Project 3 - Tennis match

WISB134 - Modellen en Simulatie

Due date: 15 April 2016

We consider a tennis match between two players A and B. We assume that, at any moment in the match, player A has a chance p of winning a service, and player B a chance $q = 1 - p$. A tennis *match* consists of a number of *sets*, each of which in turn consists of a number of *games*.

The following scores are the possible states in a *game*: 0–0, 0–15, 15–0, 0–30, 15–15, 30–0, 0–40, 15–30, 30–15, 40–0, 15–40, 30–30, 40–15, 30–40, 40–30, Advantage B, Deuce, Advantage A, Game B, Game A. For the purposes of this exercise, we can group some states together: {30–30, Deuce}, {30–40, Advantage B} en {40–30, Advantage A}. All these states are indicated in the figure on the following page.

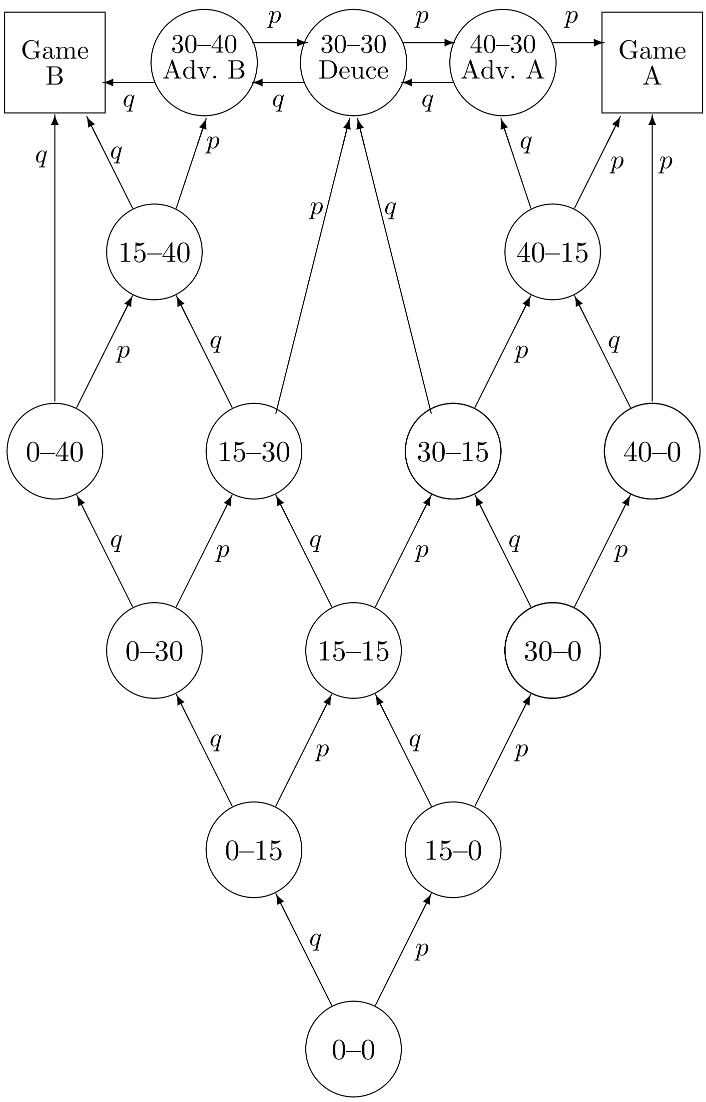
Looking at the figure, we remark that a game is comprised of two phases. The first phase consists of four services if one of the players wins in precisely four services, or five services if this is not so. The second phase is a process with five states of which the length is unknown a priori. This second phase can be modeled as a Markov chain. These five states are the five at the top of the figure, connected by horizontal arrows. We number the states from 1 to 5. Let $\mathbb{P} = (p_{ij})$ be the 5×5-matrix describing the transition likelihoods: p_{ij} is the probability of transitioning from state j to state i.

(a) Construct the matrix $\mathbb{P}$.

Starting from the first service (0–0), we can derive the probability of reaching state i in five services (or four services if one player wins in four). The probabilities are:

$$\mathbf{v} = \begin{pmatrix} q^4 + 4pq^4 \\ 10p^2q^3 \\ 0 \\ 10p^3q^2 \\ p^4 + 4qp^4 \end{pmatrix}.$$

The probability that A wins after four services is of course p^4. The probability that A wins after precisely 5 services is $4qp^4$. This is derived by following all paths of length 5 from (0–0) to "Game A" and adding up the probabilities of all such paths (that is, the product of the transition likelihoods along the path): $4qp^4 = qp^4 + pqp^3 + p^2qp^2 + p^3qp$. Adding these together gives $p^4 + 4qp^4$, i.e. the last element of $\mathbf{v}$.



(b) Check that all other elements of $\mathbf{v}$ are correct.

Let $\tilde{\mathbb{P}}$ be the sub-stochastic matrix that we obtain from $\mathbb{P}$ by changing the ones in the first and last columns to zeros. The likelihood that A wins after precisely $n > 5$ services, is the fifth element of the vector $\tilde{\mathbb{P}}^{n-5}\mathbf{v}$.

(c) Confirm this.

(d) Prove that the probability $p_{\text{Game A}}$ that player A wins a game is equal to the fifth element of $(I - \tilde{\mathbb{P}})^{-1}\mathbf{v}$.

(e) Let g be the sum of the first and fifth elements of $5\tilde{\mathbb{P}}(I-\tilde{\mathbb{P}})^{-1}\mathbf{v}+\tilde{\mathbb{P}}(I-\tilde{\mathbb{P}})^{-2}\mathbf{v}$, and define $g_0 = 4(p^4+q^4)+5\cdot 4(pq^4+qp^4)$. Prove that a game lasts on average $N_{\text{Game}} = g_0 + g$ services.

(f) Make a Mathematica notebook that computes $p_{\text{Game A}}(p)$ and $N_{\text{Game}}(p)$ as a function of p. Plot the graphs of these. Determine the exact values of $p_{\text{Game A}}(p)$ and $N_{\text{Game}}(p)$ for $p=0$, $p=1/2$, and $p=1$, and compute the approximate values for $p=0.51$, 0.55 and 0.6. What do you notice? If the players are evenly matched, we can assume $p\approx 1/2$. Determine the derivatives $p'_{\text{Game A}}(p)$ and $N'_{\text{Game}}(p)$ at $p=1/2$. Interpret your results.

To win a *set*, one has to win at least six games, but always with a difference of at least two games. To determine the probability of winning a set and the average length of a set, we can proceed as above for games. Again there are two phases:

Phase 1: Starting phase in which at least one player is more than two games away from winning the set.

Phase 2: End phase, in which the game count takes one of five possible states: “Set B”, “Set Point B”, “Even”, “Set Point A” and “Set A”.

(g) Show that Phase 1 is of finite duration, for example by enumerating all possible states. What is the maximum duration of Phase 1?

(h) Make a figure like the one above for the games in a set. Let P, Q be the probabilities that player A, respectively B, wins a game. Let Π the transition matrix associated with Phase 2 and $\tilde{\Pi}$ be the associated substochastic matrix.

(i) How many possible game scenarios (paths in the graph) result in a 6–2 (set) win for A? What is the probability that A wins a set with 6–2?

(j) Show that the probability vector in which Phase 1 ends is given by

$$\mathbf{w} = \begin{pmatrix} Q^6+6PQ^6+21P^2Q^6+56P^3Q^6 \\ 56P^4Q^5 \\ 2\cdot 35P^4Q^4 \\ 56Q^4P^5 \\ P^6+6QP^6+21Q^2P^6+56Q^3P^6 \end{pmatrix}.$$

(k) Find the probability that A wins a set.

$$p_{\text{Set A}}(P) = \left[\sum_{n=10}^{\infty}\tilde{\Pi}^{n-9}\mathbf{w}+\mathbf{w}\right]_5 = \left[\sum_{k=0}^{\infty}\tilde{\Pi}^{k}\mathbf{w}\right]_5 = \left[(I-\tilde{\Pi})^{-1}\mathbf{w}\right]_5.$$

(l) What is the average duration of a set?

(m) Extend your Mathematica notebook with implementations and plots of the functions $p_{\text{Set A}}(P)$ and $N_{\text{Set}}(P)$ from the previous two parts, analogous to part (f). Define composite functions $p_{\text{Set A}}(p)$ and $N_{\text{Set}}(p)$ by expressing P as a function of p, and answer all of the questions from (f) for the composite functions of p.

(n) What do you notice about the probability for winning games as opposed to that for winning sets?

The player who wins the most out of three sets, wins the *match*.

(o) Determine that chance that player A wins the match, as well as the average duration of a match.

(p) Extend your Mathematica notebook again with implementations and plots of the functions $p_{\text{Match A}}(P)$ and $N_{\text{Match}}(P)$ and address again the questions from part (f). How do the likelihoods of winning a game, a set and a match compare for $p = 1/2+\varepsilon$, where $0 < \varepsilon \ll 1$?

(q) Discuss a number of ways in which the model above could be made more realistic.