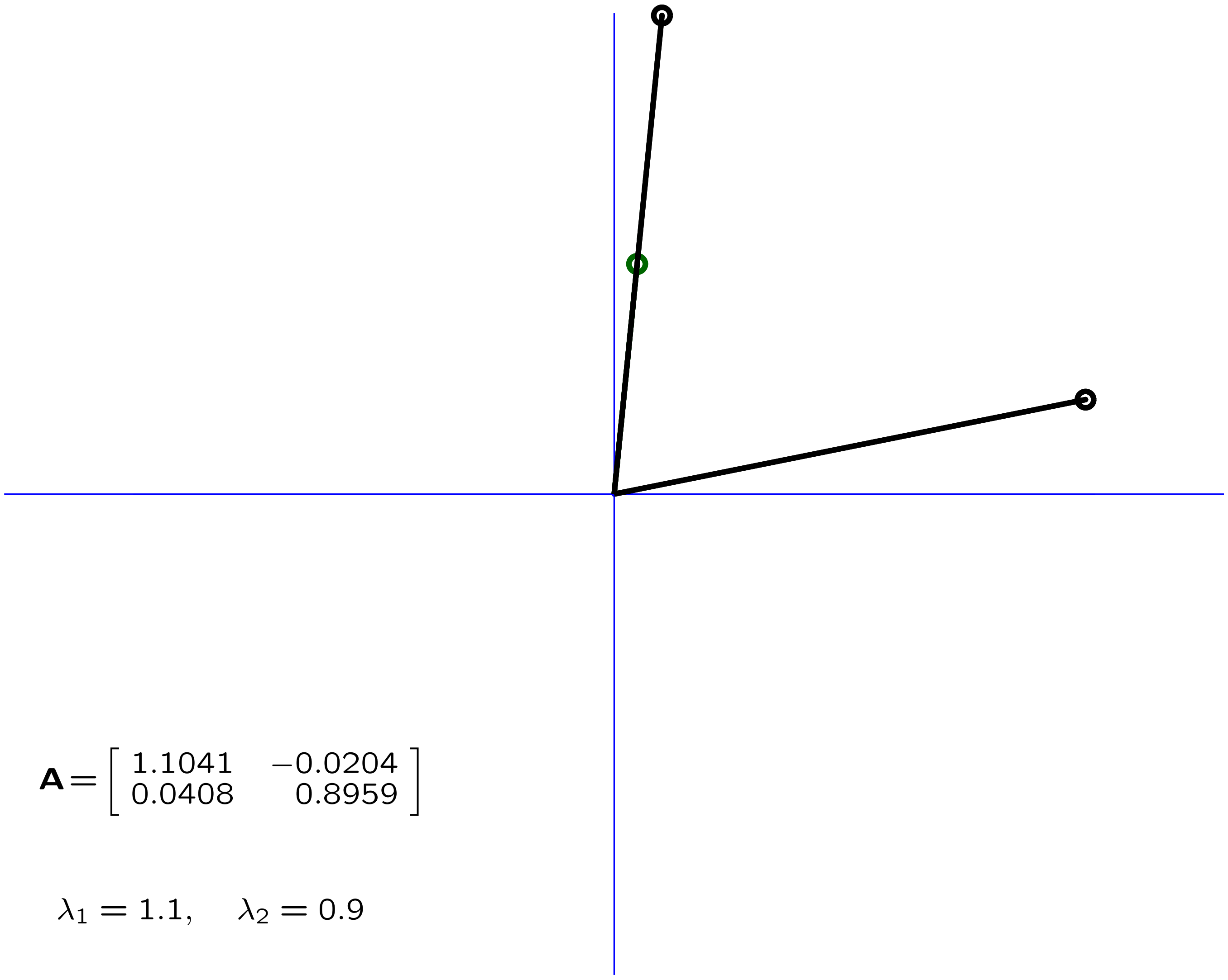
# Iteratie in beeld



$$\mathbf{A} = \begin{bmatrix} 1.1041 & -0.0204 \\ 0.0408 & 0.8959 \end{bmatrix}$$

$$\lambda_1 = 1.1, \quad \lambda_2 = 0.9$$

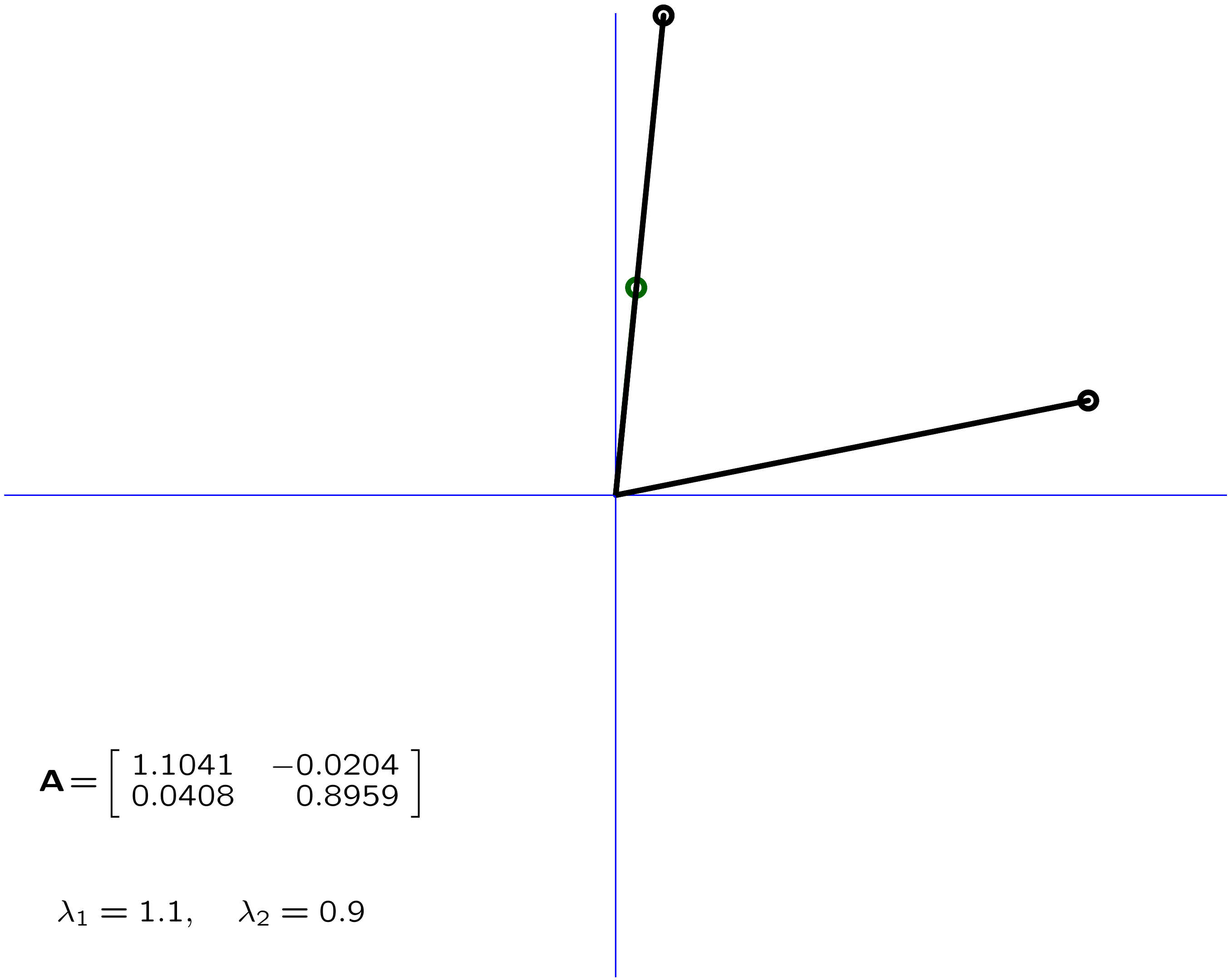
# Iteratie in beeld



$$\mathbf{A} = \begin{bmatrix} 1.1041 & -0.0204 \\ 0.0408 & 0.8959 \end{bmatrix}$$

$$\lambda_1 = 1.1, \quad \lambda_2 = 0.9$$

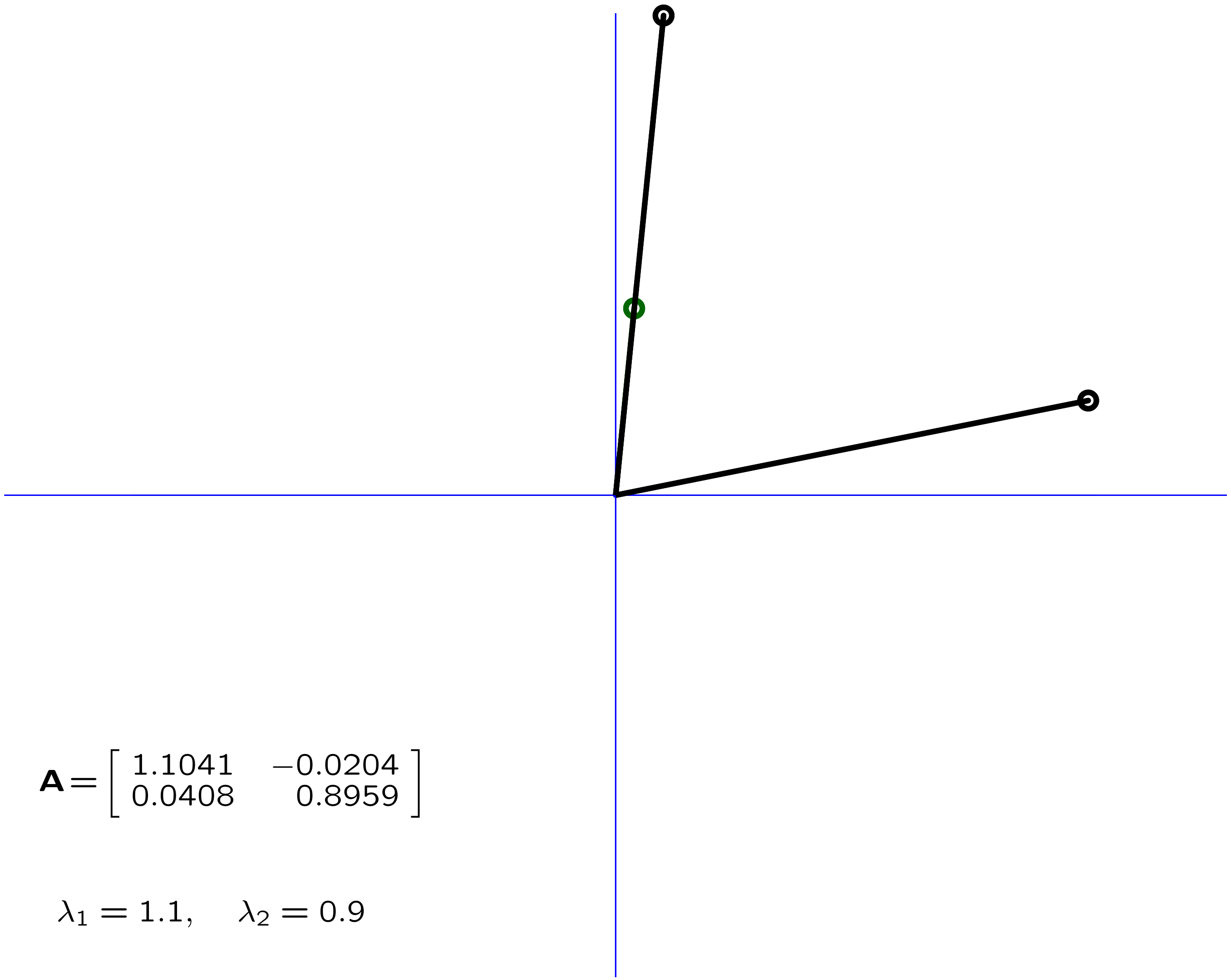
# Iteratie in beeld



$$\mathbf{A} = \begin{bmatrix} 1.1041 & -0.0204 \\ 0.0408 & 0.8959 \end{bmatrix}$$

$$\lambda_1 = 1.1, \quad \lambda_2 = 0.9$$

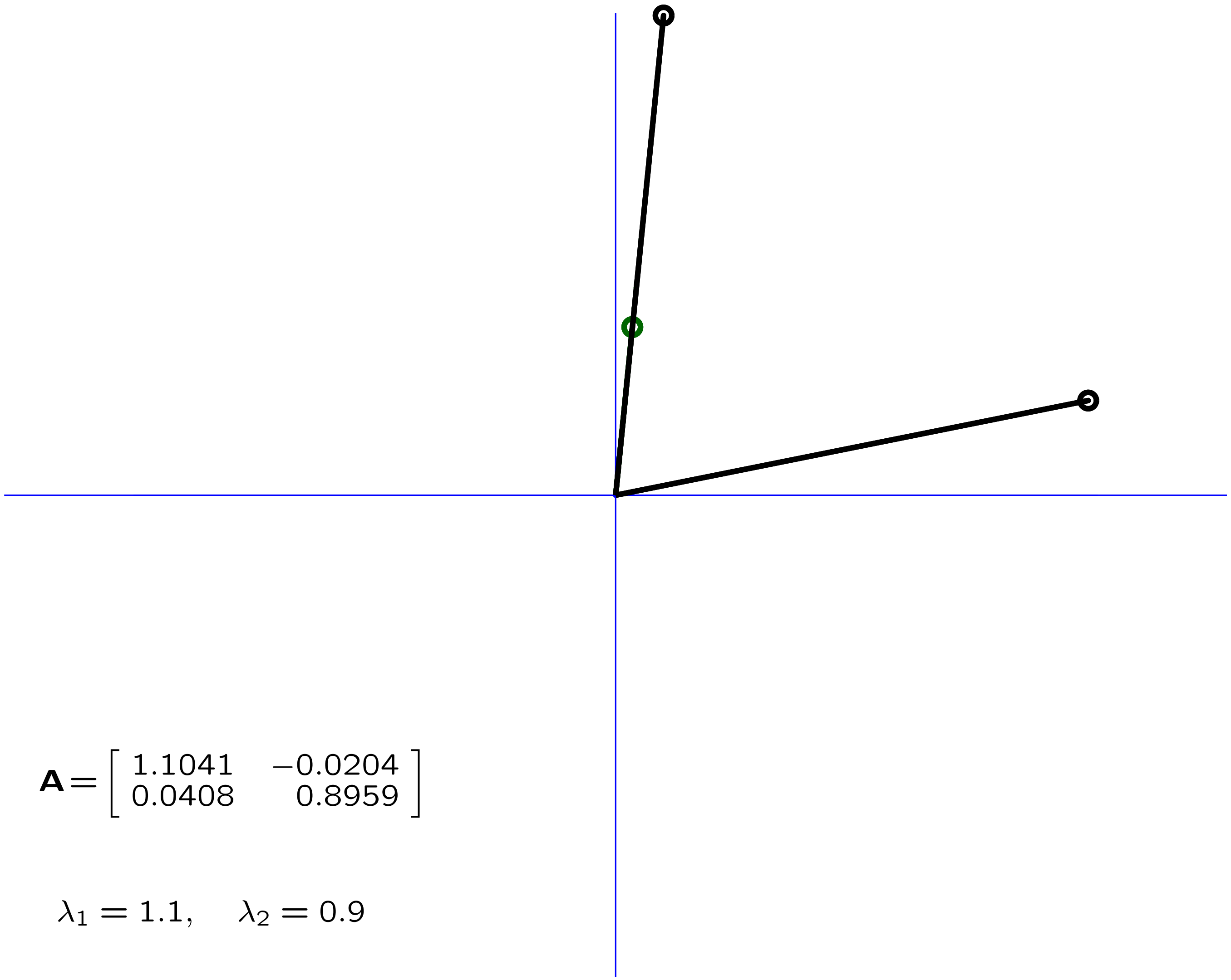
# Iteratie in beeld



$$\mathbf{A} = \begin{bmatrix} 1.1041 & -0.0204 \\ 0.0408 & 0.8959 \end{bmatrix}$$

$$\lambda_1 = 1.1, \quad \lambda_2 = 0.9$$

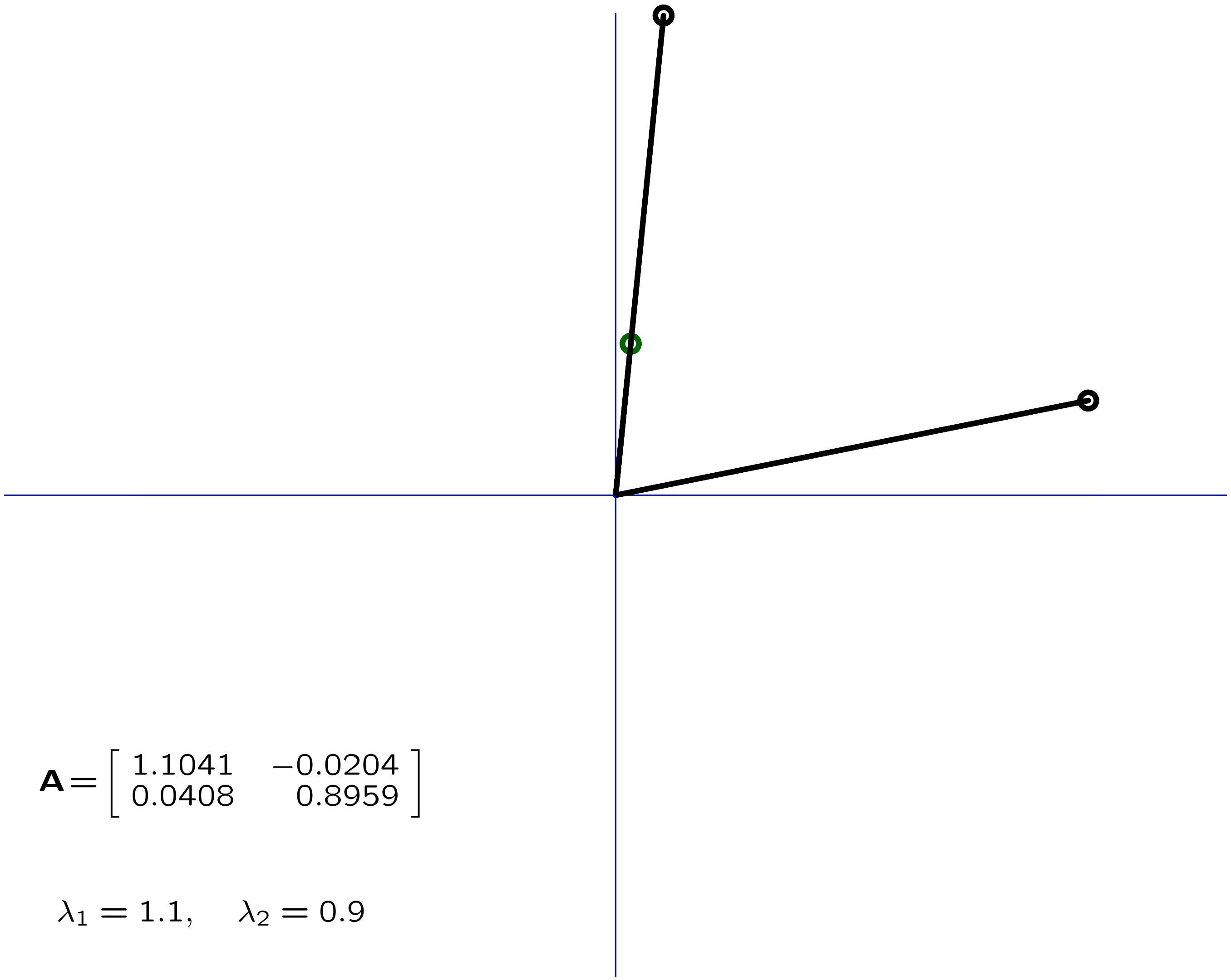
# Iteratie in beeld



$$\mathbf{A} = \begin{bmatrix} 1.1041 & -0.0204 \\ 0.0408 & 0.8959 \end{bmatrix}$$

$$\lambda_1 = 1.1, \quad \lambda_2 = 0.9$$

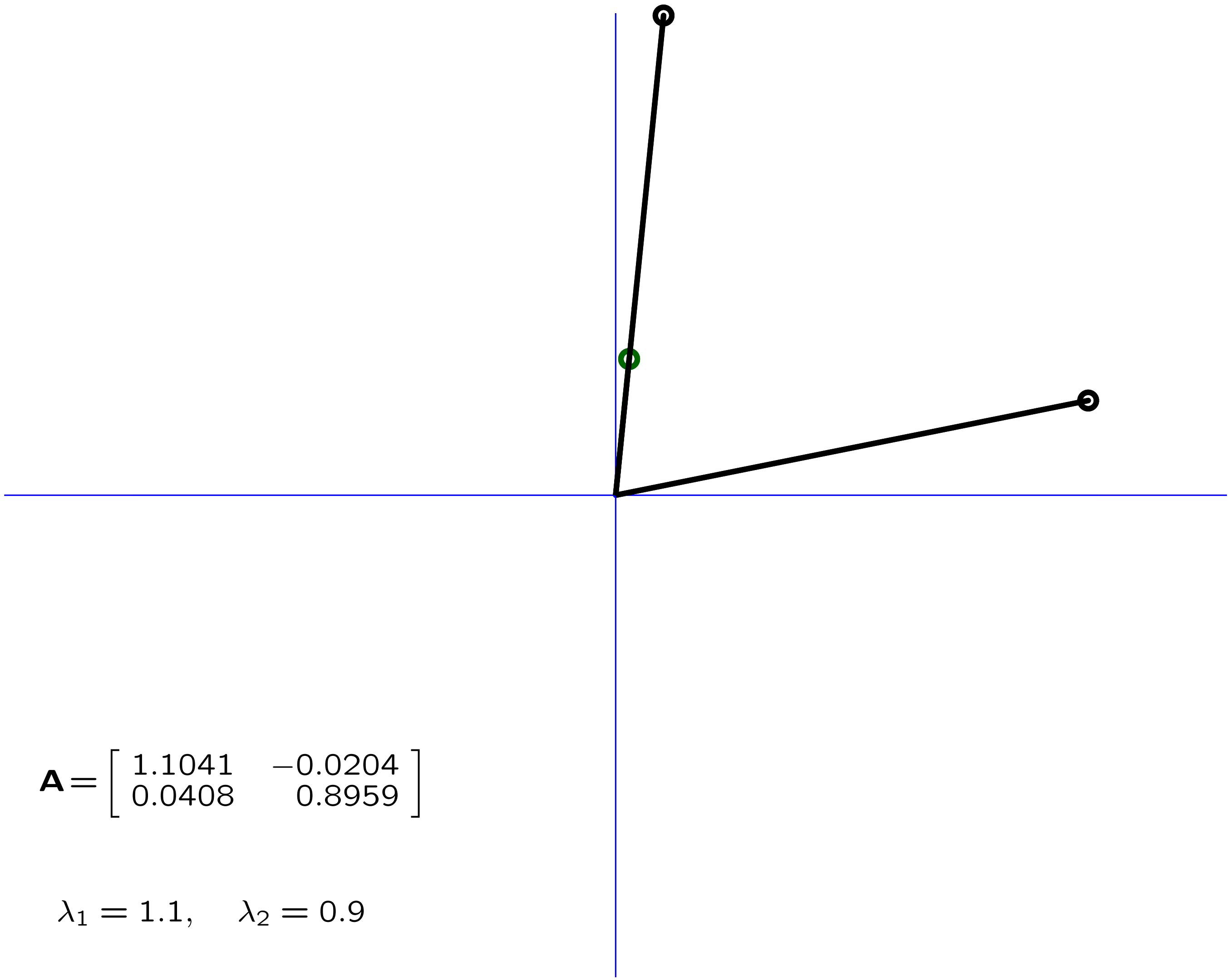
# Iteratie in beeld



$$\mathbf{A} = \begin{bmatrix} 1.1041 & -0.0204 \\ 0.0408 & 0.8959 \end{bmatrix}$$

$$\lambda_1 = 1.1, \quad \lambda_2 = 0.9$$

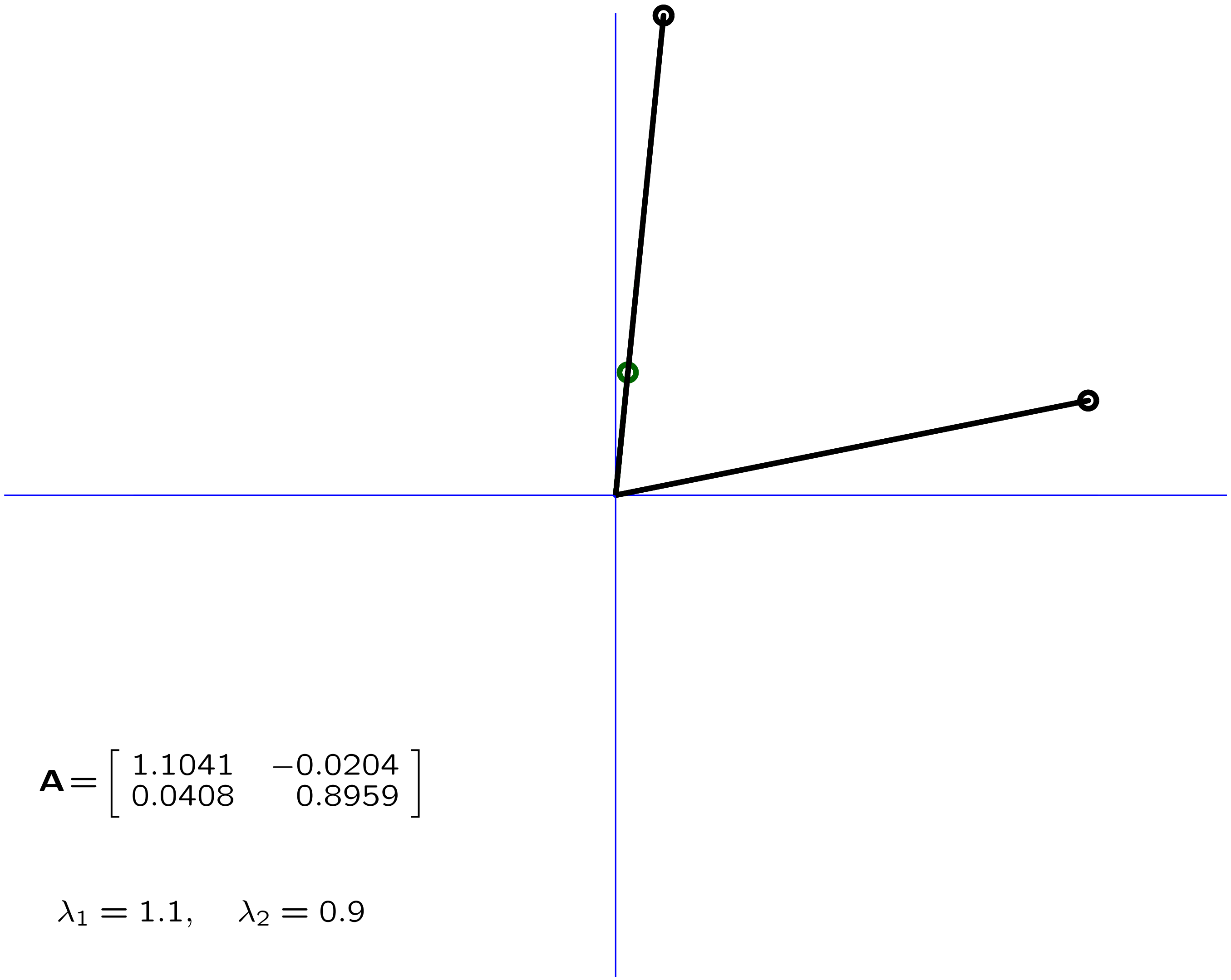
# Iteratie in beeld



$$\mathbf{A} = \begin{bmatrix} 1.1041 & -0.0204 \\ 0.0408 & 0.8959 \end{bmatrix}$$

$$\lambda_1 = 1.1, \quad \lambda_2 = 0.9$$

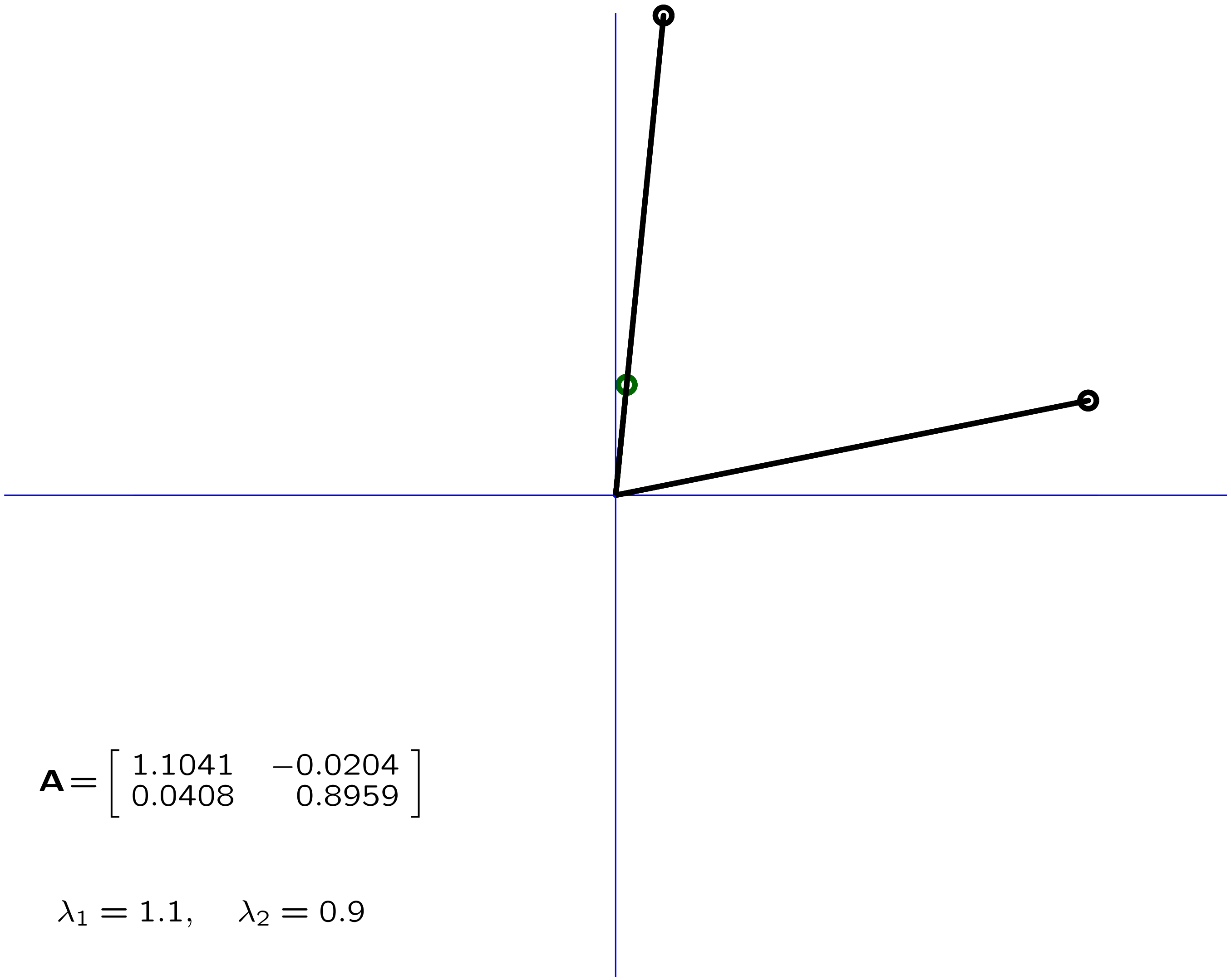
# Iteratie in beeld



$$\mathbf{A} = \begin{bmatrix} 1.1041 & -0.0204 \\ 0.0408 & 0.8959 \end{bmatrix}$$

$$\lambda_1 = 1.1, \quad \lambda_2 = 0.9$$

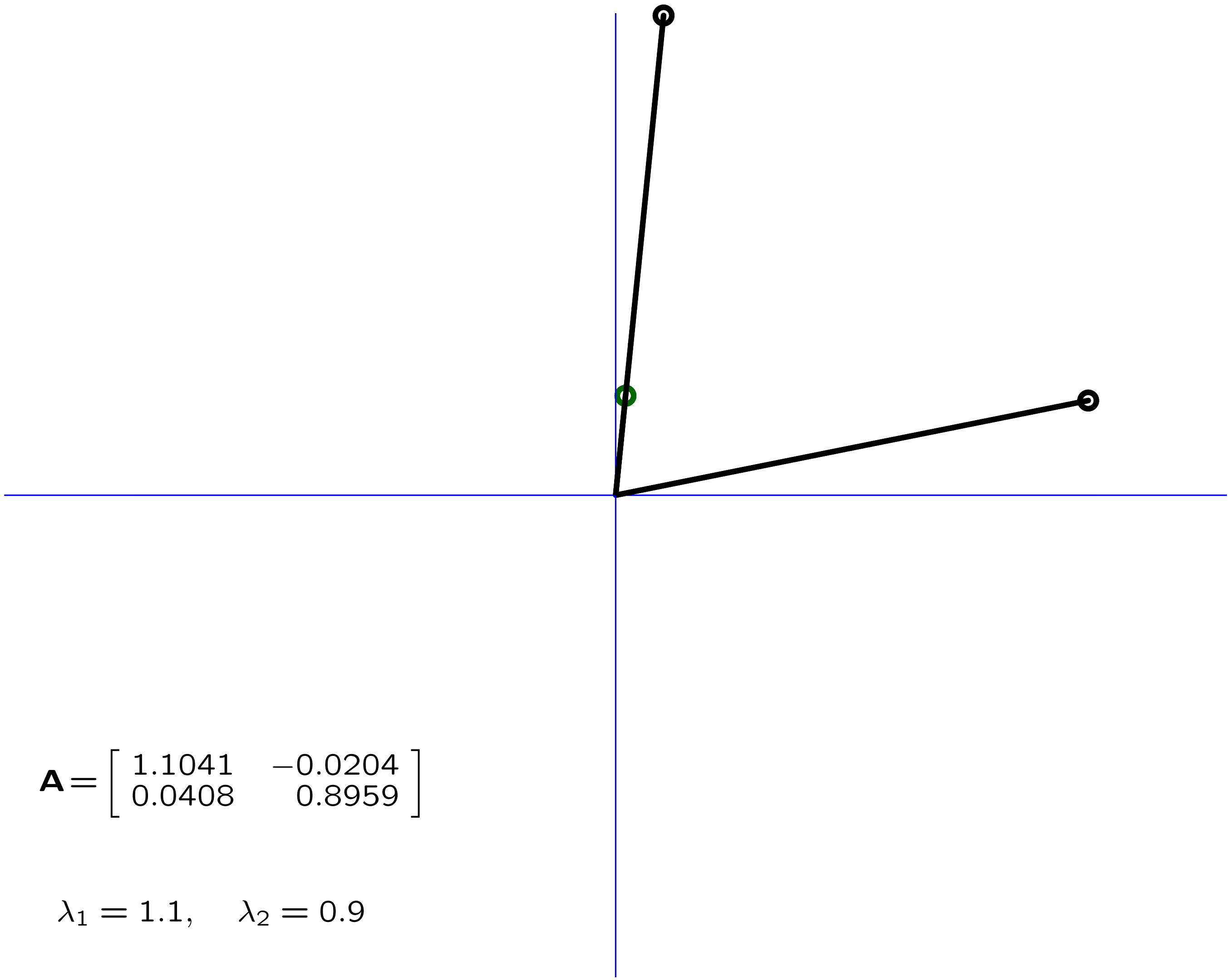
# Iteratie in beeld



$$\mathbf{A} = \begin{bmatrix} 1.1041 & -0.0204 \\ 0.0408 & 0.8959 \end{bmatrix}$$

$$\lambda_1 = 1.1, \quad \lambda_2 = 0.9$$

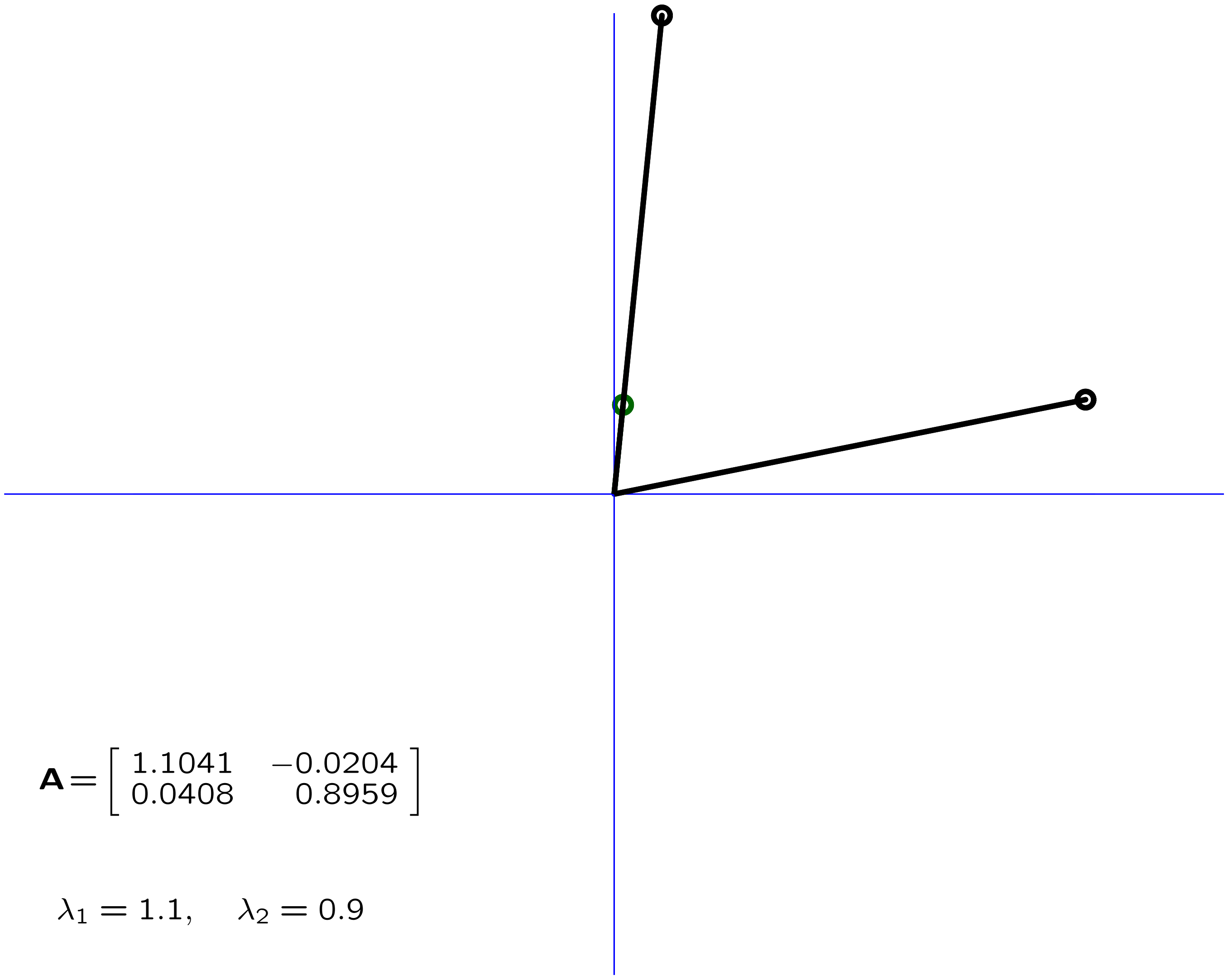
# Iteratie in beeld



$$\mathbf{A} = \begin{bmatrix} 1.1041 & -0.0204 \\ 0.0408 & 0.8959 \end{bmatrix}$$

$$\lambda_1 = 1.1, \quad \lambda_2 = 0.9$$

# Iteratie in beeld

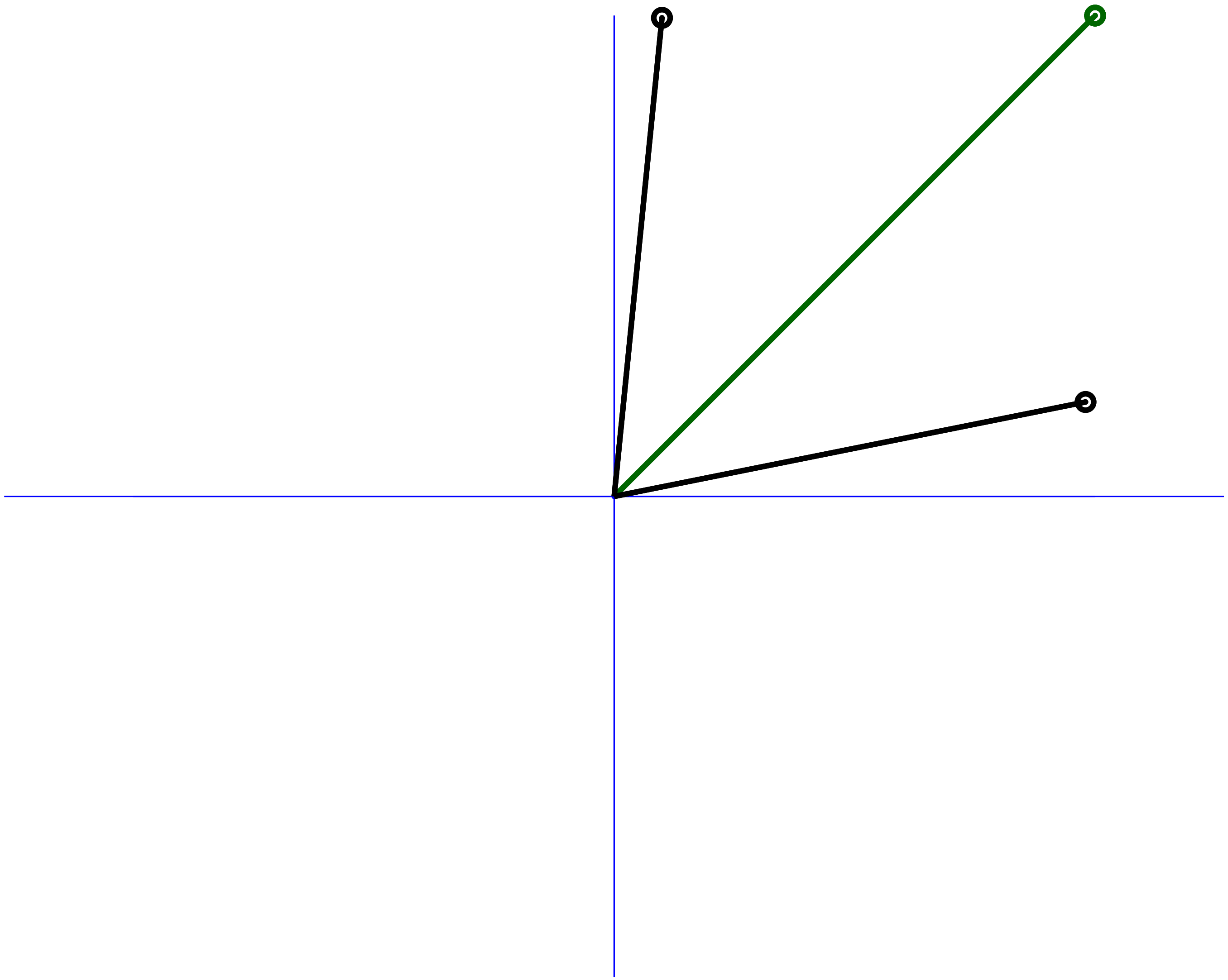


$$\mathbf{A} = \begin{bmatrix} 1.1041 & -0.0204 \\ 0.0408 & 0.8959 \end{bmatrix}$$

$$\lambda_1 = 1.1, \quad \lambda_2 = 0.9$$

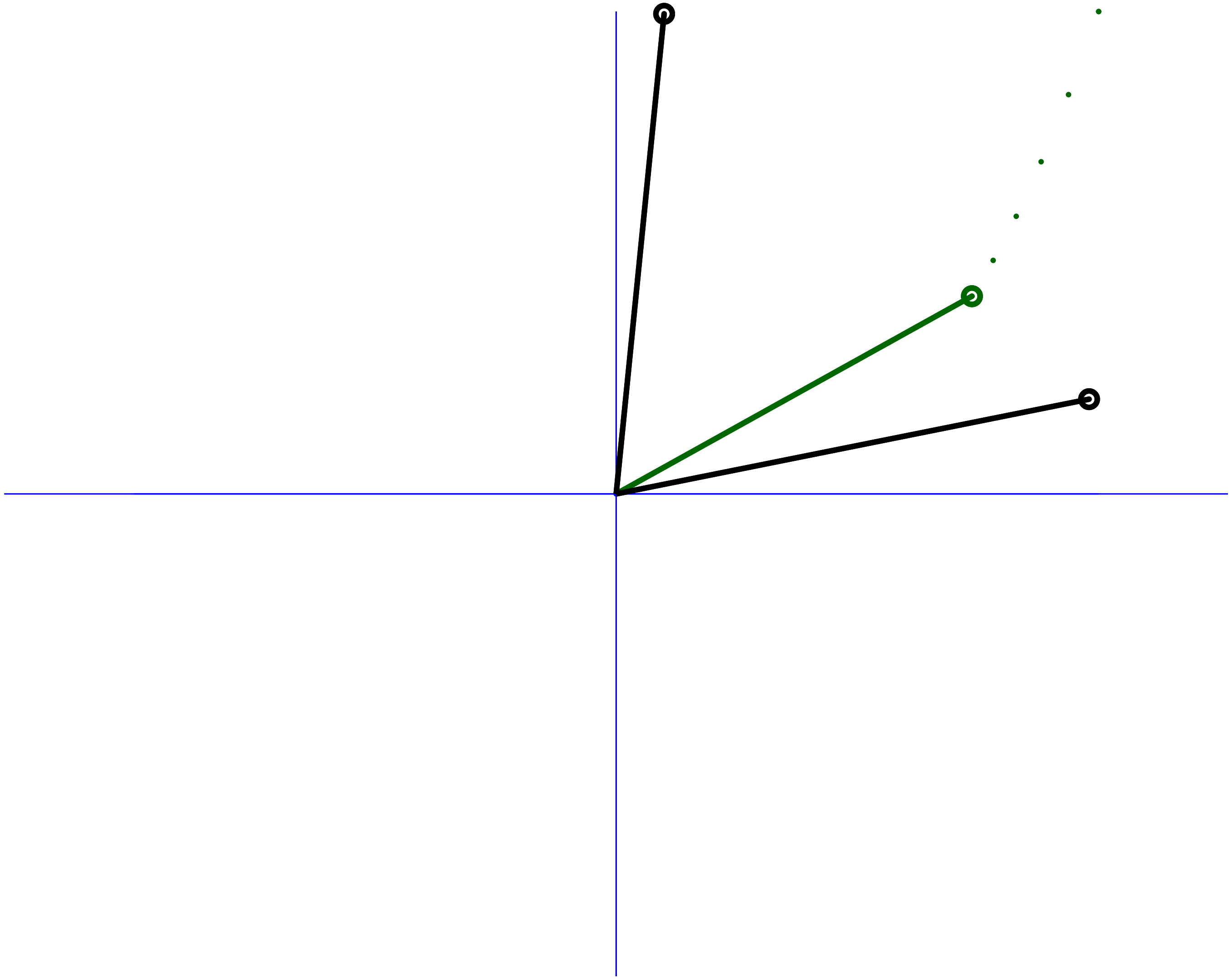
# Iteratie in beeld

$x_0$



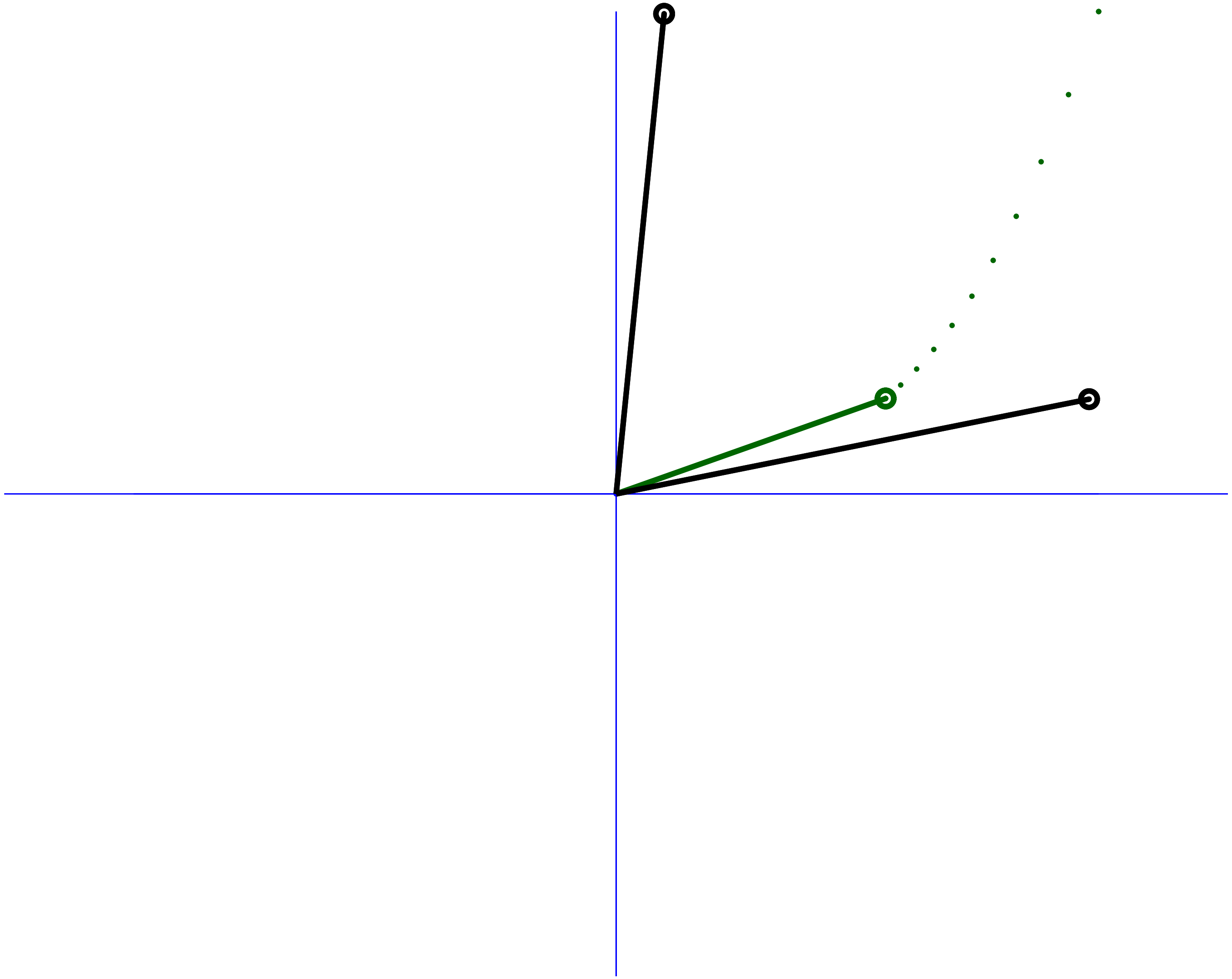
# Iteratie in beeld

$x_5$

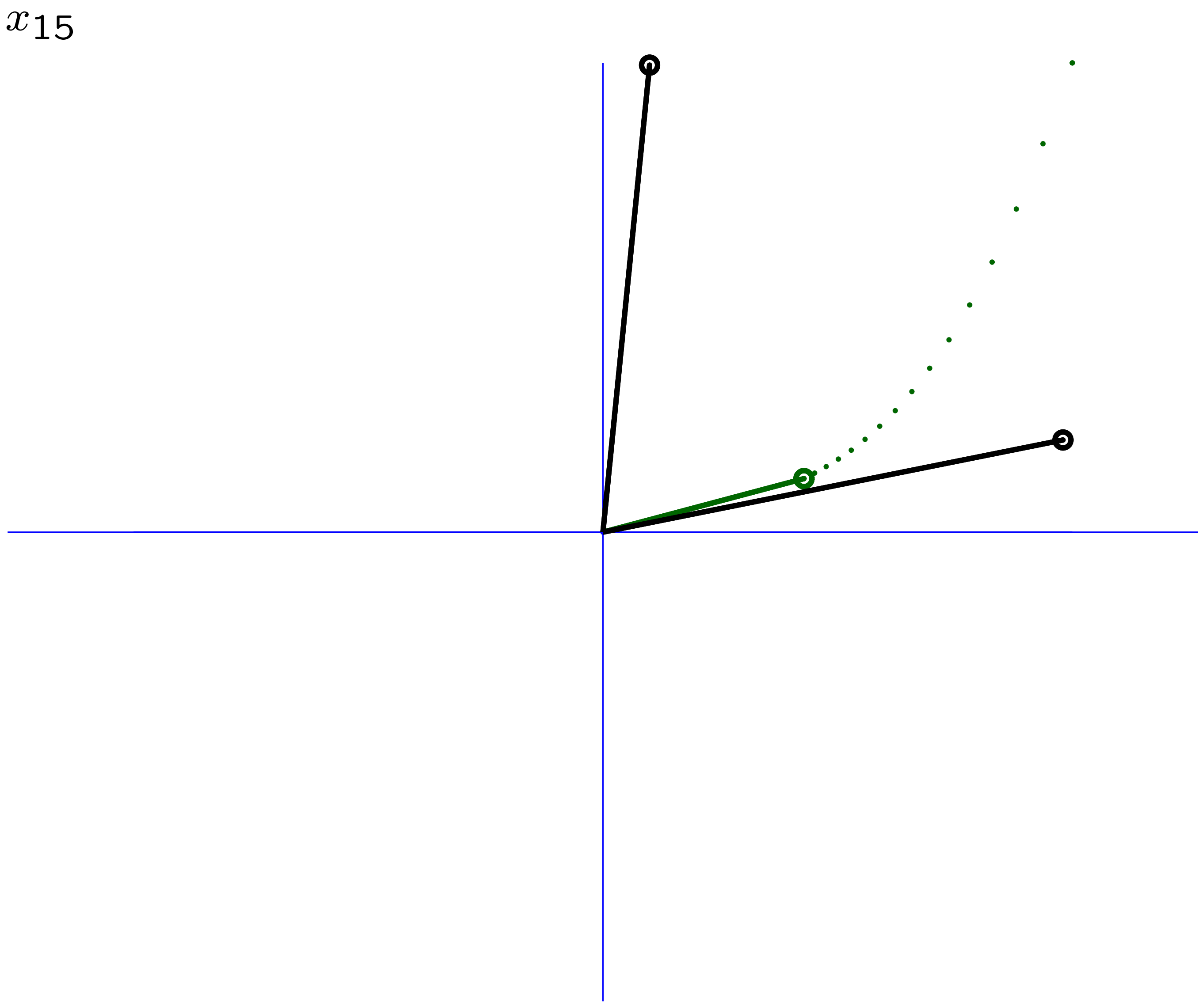


# Iteratie in beeld

$x_{10}$

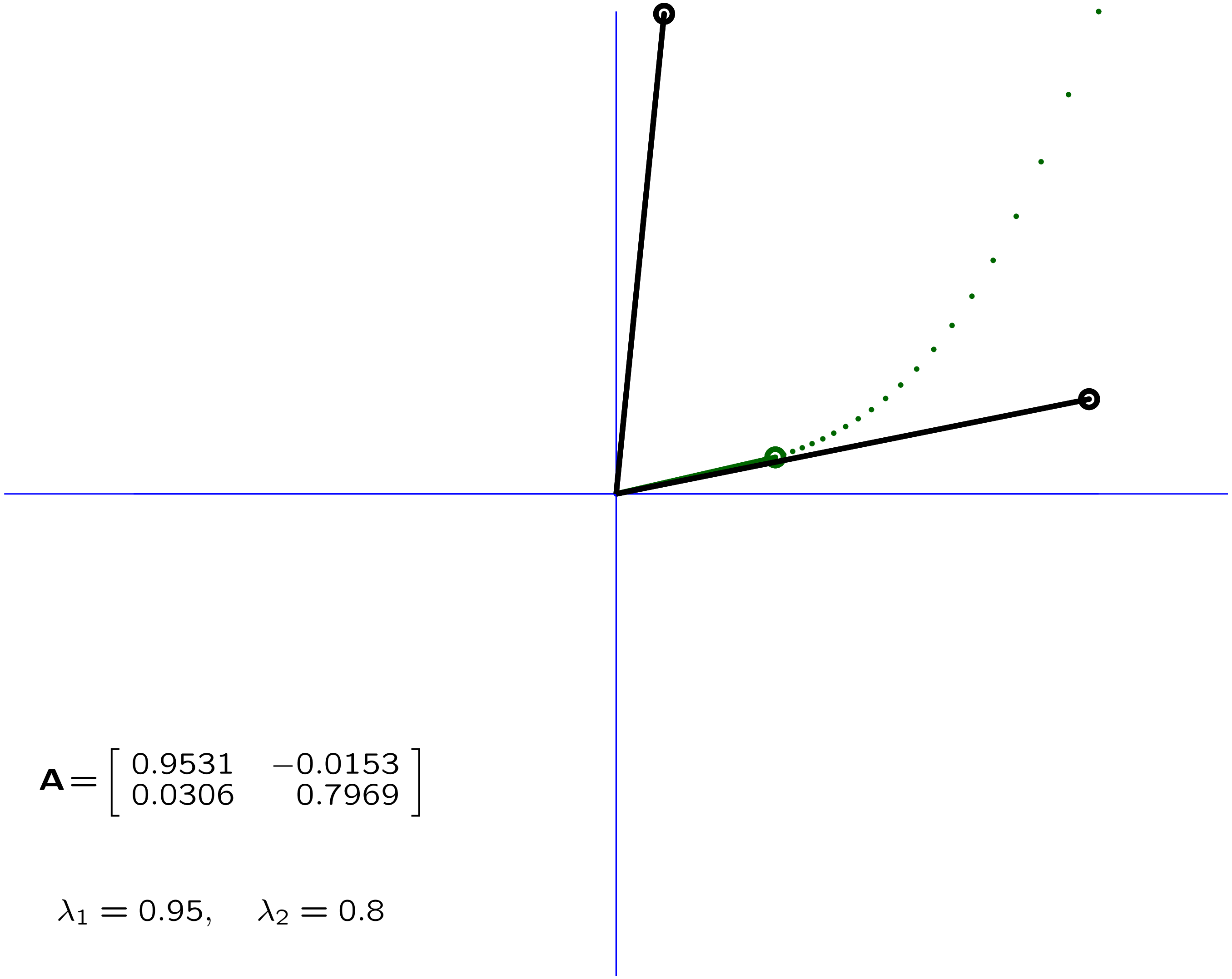


# Iteratie in beeld



# Iteratie in beeld

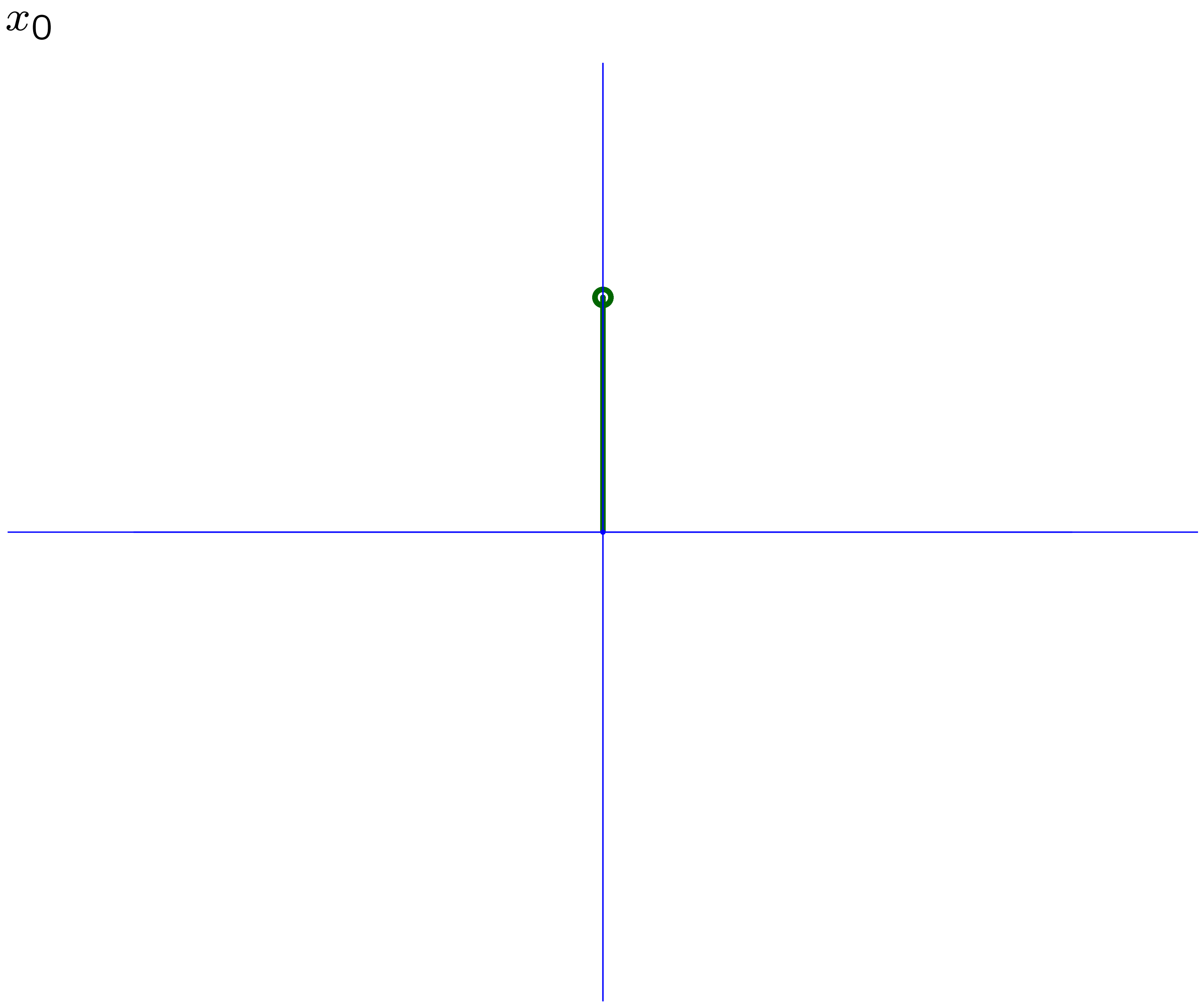
$x_{20}$



$$\mathbf{A} = \begin{bmatrix} 0.9531 & -0.0153 \\ 0.0306 & 0.7969 \end{bmatrix}$$

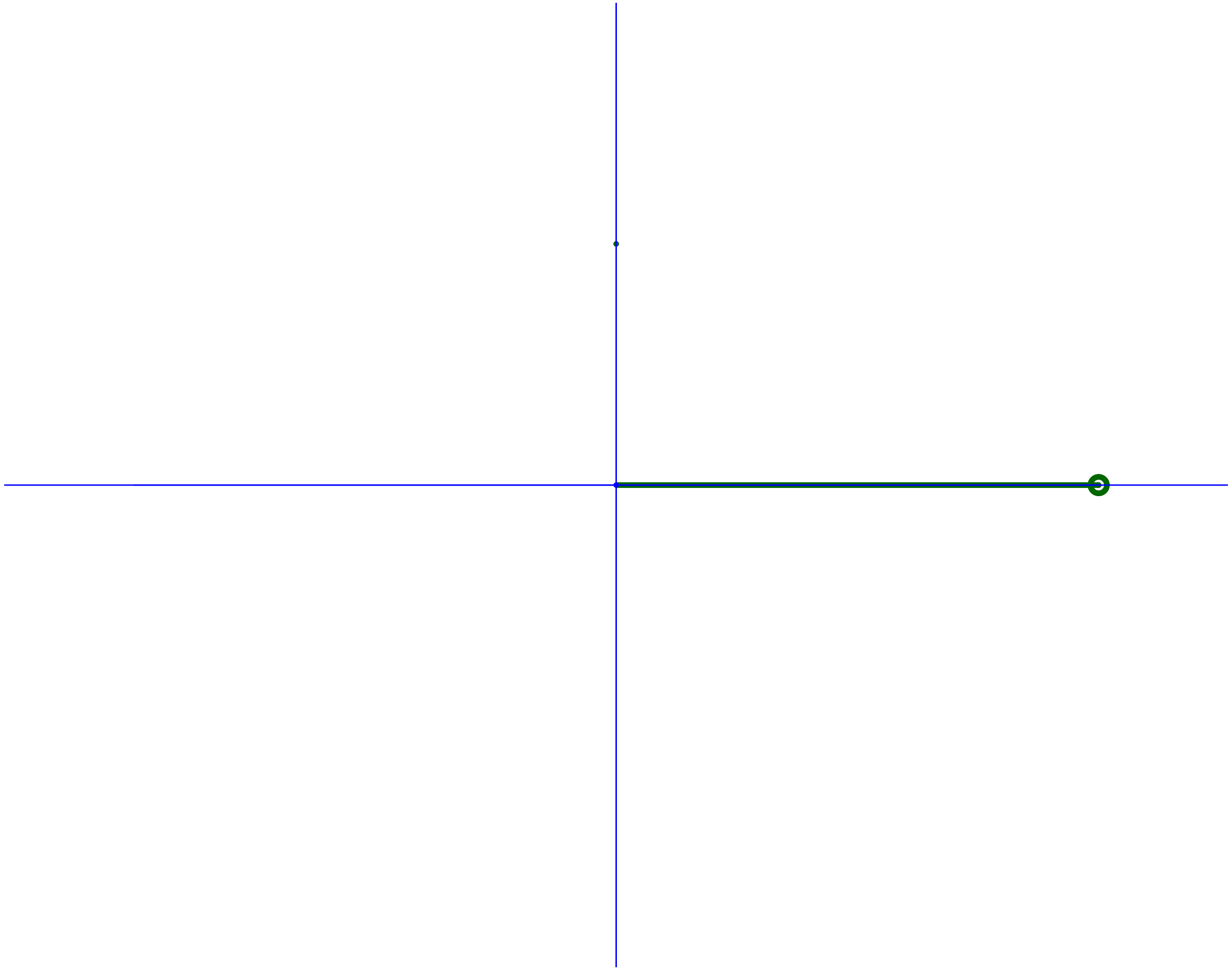
$$\lambda_1 = 0.95, \quad \lambda_2 = 0.8$$

# Iteratie in beeld

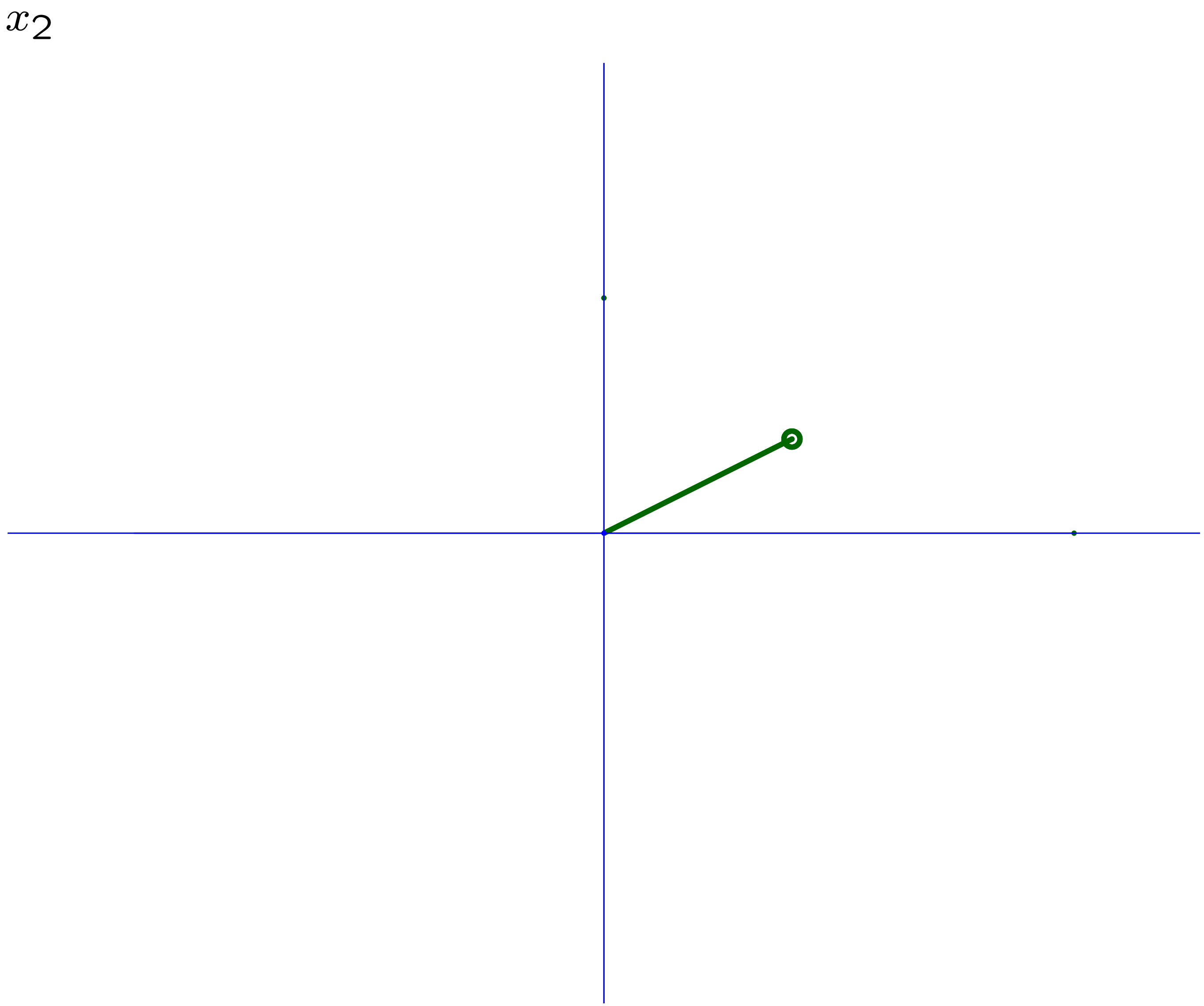


# Iteratie in beeld

$x_1$

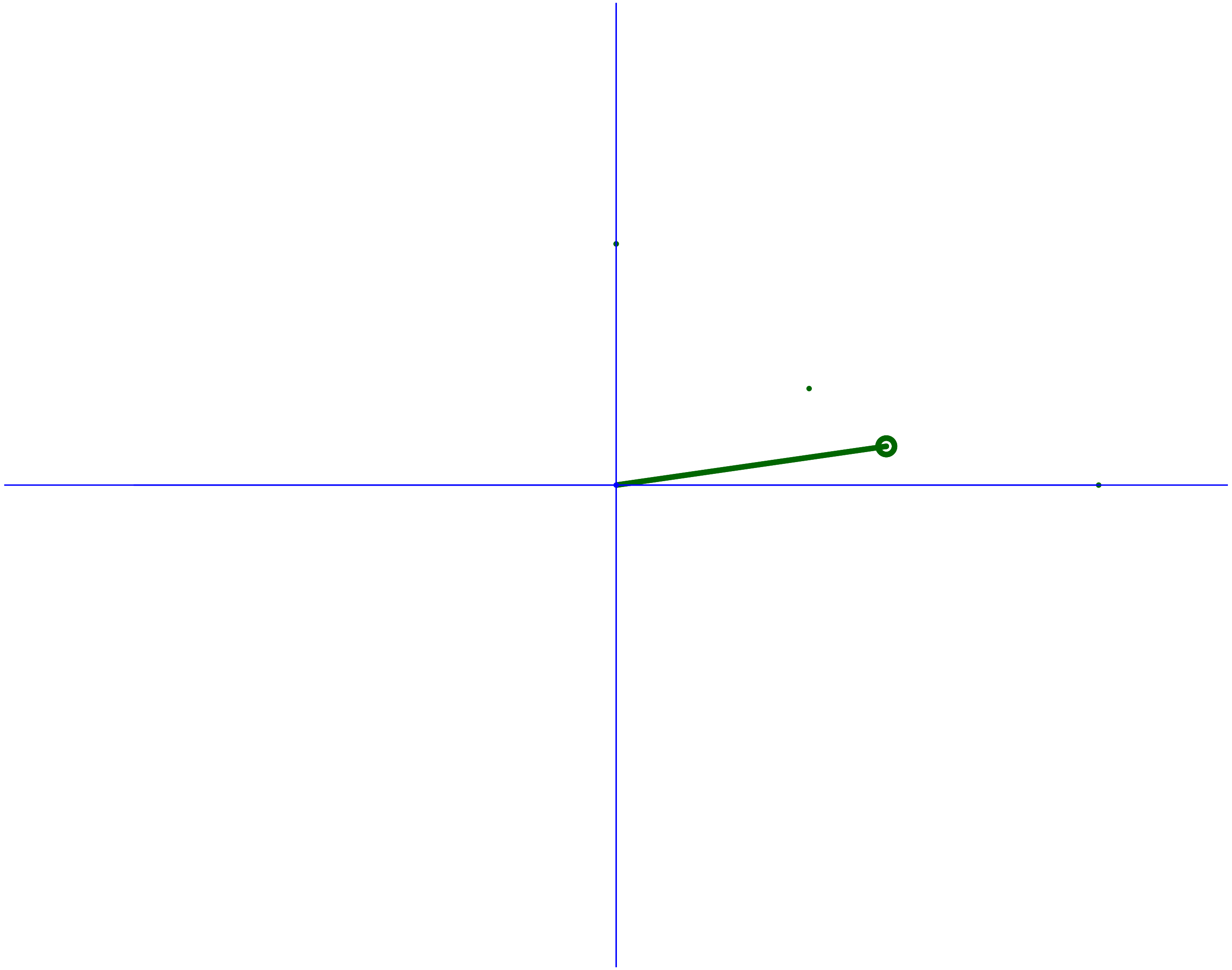


# Iteratie in beeld



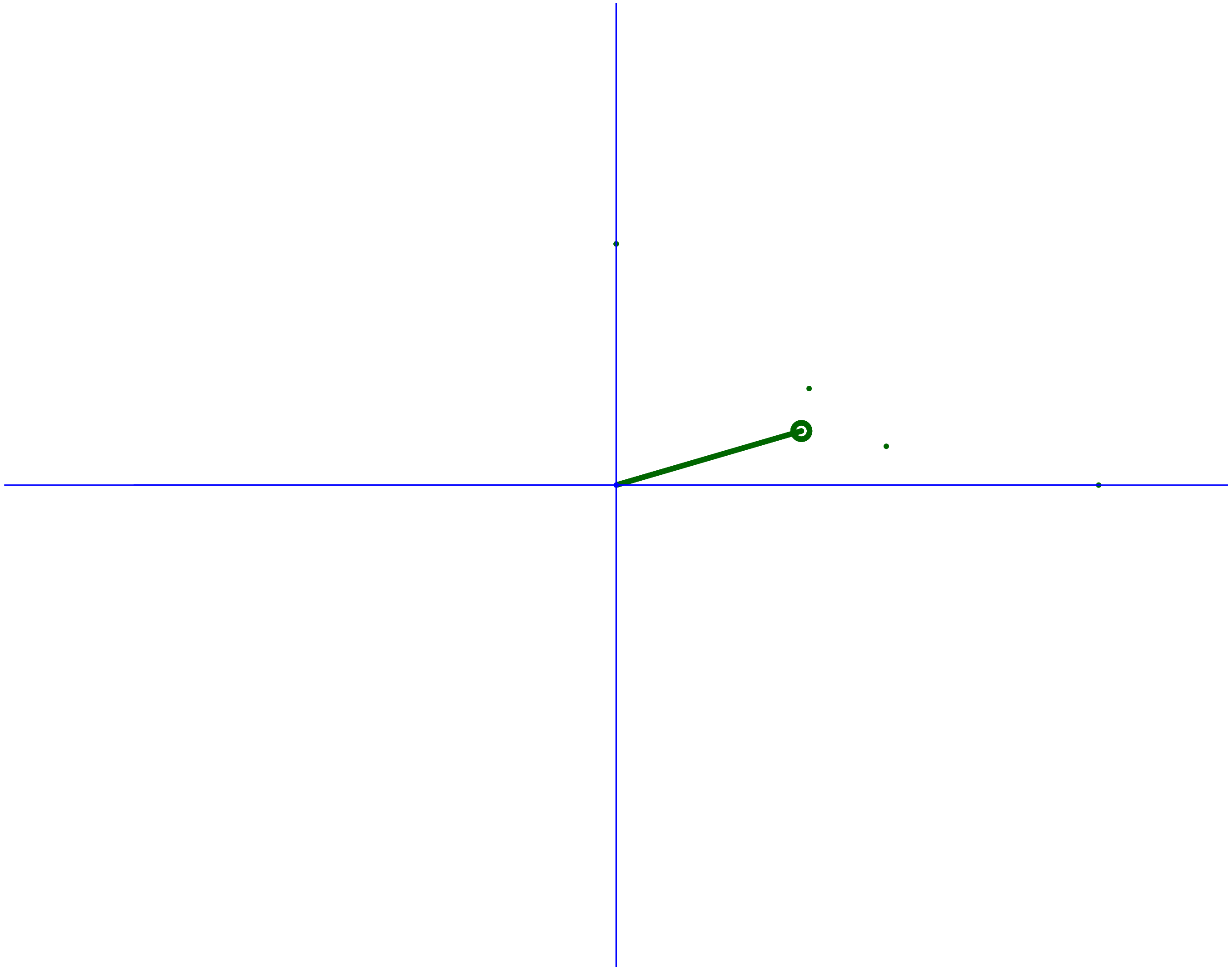
# Iteratie in beeld

$x_3$



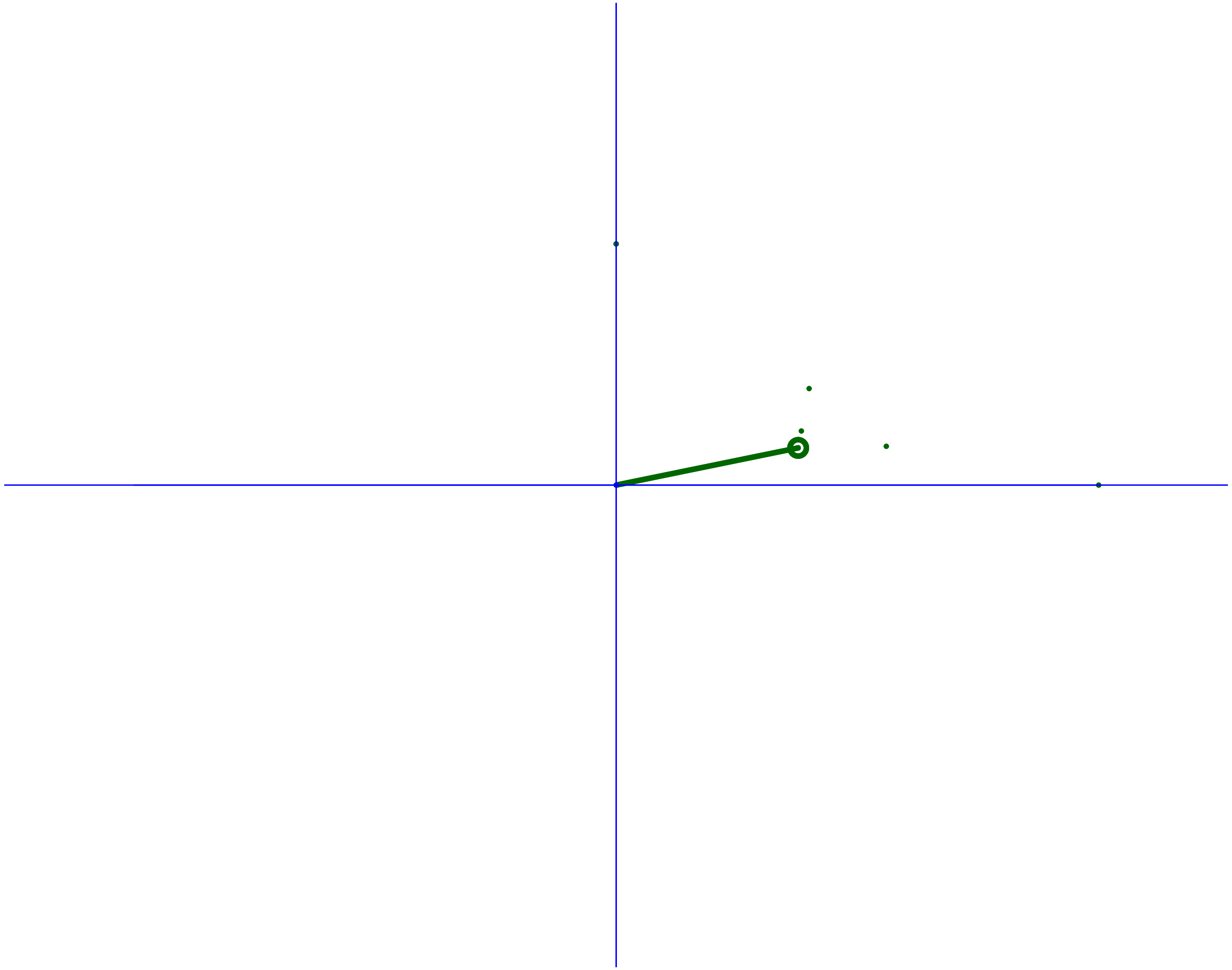
# Iteratie in beeld

$x_4$



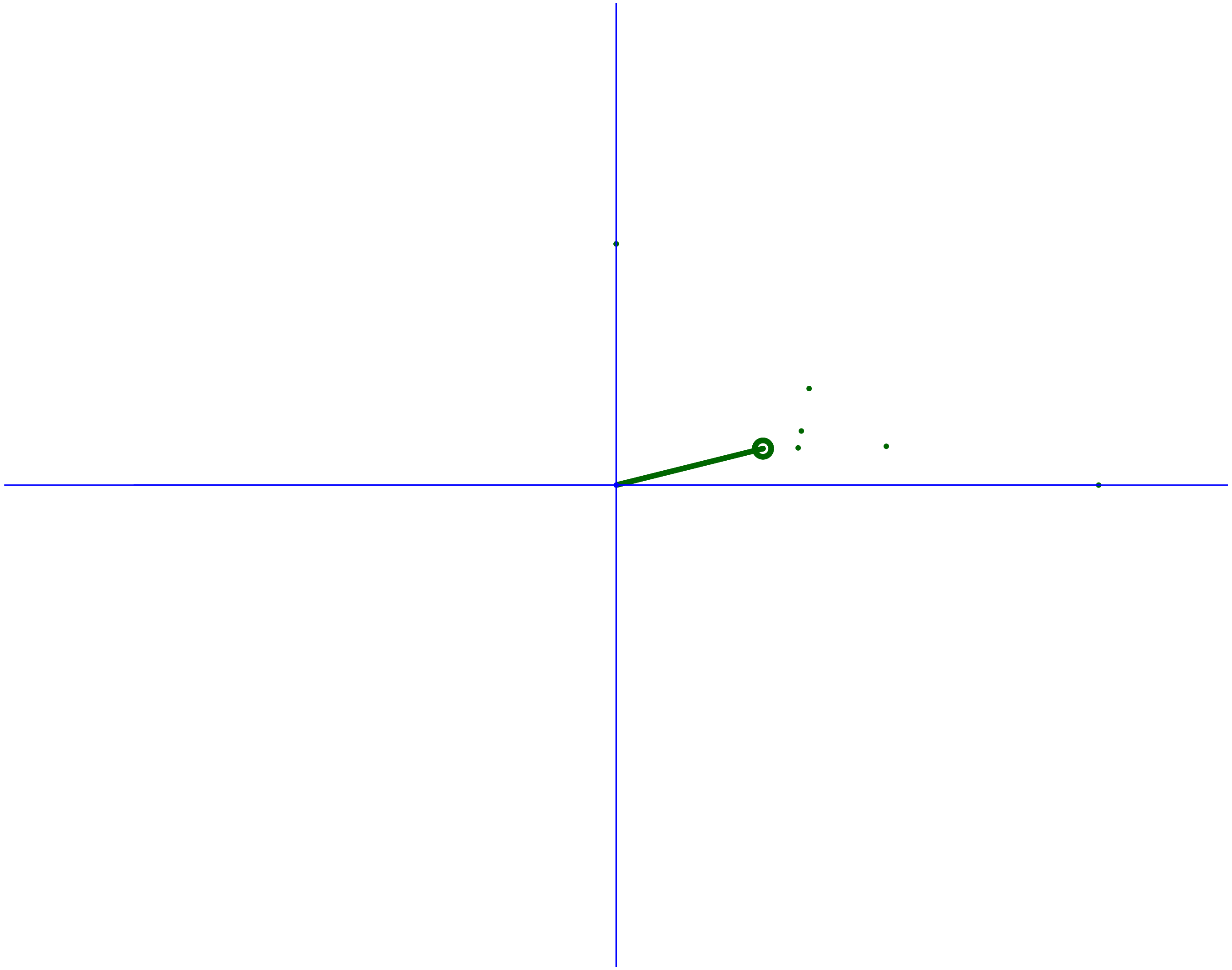
# Iteratie in beeld

$x_5$



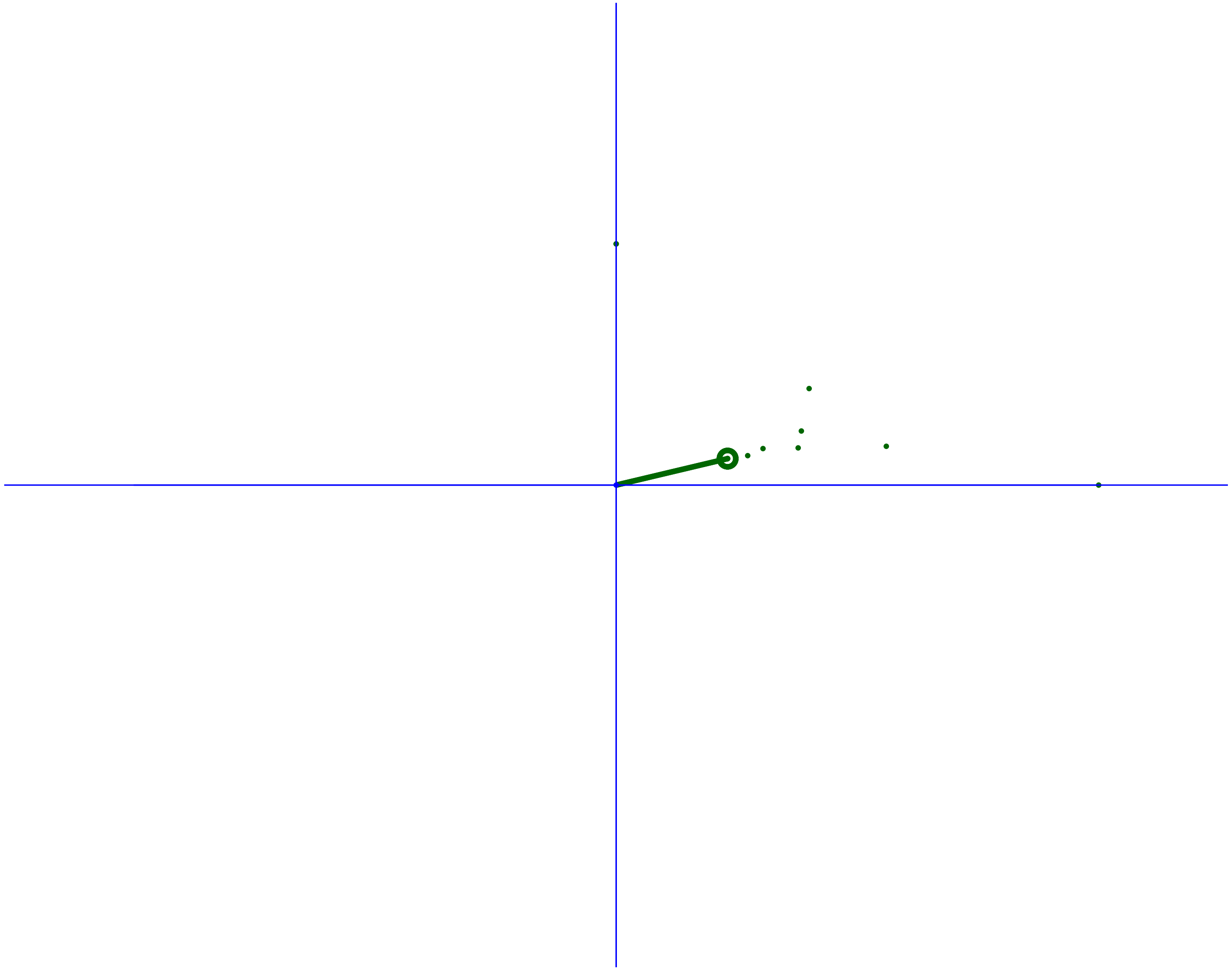
# Iteratie in beeld

$x_6$



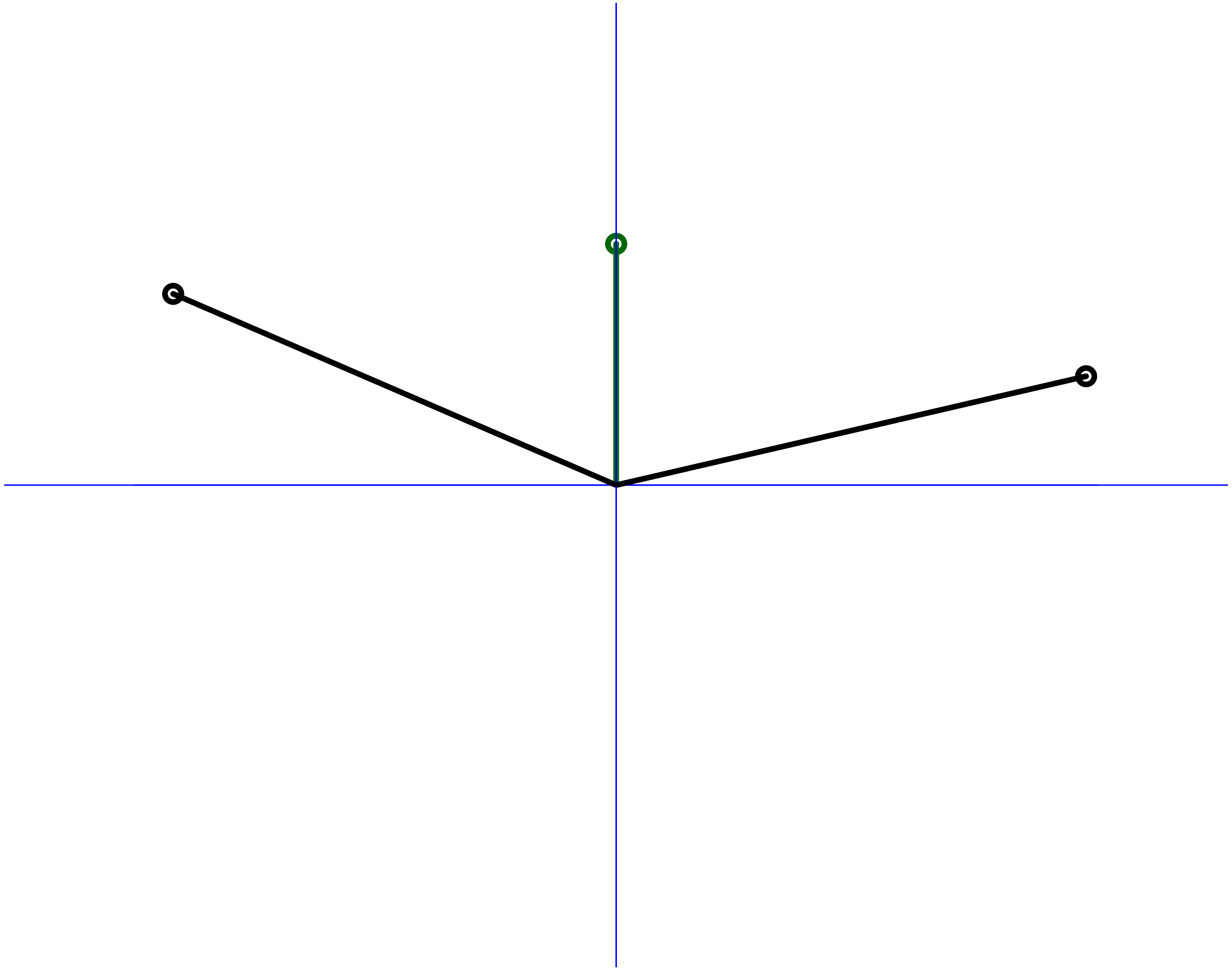
# Iteratie in beeld

$x_8$

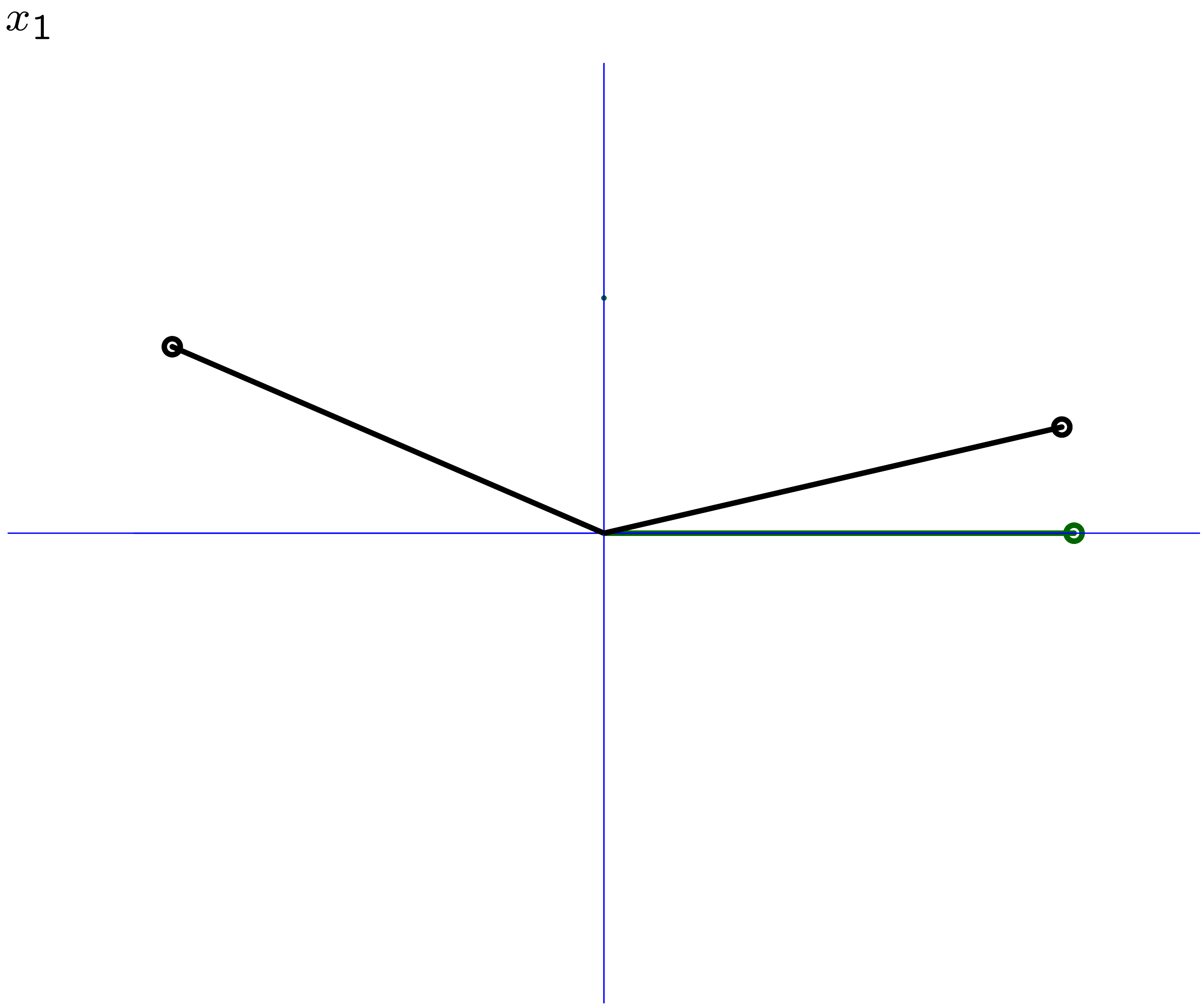


# Iteratie in beeld

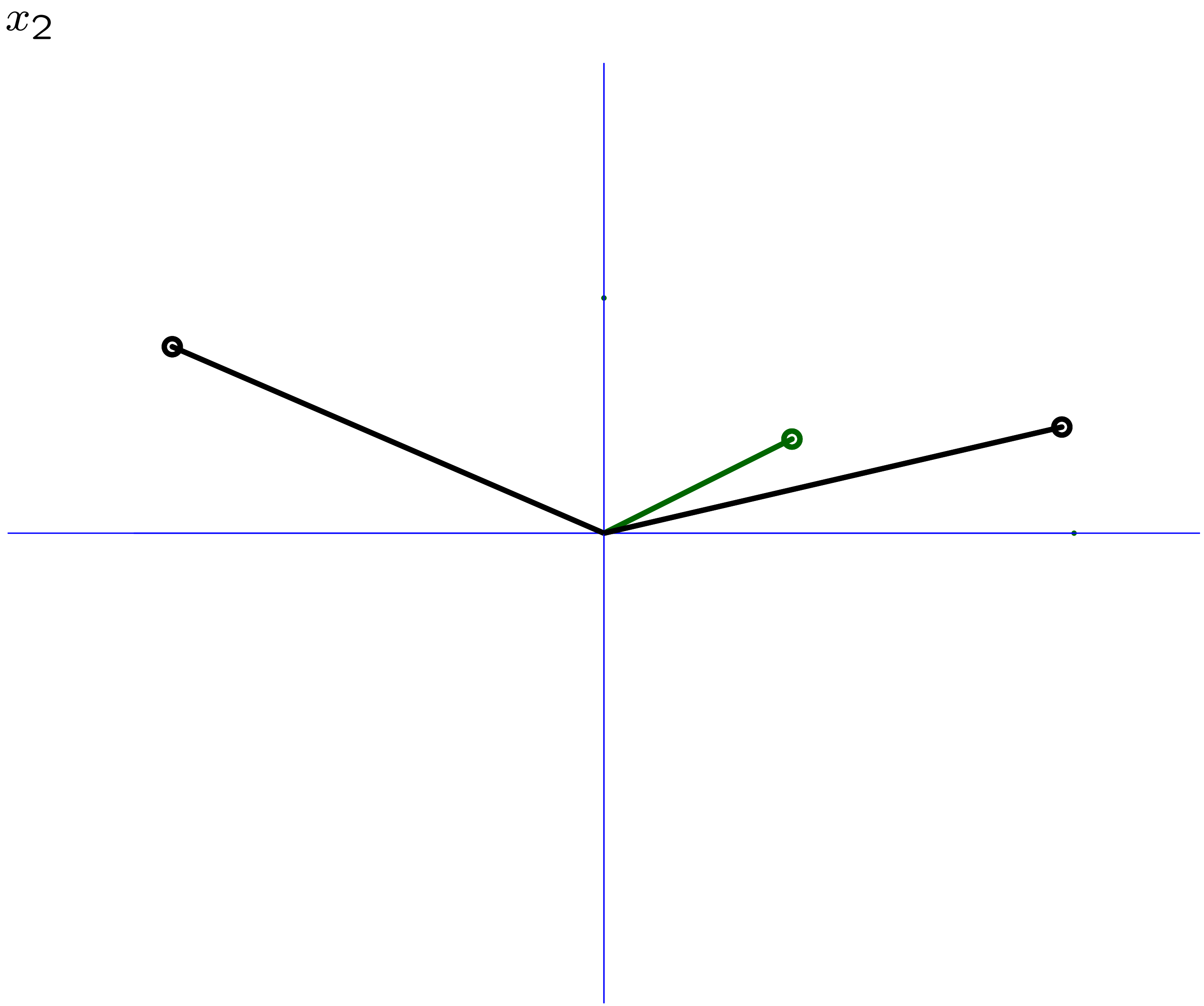
$x_0$



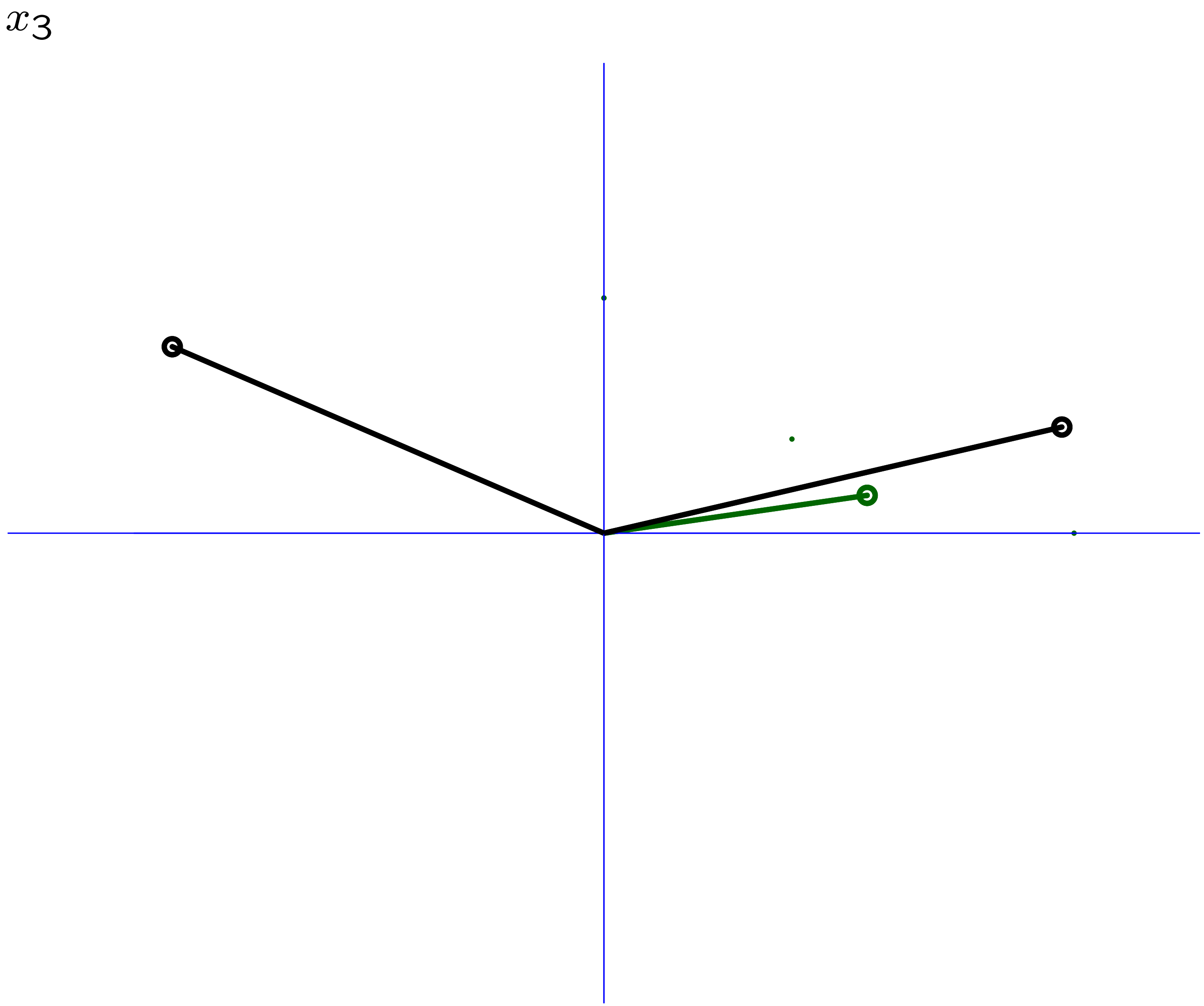
# Iteratie in beeld



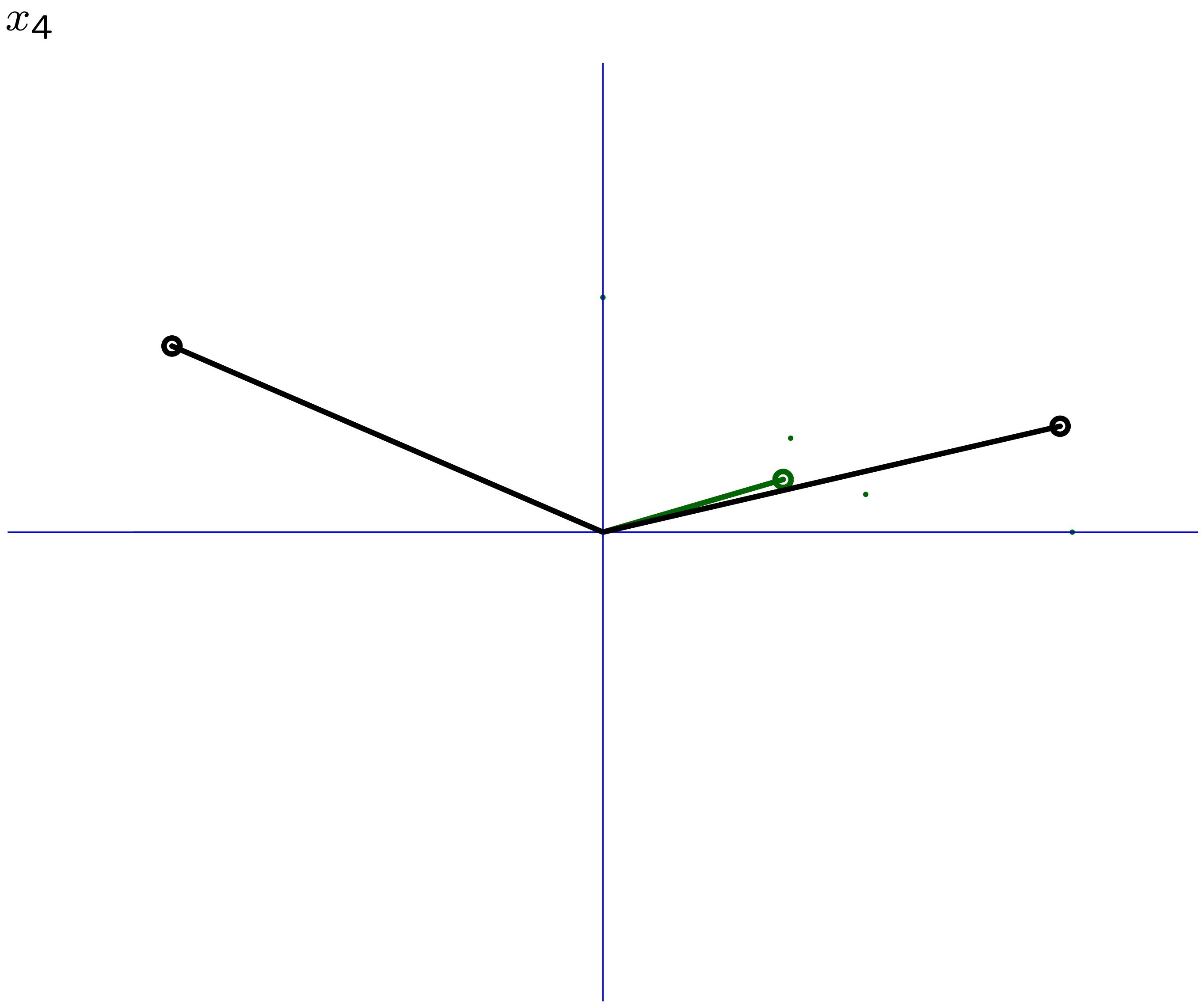
# Iteratie in beeld



# Iteratie in beeld

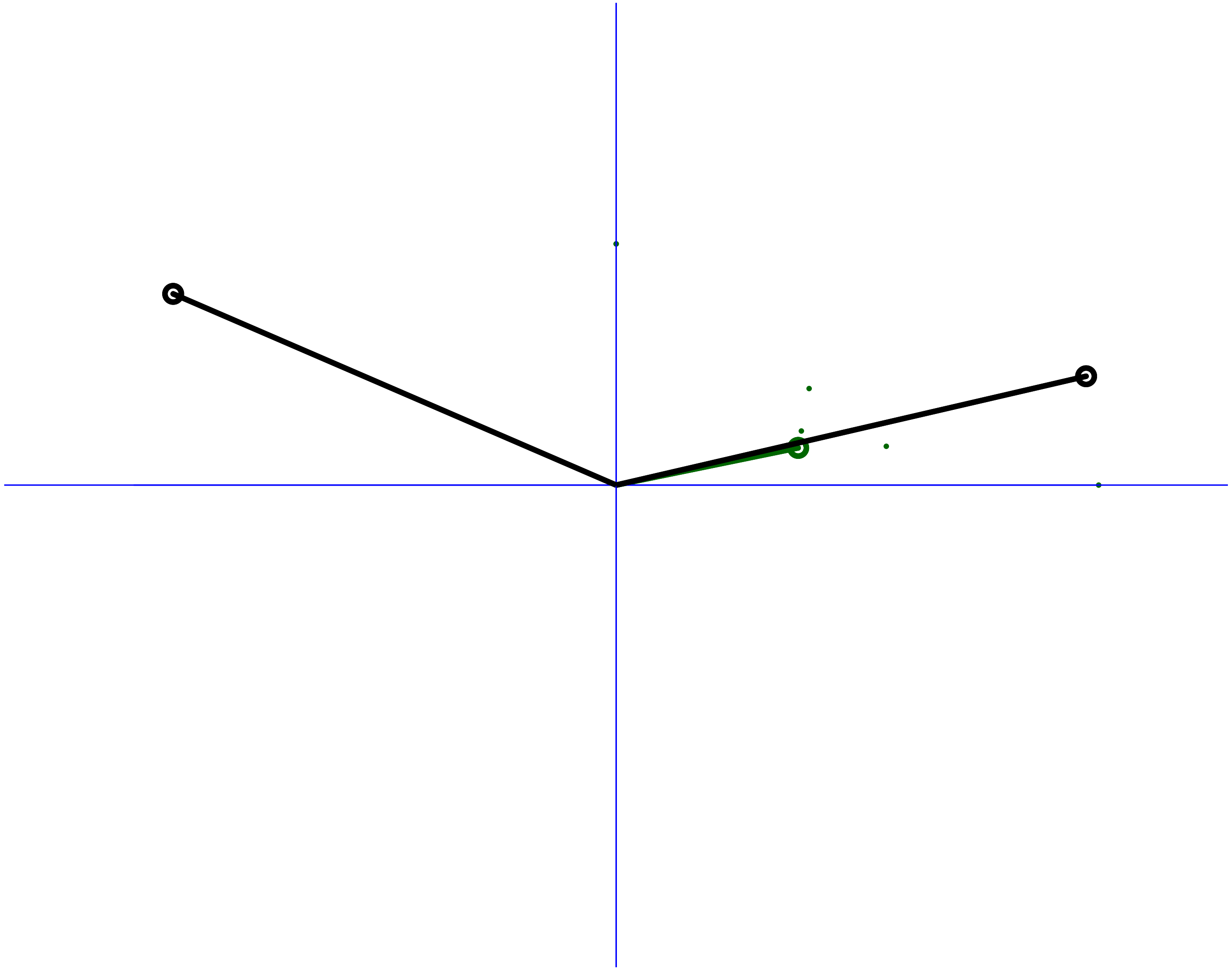


# Iteratie in beeld



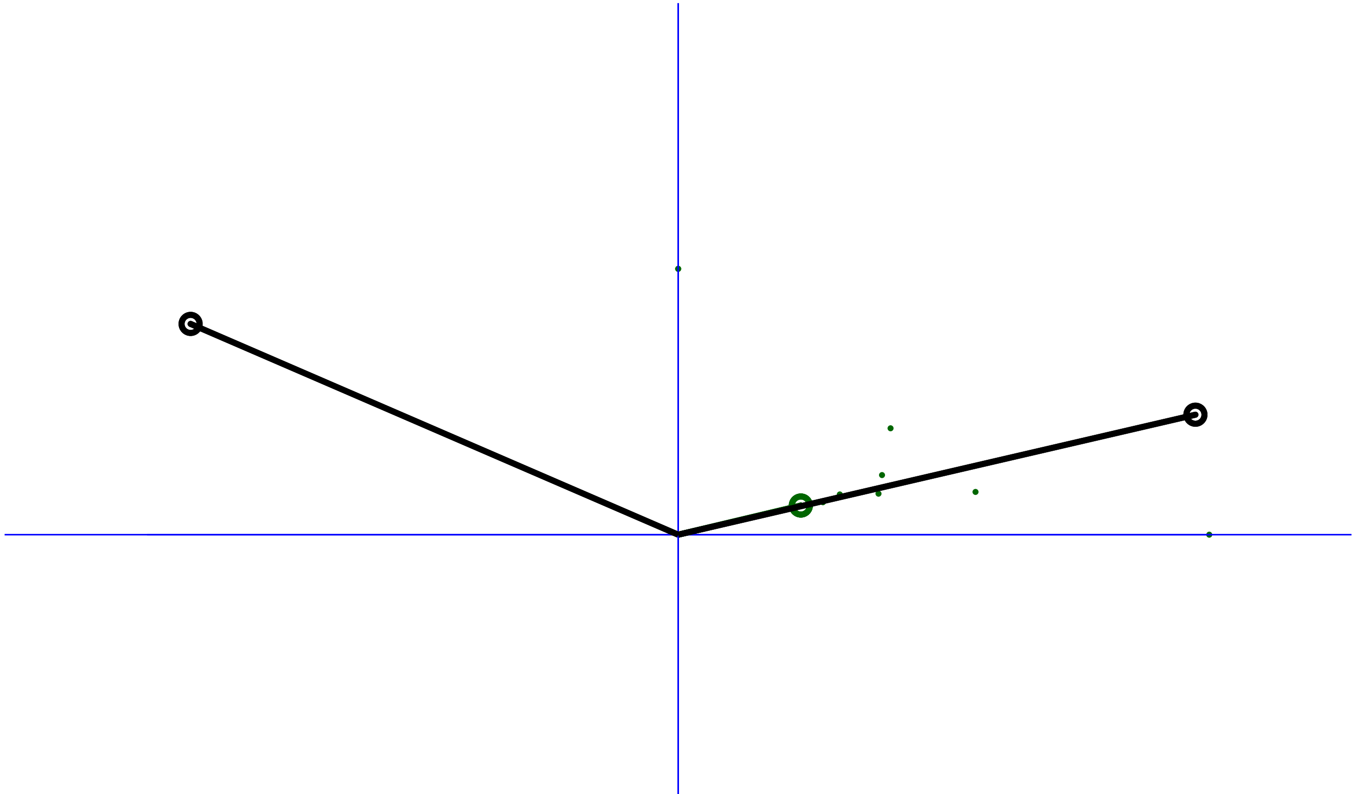
# Iteratie in beeld

$x_5$



# Iteratie in beeld

$x_8$

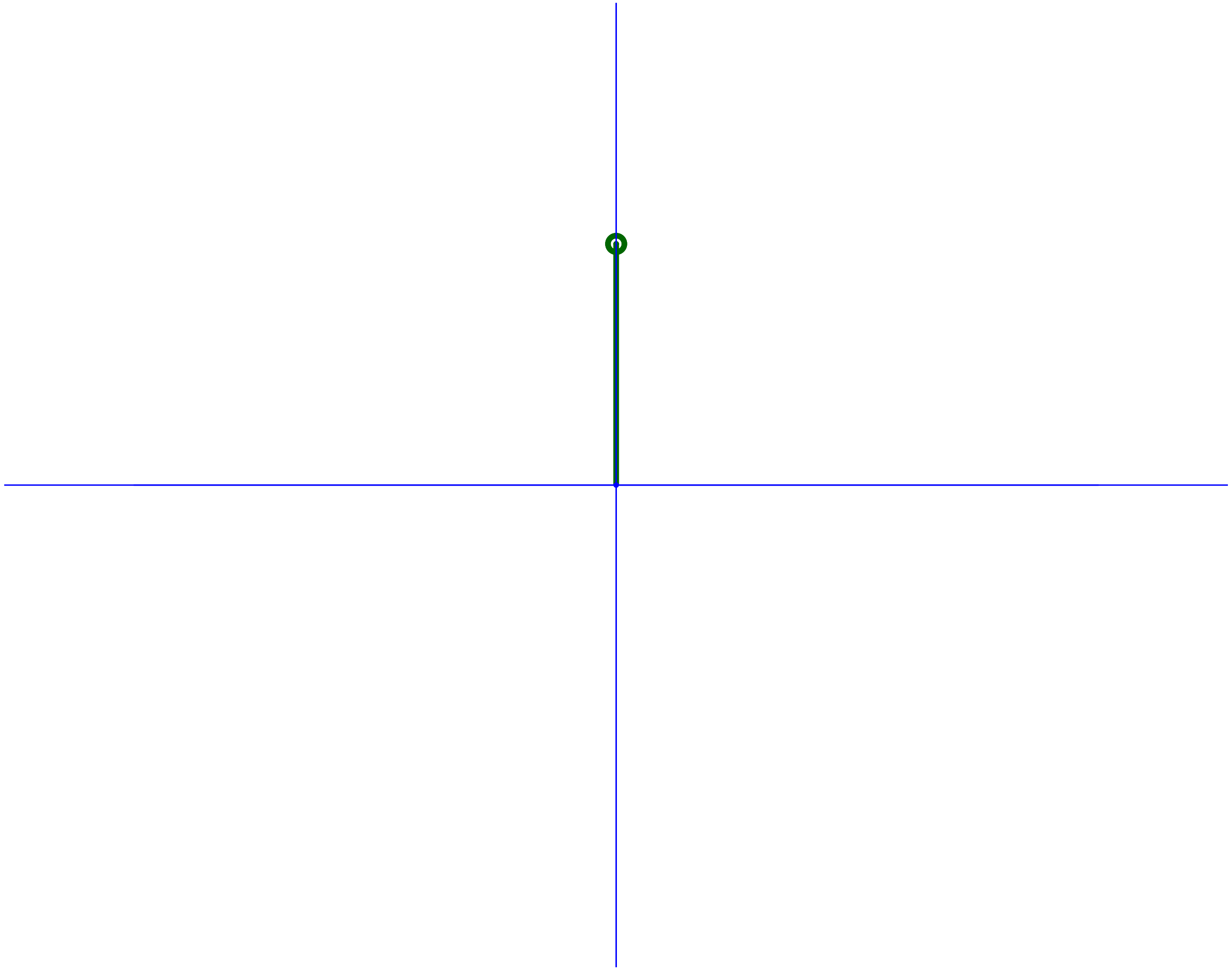


$$\mathbf{A} = \begin{bmatrix} 0.4000 & 2.0000 \\ 0.2000 & 0 \end{bmatrix}$$

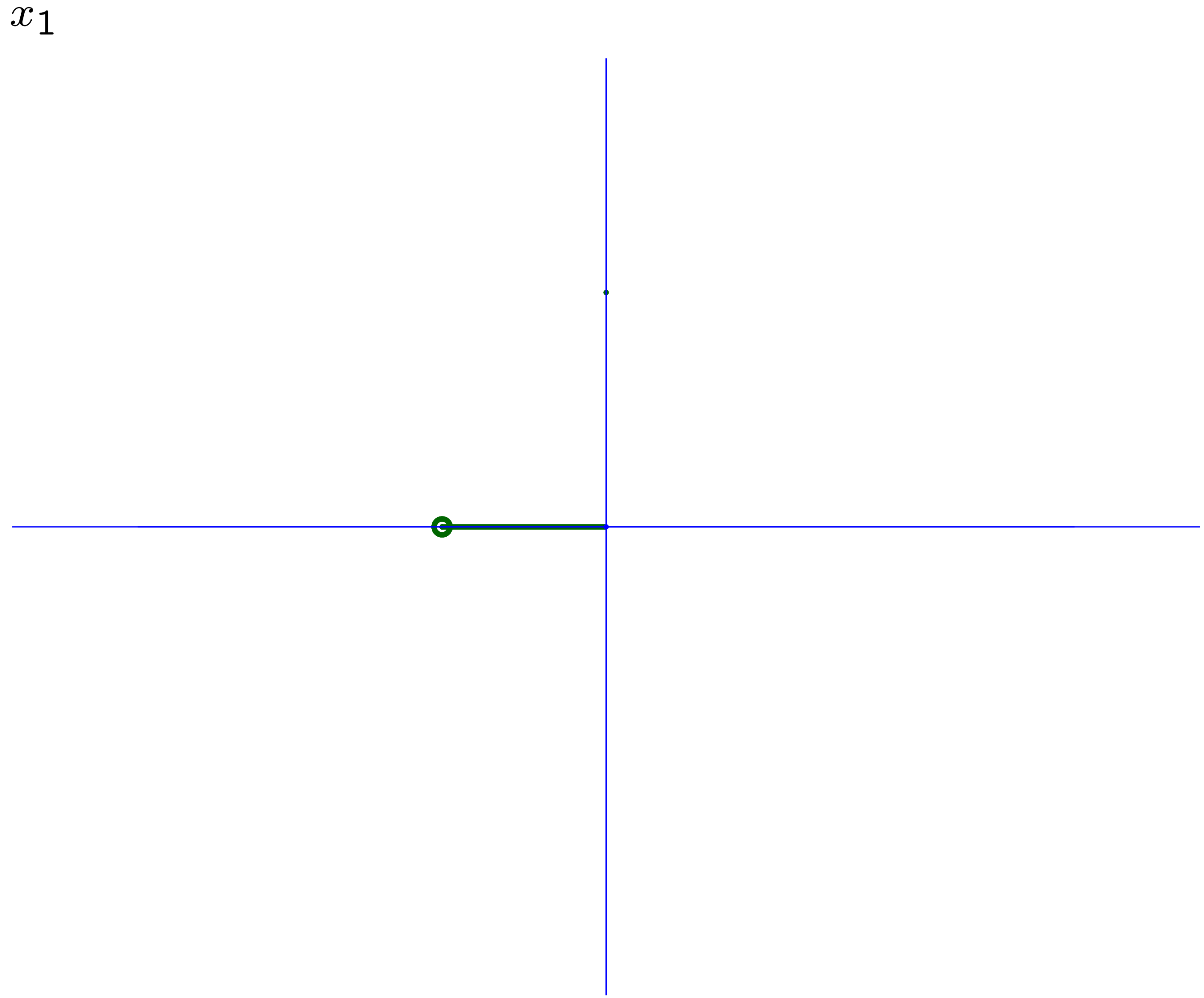
$\lambda_2 = -0.4633 \quad \lambda_1 = 0.8633$

# Iteratie in beeld

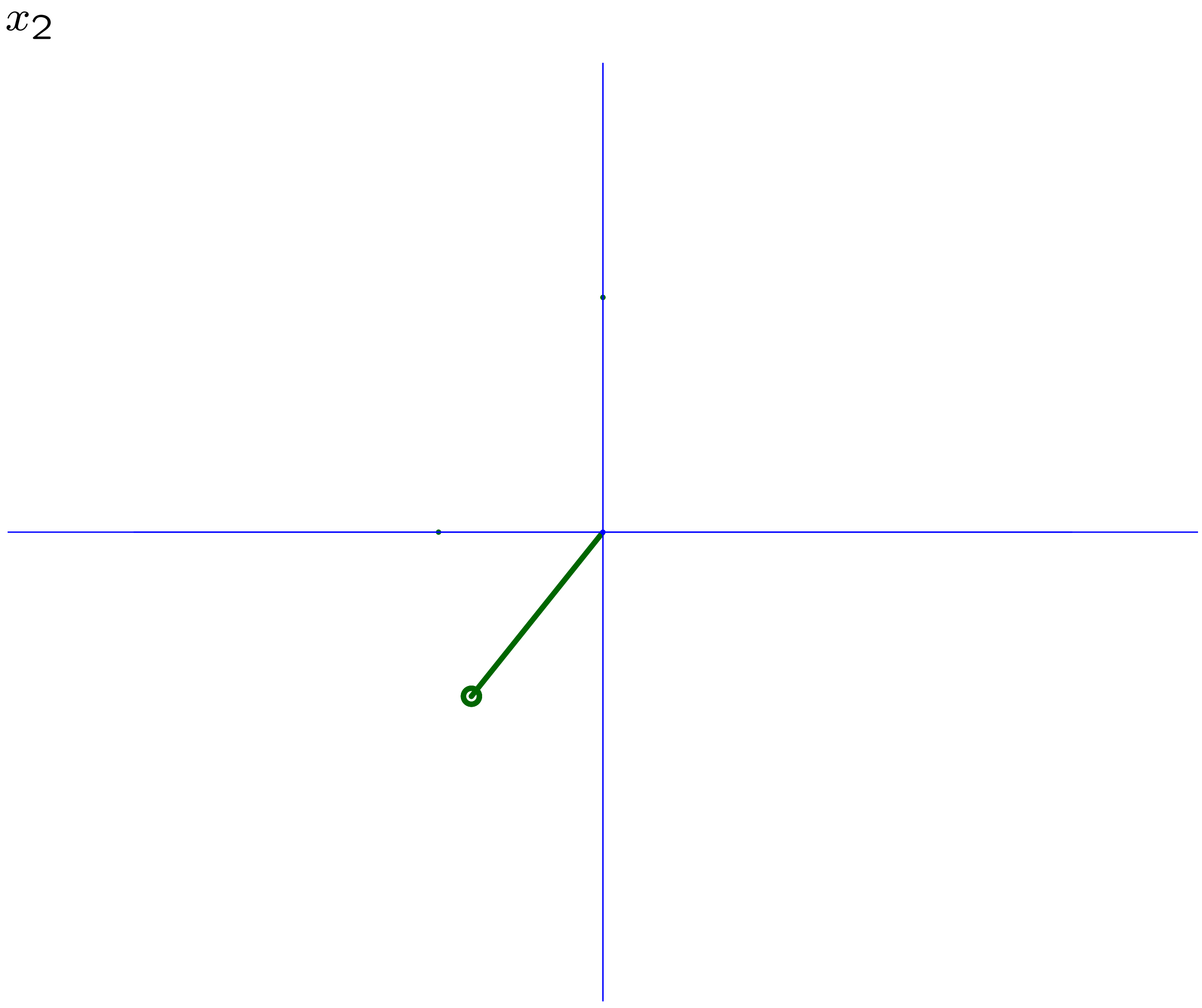
$x_0$



# Iteratie in beeld

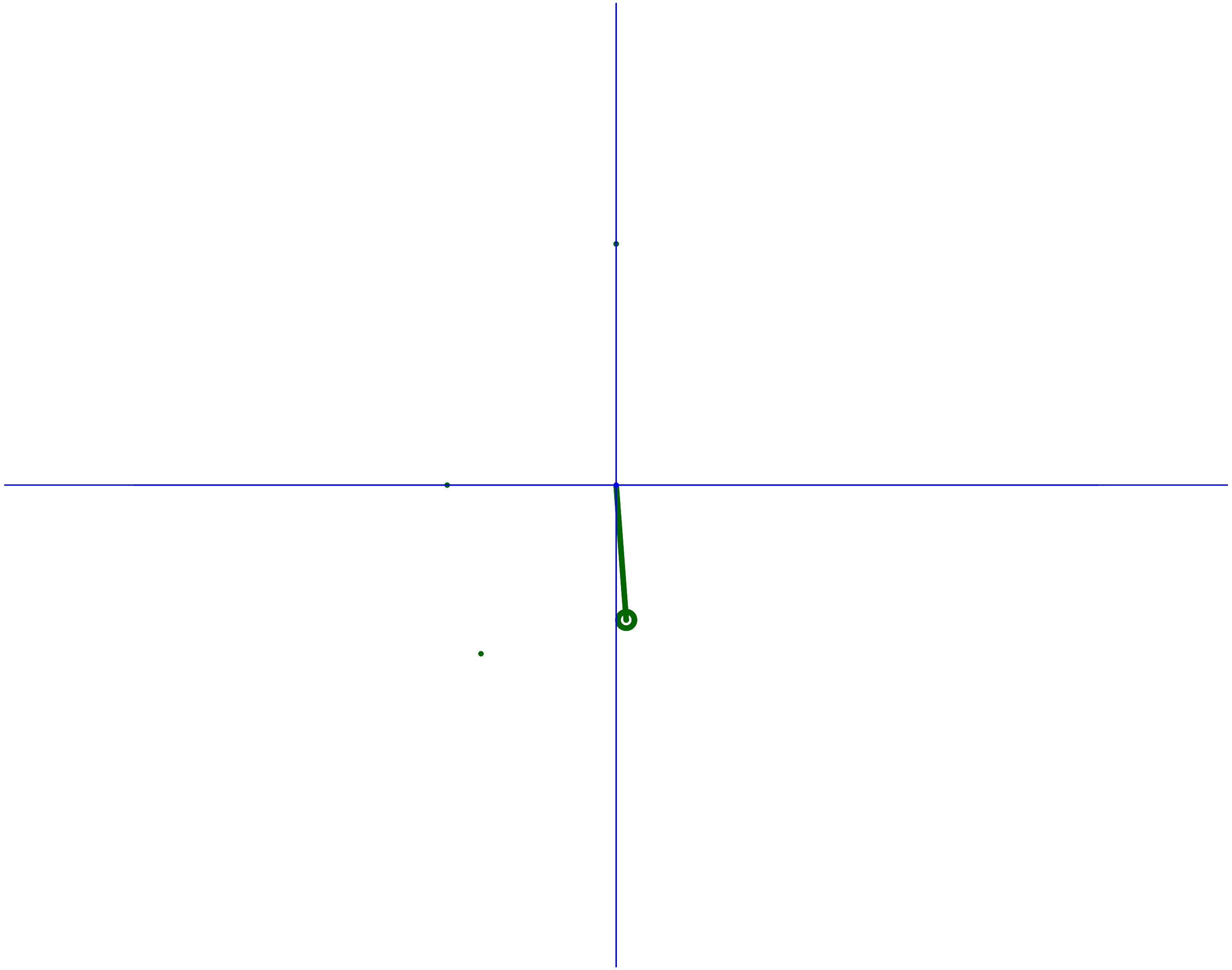


# Iteratie in beeld



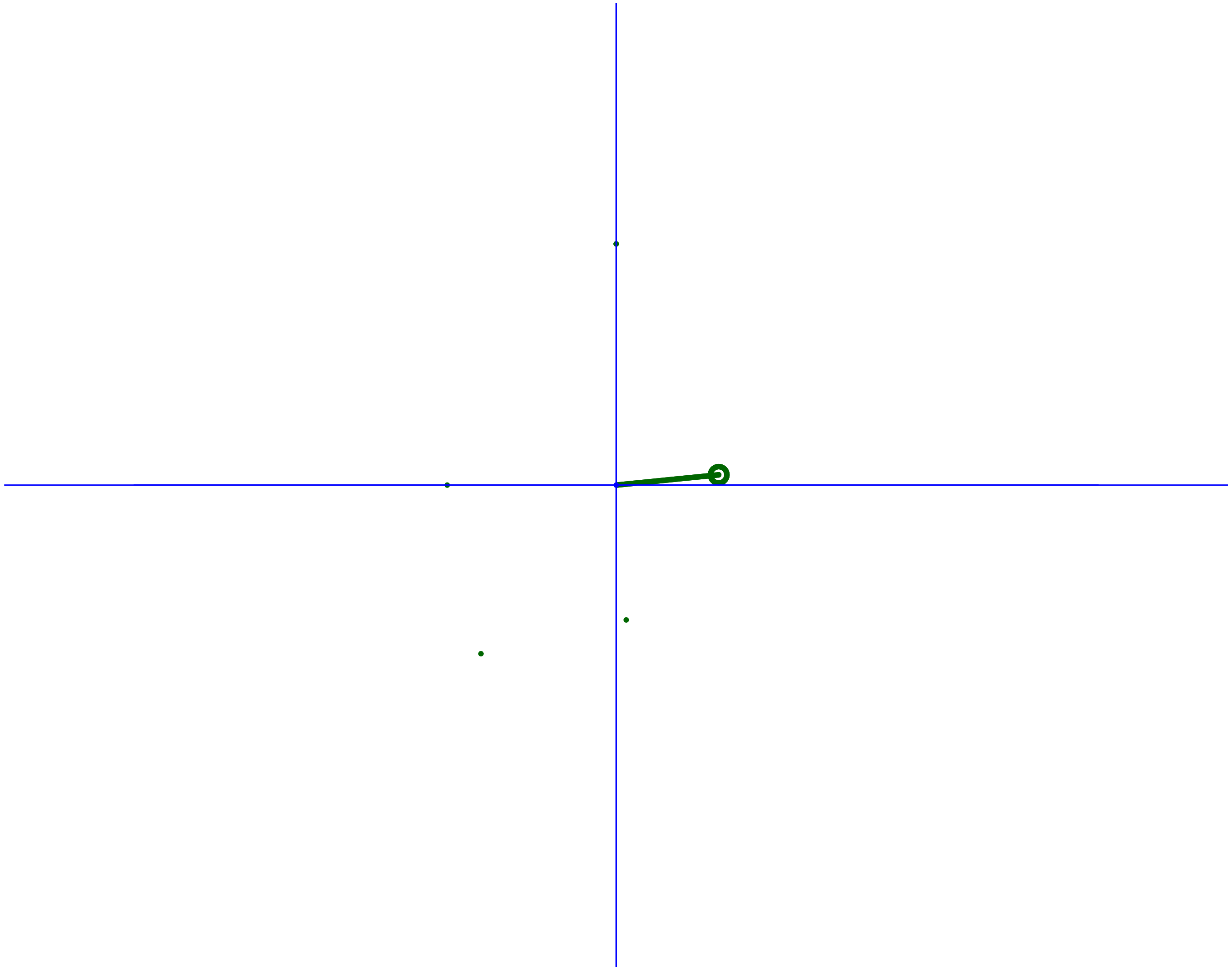
# Iteratie in beeld

$x_3$



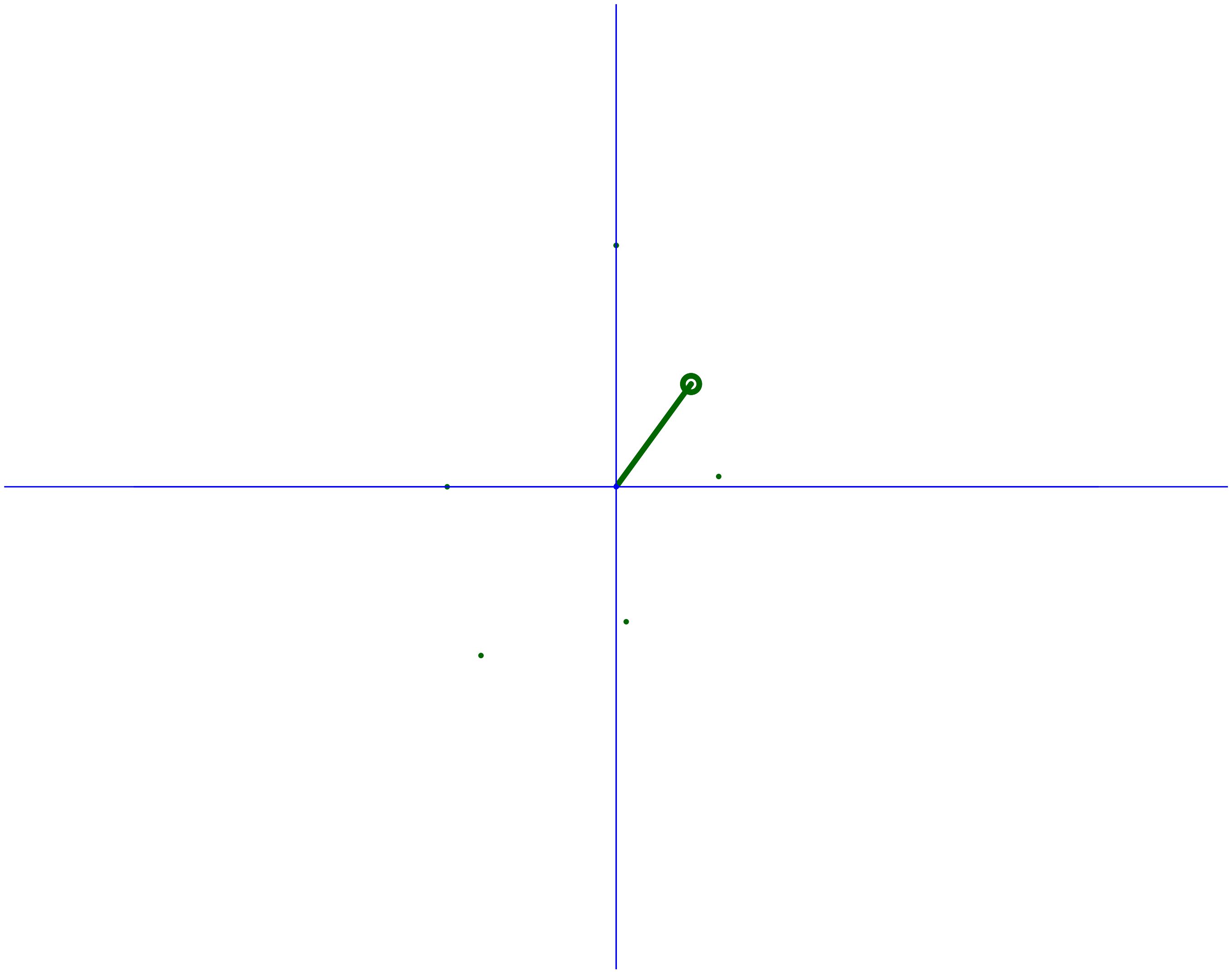
# Iteratie in beeld

$x_4$



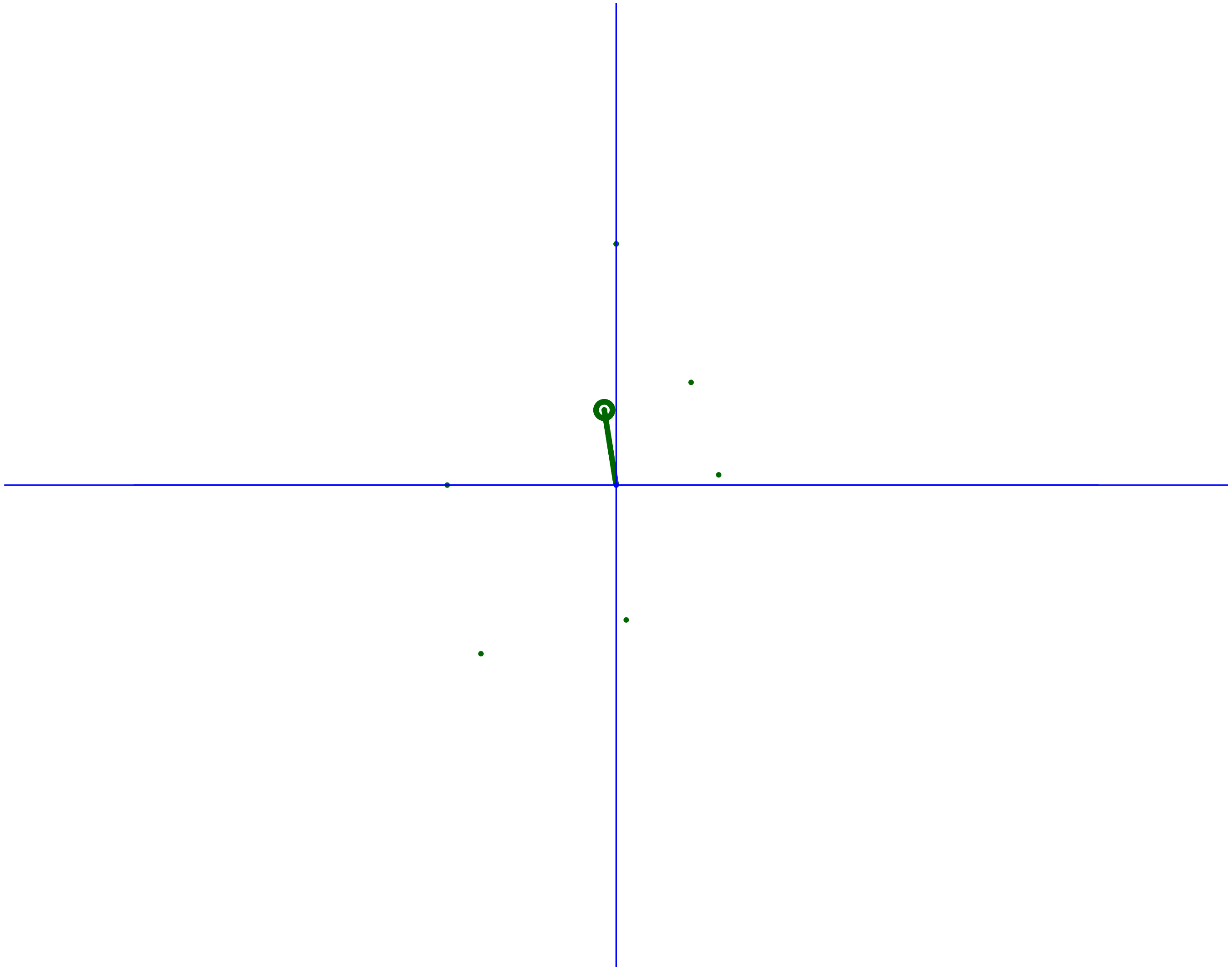
# Iteratie in beeld

$x_5$



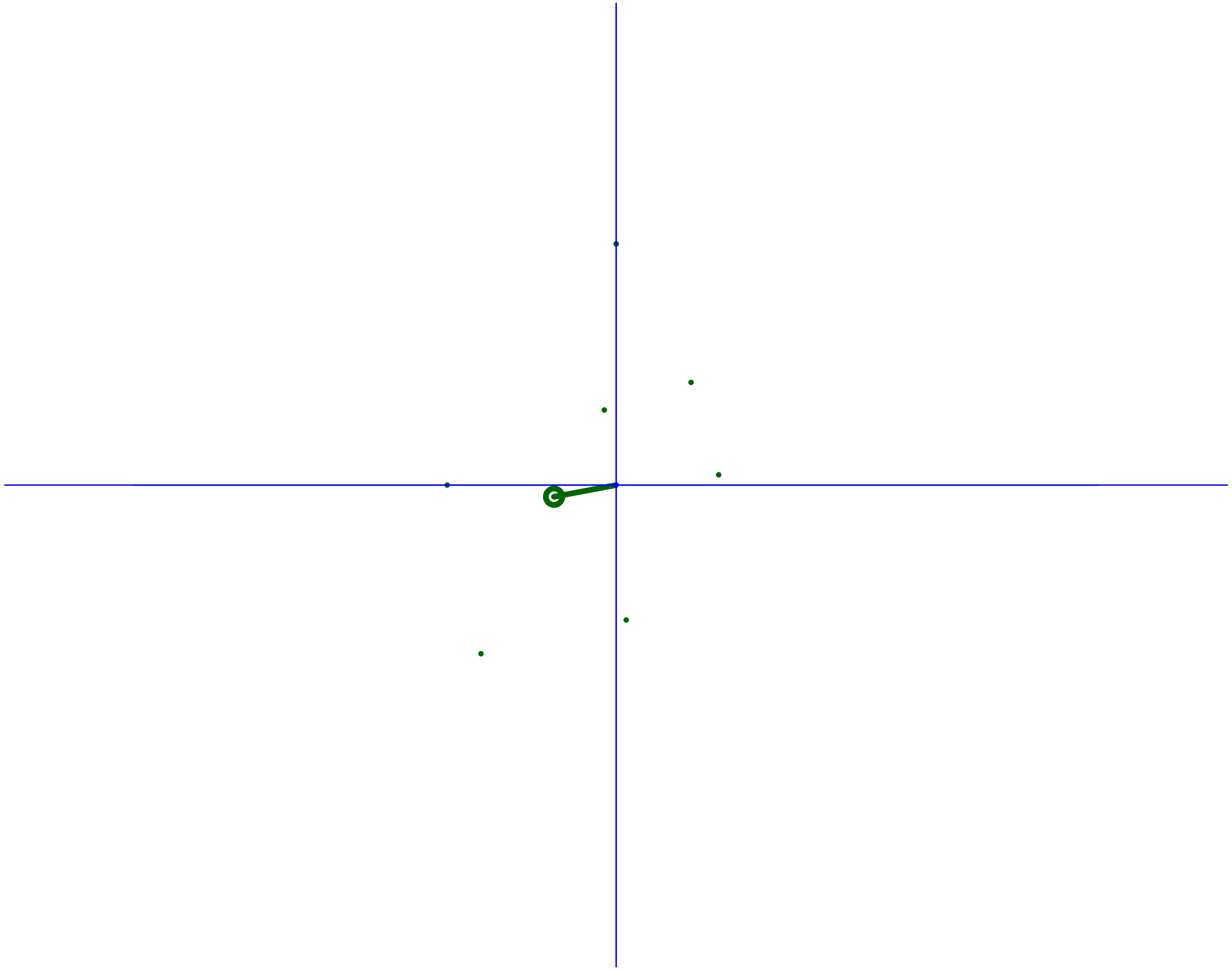
# Iteratie in beeld

$x_6$

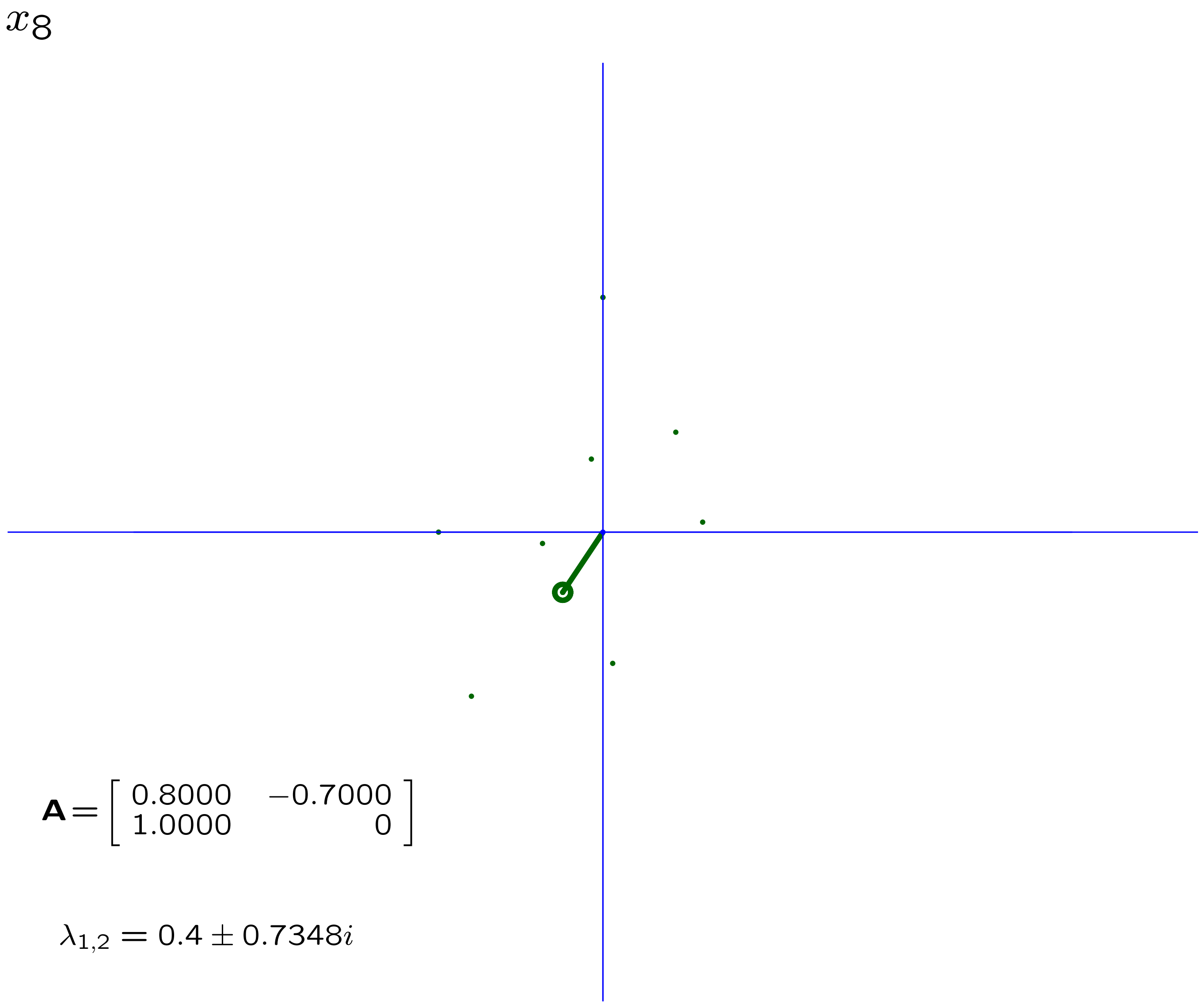


# Iteratie in beeld

$x_7$



# Iteratie in beeld



$$\mathbf{A} = \begin{bmatrix} 0.8000 & -0.7000 \\ 1.0000 & 0 \end{bmatrix}$$

$$\lambda_{1,2} = 0.4 \pm 0.7348i$$

## Complexe eigenwaarden

Stel $\mathbf{A}$ alleen reële coëfficiënten.

$\mathbf{A}\mathbf{v}_1 = \lambda_1 \mathbf{v}_1$. Dus $\mathbf{A}\bar{\mathbf{v}}_1 = \bar{\mathbf{A}}\bar{\mathbf{v}}_1 = \overline{\mathbf{A}\mathbf{v}_1} = \overline{\lambda_1 \mathbf{v}_1} = \bar{\lambda}_1 \bar{\mathbf{v}}_1$.

Als $\lambda_1 \in \mathbb{C}\backslash\mathbb{R}$, dan $\lambda_2 = \bar{\lambda}_1$ en $\mathbf{v}_2 = \bar{\mathbf{v}}_1$.

Schrijf $\mathbf{v}_1 = \mathbf{w}_1 + i\mathbf{w}_2$ met $\mathbf{w}_1, \mathbf{w}_2$ reële vectoren.
Omdat $\mathbf{x}_0$ een reële vector is, zijn er $\alpha, \beta \in \mathbb{R}$ zodat

$$\mathbf{x}_0 = \alpha \mathbf{w}_1 + \beta \mathbf{w}_2 = \mathrm{Re}((\alpha - i\beta)(\mathbf{w}_1 + i\mathbf{w}_2))$$

Schrijf $\gamma \equiv \alpha - i\beta$. Dan

$$\mathbf{x}_0 = \mathrm{Re}(\gamma \mathbf{v}_1) = \frac{1}{2}(\gamma \mathbf{v}_1 + \bar{\gamma}\bar{\mathbf{v}}_1) = \frac{1}{2}(\gamma \mathbf{v}_1 + \bar{\gamma}\mathbf{v}_2)$$

Dus

$$\mathbf{x}_n = \mathbf{A}^n \mathbf{x}_0 = \frac{1}{2}(\gamma \lambda_1^n \mathbf{v}_1 + \bar{\gamma}\bar{\lambda}_1^n \mathbf{v}_2)$$

Met $\lambda_1 = |\lambda_1|\, e^{i\phi}$ en $\gamma = |\gamma|\, e^{i\psi}$ is

$$\mathbf{x}_n = |\gamma|\, |\lambda_1|^n \,(\cos(\psi + n\phi)\mathbf{w}_1 - \sin(\psi + n\phi)\mathbf{w}_2)$$

# Leslie matrices

$m$ leeftijdsgroepen (tijdvak bv. 1 jaar)

$N_k(n)$ aantal individuen van leeftijd $k-1$ begin $n$-de tijdvak

**Aanname** [Leslie]:
- In elk tijdvak overleeft de fractie $s_k$ van leeftijd $k-1$
- Aantal 0-jarige hangt lineair af
  van de vruchtbaarheid van de diverse leeftijdsgroepen

**Model**: $N_{k+1}(n+1) = s_k\, N_k(n)$ &

$$N_1(n+1) = g_1\, N_1(n) + \ldots + g_m\, N_m(n)$$

In matrix taal

$$\begin{bmatrix} N_1(n+1) \\ N_2(n+1) \\ N_3(n+1) \\ \vdots \\ N_m(n+1) \end{bmatrix} = \begin{bmatrix} g_1 & g_2 & \ldots & g_{m-1} & g_m \\ s_1 & 0 & \ldots & 0 & 0 \\ 0 & s_2 & \ldots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \ldots & s_{m-1} & 0 \end{bmatrix} \begin{bmatrix} N_1(n) \\ N_2(n) \\ N_3(n) \\ \vdots \\ N_m(n) \end{bmatrix}$$

# Leslie matrices

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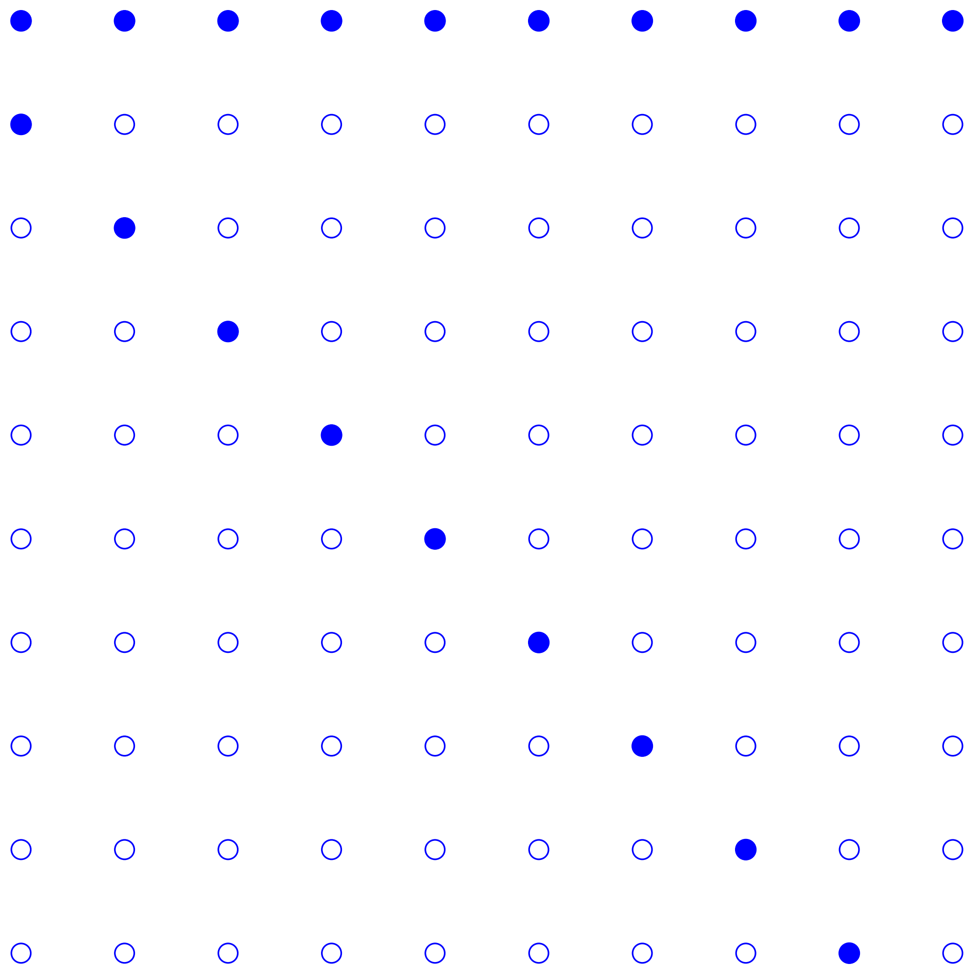
$$N_1(n+1) = g_1\, N_1(n) + \ldots + g_m\, N_m(n)$$

In matrix taal $\mathbf{x}_{n+1} = \mathbf{A}\,\mathbf{x}_n$

met $\mathbf{A} \equiv \begin{bmatrix} g_1 & g_2 & \cdots & g_{m-1} & g_m \\ s_1 & 0 & \ldots & 0 & 0 \\ 0 & s_2 & \ldots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \ldots & s_{m-1} & 0 \end{bmatrix}$ en $\mathbf{x}_n \equiv \begin{bmatrix} N_1(n) \\ N_2(n) \\ N_3(n) \\ \vdots \\ N_m(n) \end{bmatrix}$

**Leslie matrix**

# Leslie matrix



**Evenwicht** als

voor alle $n$: $\quad N_j(n+1) = N_j(n) = \alpha_j \quad$ voor $j = 1, \ldots, m$

$$\vec{\alpha} \equiv \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{bmatrix} = \mathbf{A}\,\vec{\alpha} \quad \Leftrightarrow \quad 1 \text{ eigenwaarde } \mathbf{A}$$

**Evenwichtige opbouw in leeftijd** als voor alle $n$:

$N_i(n+1)/N_j(n+1) = N_i(n)/N_j(n)$ voor alle $i, j = 1, \ldots, m$.

$$\lambda\,\vec{\beta} = \mathbf{A}\,\vec{\beta} \quad \Leftrightarrow \quad \lambda \text{ eigenwaarde } \mathbf{A}$$

*Wanneer is evenwicht stabiel?*

*Wanneer is evenwichtige leeftijdsopbouw stabiel?*

Beschouw de **iteratie** in $\mathbb{R}^m$

$$\mathbf{x}_{n+1} \;=\; \mathbf{A}\,\mathbf{x}_n$$

Dan $\qquad \mathbf{x}_n = \mathbf{A}\,\mathbf{x}_{n-1} = \ldots = \mathbf{A}^n\,\mathbf{x}_0$

**Eigenwaarden** en **eigenvectoren** van **A**:

$$\mathbf{A}\mathbf{v}_j = \lambda_j\,\mathbf{v}_j \quad (j = 1, \ldots, m).$$

$$\mathbf{x}_0 = \gamma_1\mathbf{v}_1 + \ldots + \gamma_m\mathbf{v}_m \quad \Rightarrow \quad \mathbf{x}_n = \gamma_1\,\lambda_1^n\,\mathbf{v}_1 + \ldots + \gamma_m\,\lambda_m^n\,\mathbf{v}_m$$

$$\mathbf{x}_n = \gamma_1\,\lambda_1^n\left(\mathbf{v}_1 + \frac{\gamma_2}{\gamma_1}\left(\frac{\lambda_2}{\lambda_1}\right)^n \mathbf{v}_2 + \ldots + \frac{\gamma_m}{\gamma_1}\left(\frac{\lambda_m}{\lambda_1}\right)^n \mathbf{v}_m\right)$$

Als $|\lambda_j| < |\lambda_1|$ alle $j > 1$ en $\gamma_1 \neq 0$,

dan $\mathbf{x}_n \approx \gamma_1\,\lambda_1^n\,\mathbf{v}_1$ voor grote $n$.

# Lineaire differentiaalvergelijkingen

**Dynamica op $R^d$**

- Tijdsreeks en fase ruimte
- Lineaire dynamica:
  - Lineaire recursies
  - Lineaire DVs
- Niet-lineaire recursies
- Niet-lineaire DVs
- Stabiliteit van numerieke methoden

We beschouwen een lineaire DV:

$$\frac{dy}{dt} = Ay, \quad y(t) \in \mathbf{R}^d, \quad A \in \mathbf{R}^{d\times d}$$

Stel dat A een basis van d onafhankelijke eigenvectoren bezit: $Av_i = \lambda_i v_i, \quad i = 1, \ldots, d$

Elke vector in $\boldsymbol{R}^d$ kan op unieke wijze geschreven zijn als een lineaire combinatie van de $v_i$:

$$y(t) = \alpha_1(t)v_1 + \alpha_2(t)v_2 + \cdots + \alpha_d(t)v_d$$

$$\dot{y}(t) = \dot{\alpha}_1(t)v_1 + \dot{\alpha}_2(t)v_2 + \cdots + \dot{\alpha}_d(t)v_d$$

$$y(0) = \alpha_1(0)v_1 + \alpha_2(0)v_2 + \cdots + \alpha_d(0)v_d$$

# Lineaire differentiaalvergelijkingen

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Stel dat A een basis van d onafhankelijke eigenvectoren bezit: $Av_i = \lambda_i v_i, \quad i = 1, \ldots, d$

De rechter kant van onze DV wordt

$$\begin{aligned} Ay(t) &= \alpha_1(t)Av_1 + \alpha_2(t)Av_2 + \cdots + \alpha_d(t)Av_d \\ &= \alpha_1(t)\lambda_1 v_1 + \alpha_2(t)\lambda_2 v_2 + \cdots + \alpha_d(t)\lambda_d v_d \end{aligned}$$

Trek deze relatie af van die voor $\dot{y}$:

$$\dot{y}(t) = \dot{\alpha}_1(t)v_1 + \dot{\alpha}_2(t)v_2 + \cdots + \dot{\alpha}_d(t)v_d$$

$$\dot{y}(t) - Ay(t) = 0 = (\dot{\alpha}_1 - \lambda_1\alpha_1)v_1 + \cdots + (\dot{\alpha}_d - \lambda_d\alpha_d)v_d$$

# Lineaire differentiaalvergelijkingen

**Dynamica op $R^d$**
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- Niet-lineaire DVs
- Stabiliteit van numerieke methoden

Omdat de eigenvectoren onafhankelijk zijn moeten alle coefficienten nul zijn:

$$\dot{y}(t) - Ay(t) = 0 = (\dot{\alpha}_1 - \lambda_1\alpha_1)v_1 + \cdots + (\dot{\alpha}_d - \lambda_d\alpha_d)v_d$$

Dus:

$$\begin{aligned}
\dot{\alpha}_1(t) &= \lambda_1\alpha_1(t) \quad \Rightarrow \quad \alpha_1(t) = e^{\lambda_1 t}\alpha_1(0) \\
\dot{\alpha}_2(t) &= \lambda_2\alpha_2(t) \quad \Rightarrow \quad \alpha_2(t) = e^{\lambda_2 t}\alpha_2(0) \\
&\vdots \\
\dot{\alpha}_d(t) &= \lambda_d\alpha_d(t) \quad \Rightarrow \quad \alpha_d(t) = e^{\lambda_d t}\alpha_d(0)
\end{aligned}$$

De algemene oplossing voor $y(t)$ is dus

$$y(t) = \alpha_1(0)e^{\lambda_1 t}v_1 + \alpha_2(0)e^{\lambda_2 t}v_2 + \cdots + \alpha_d(0)e^{\lambda_d t}v_d$$

# Lineaire differentiaalvergelijkingen

De algemene oplossing voor *y(t)* is dus

$$y(t) = \alpha_1(0)e^{\lambda_1 t}v_1 + \alpha_2(0)e^{\lambda_2 t}v_2 + \cdots + \alpha_d(0)e^{\lambda_d t}v_d$$

Stel voor de eigenwaarden geldt: $\mathrm{Re}\,\lambda_1 \geq \mathrm{Re}\,\lambda_2 \geq \cdots \geq \mathrm{Re}\,\lambda_d$

Dan kan de oplossing geschreven worden als

$$y(t) = \alpha_1(0)e^{\lambda_1 t}\left[v_1 + \frac{\alpha_2(0)}{\alpha_1(0)}e^{(\lambda_2-\lambda_1)t}v_2 + \cdots + \frac{\alpha_d(0)}{\alpha_1(0)}e^{(\lambda_d-\lambda_1)t}v_d\right]$$

Voor twee complexe getallen

$$\lambda_1 = \rho_1 + i\nu_i, \quad \lambda_2 = \rho_2 + i\nu_i, \qquad \rho_i, \nu_i \in \mathbf{R}$$

geldt:

$$\rho_1 > \rho_2 \quad \Rightarrow \quad e^{(\lambda_2-\lambda_1)t} = e^{(\rho_2-\rho_1)t}e^{i(\nu_2-\nu_1)t}$$

$$|e^{(\lambda_2-\lambda_1)t}| = |e^{(\rho-2-\rho_1)t}| < 1$$

$$\lim_{t\to\infty} |e^{(\lambda_2-\lambda_1)t}| = 0$$

# Lineaire differentiaalvergelijkingen

Dus als

$$\operatorname{Re}\lambda_1 > \operatorname{Re}\lambda_2 \geq \cdots \geq \operatorname{Re}\lambda_d$$

(strict), dan geldt voor

$$y(t) = \alpha_1(0)e^{\lambda_1 t}\left[v_1 + \frac{\alpha_2(0)}{\alpha_1(0)}e^{(\lambda_2-\lambda_1)t}v_2 + \cdots + \frac{\alpha_d(0)}{\alpha_1(0)}e^{(\lambda_d-\lambda_1)t}v_d\right]$$

voor grote $t$:

$$y(t) \approx \alpha_1(0)e^{\lambda_1 t}v_1$$

Stelling. Het evenwicht $\boldsymbol{y^* = 0}$ is asymptotisch stabiel als

$$\operatorname{Re}\lambda_i < 0, \qquad i = 1, \ldots, d$$