Project 2 - Population dynamics

WISB134 - Modellen en Simulatie

Due date: 30 March 2015

In this report we investigate a model for the dynamics of a population of wolves (predators) and a population of rabbits (prey). The model has two advanced properties. The first is a saturation effect that takes into account the carrying capacity of the environment (for example, a limited supply of carrots for the rabbits). The second characteristic models the 'full tummy' effect of predators which takes into account that wolves prefer to let their last meal settle before going in search of new juicy rabbits. Let $K(t)$ denote the number of rabbits, and $W(T)$ the number of wolves at time t. The model is

$$\frac{dK}{dt} = rK\left(1 - \frac{K}{K_d}\right) - \frac{\alpha K}{1 + \tau\alpha K}W, \tag{1}$$

$$\frac{dW}{dt} = \varepsilon\frac{\alpha K}{1 + \tau\alpha K}W - \delta W. \tag{2}$$

The following parameters (all nonnegative) appear in the model: r is the unchecked birthrate of prey, δ is the death rate of predators, and K_d is the carrying capacity of the environment. We assume that the chance that a predator catches and eats a prey grows proportional to the number of prey, with proportionality constant α. The efficiency factor with which eaten prey lead to growth in the predator population is denoted by ε. Furthermore, τ is the amount of time that passes before a predator becomes hungry again.

In this project we want to investigate how the carrying capacity K_d influences the population dynamics. We choose fixed values for the other parameters r, α, τ, ε and δ.

(a) Explain the model: Give an explanation for the factor $(1 - K/K_d)$? For example, what happens when $K \ll K_d$, and when $K \to K_d$? Also provide an explanation for the factor $\alpha K/(1 + \tau\alpha K)$. What happens when $\tau = 0$?

(b) Show that in the absence of wolves, the rabbit population will necessarily converge to a constant number.

(c) Discuss the equilibria (K^*, W^*) of the population model. Which equilibria make sense biologically and which not? Under what conditions?

(d) Construct the Jacobian matrix $J(K^*, W^*)$ of the population model. To do so, it is handy to make use of the function $g(K) = \alpha K/(1 + \tau\alpha K)$.

(e) One equilibrium state consists only of rabbits (the "wolf-free state"). What condition on the carrying capacity K_d ensures this favorable (for the rabbits) equilibrium is stable?

(f) For what condition on K_d does there exist a stable equilibrium with positive numbers of both rabbits and wolves? (Hints: (1) to avoid excess algebra, do not directly substitute the formulas for K^* and W^* into the Jacobian, rather work with a generic equilibrium (K^*, W^*) as long as possible and subsequently use the relations that define the equilibria to simplify the Jacobian, (2) use the stability diagram based on the trace and determinant of the Jacobian to analyse stability.)

Using Mathematica we investigate further how the population dynamics change as a function of K_d. To do so we take the following values for the other parameters:

$$r = 1, \quad \alpha = 1/10, \quad \tau = 1/100, \quad \varepsilon = 1/50, \quad \delta = 1/2.$$

Remember that the populations R and W express averages (for example, wolves per square meter), so the precise values may not appear realistic.

(g) Plot the trace s and determinant d of $J(K^*, W^*)$ as well as $s^2/4$ as functions of K_d for $0 < K_d < 3000$ for the nontrivial equilibrium $(K^* > 0, W^* > 0)$. What is the significance of the points where the graphs of s and d change sign?

(h) Denote by $\tilde{K}_d$ the value of K_d for which the curves for $s^2/4$ and d intersect. For $d > s^2/4$ the eigenvalues of the Jacobian are complex, for $d \leq s^2/4$ real. Graph the vector field in the phase space, for some $K_d > \tilde{K}_d$ and $K_d < \tilde{K}_d$. (For this, you can use the function `StreamPlot` in Mathematica.) Can you a discern a difference in the equilibria?

(i) In parts (e) and (f) you determined some critical values of K_d. Show how the solutions behave for values above and below these critical values (or at the critical value itself, if it seems interesting). Consider solutions that start near an equilibrium and those that start far away from it. Show the solutions both as functions of time (time series) and in the phase plane. Discuss your results.

(j) Think now about how environmental changes influence animal populations. What happens when an area with a low carrying capacity for a prey population becomes more nutrient rich, for example through climate change or human influence? Is such an enrichment necessarily beneficial for the ecosystem? Remember that there are always random factors that can perturb a system. If a population becomes very small, such a perturbation—a disease for example—can lead to extinction. Does enrichment of an environment lead to more stable populations?