Exercises (WISB134)

February 16, 2016

1 Exercises for Chapter 1

Exercises for Chapter 1 consist of a Mathematica tutorial and accompanying exercises from Chapter 0 of the book by Lynch.

- Install Mathematica. See the website: `https://ict.science.uu.nl/index.php/Mathematica_%28NL%29`.
- Work through the tutorial in Chapter 0 of the book by Lynch. See `http://link.springer.com.proxy.library.uu.nl/content/pdf/10.1007%2F978-0-8176-4586-1_1.pdf`.
- Work exercises 0.2, 0.4–0.7, 0.10 at the end of Chapter 0 (of Lynch).

2 Exercises for Chapter 2

Exercise 2.1

For investments in which one expects a growth factor between $\kappa = 1.01$ and $\kappa = 1.2$, economists use the approximation

$$T = \frac{72}{100(\kappa - 1)},$$

where T is the number of years in which the investment doubles, and $100(\kappa - 1)$ is the yearly interest rate. Suppose an economist asks you for which values of T and κ the approximation is accurate. What would your answer be?

Exercise 2.2

Find the the general solution of:

(a) $\frac{dy}{dx} = \frac{1+2y}{4-x}$

(b) $\frac{dy}{dx} = -\frac{\cos x}{\sin y}$

(c) $\frac{dy}{dx} = -(1 + y^2) \cdot \ln x$

(d) $\frac{dy}{dx} = \frac{y}{x^2-3x+2}$

(e) $\frac{dy}{dx} = \frac{yx}{x^2-3x+2}$

Exercise 2.3

The lifespan of a lightbulb filament. We consider the filament in an incandescent lightbulb as a thin, straight cylindrical wire. The wire maintains a high constant temperature due to the flow of electric current. We further assume that the length ℓ of the wire is constant. As a consequence of the high temperature, there is a continual evaporation of material from the wire. In each time unit the amount of material that evaporates is proportional to the exposed surface area of the wire. A time $t = 0$ the wire has thickness d_0. The wire breaks when it becomes thinner than $0.1d_0$.

(a) Derive a differential equation for the wire thickness d.

(b) Solve this differential equation.

(c) Determine the lifespan of the wire given that its thickness at time $t = 0$ is 0.1mm and at time $t = 100$ is 0.09mm.

Solution. Let $d(t)$ denote the diameter of the wire at time t. The surface area is $S(t) = \pi d(t) \cdot \ell$, the volume is $V(t) = \pi \left(\frac{d(t)}{2}\right)^2 \cdot \ell$. We interpret the "amount of material" to mean the volume. Then in a time unit we have

$$V(t + \Delta t) - V(t) = -\lambda S(t),$$

for some $\lambda > 0$. Expanding the first term in Taylor series gives

$$V(t + \Delta t) = V(t) + \Delta t V'(t) + \frac{\Delta^2}{2!} V''(t) + \cdots$$

Assuming Δt is small, we neglect the terms that are second order or higher in Δt. Substituting the second equation in the first yields

$$\Delta t V'(t) = -\lambda S(t).$$

In terms of $d(t)$ this gives

$$\pi \ell d(t) d'(t) = -c\pi \ell d(t) \quad \Rightarrow \quad d'(t) = -c,$$

for some constant $c = \lambda/\Delta t > 0$. The solution of this ODE is $d(t) = d_0 - ct$. With $d_0 = 0.1$mm and $d(100) = 0.09$mm, we find $c = 0.0001$. The wire breaks when $d(t) = 0.01$, for which $t = 900$.

Exercise 2.4

A mothball is a spherical ball of camphor that evaporates quickly. The mass of the mothball decays due to evaporation, and the amount lost per time unit is proportional to the exposed surface area of the sphere. The mass G (in grams) is a function of time t (in weeks).

(a) Show that there exists a positive constant c such that:

$$G' = -cG^{2/3}.$$

(b) Suppose the mothball original weighs 10 grams and after ten weeks only 1 gram. After how many weeks is the ball completely evaporated?

Solution. Let $r(t)$ be the radius of the sphere at time t. Its volume is $V(t) = \frac{4}{3}\pi r(t)^3$ and surface area $S(t) = 4\pi r(t)^2$. We suppose the mass $G(t)$ is proportional to the volume: $G(t) = \kappa \frac{4}{3}\pi r(t)^3$, for some constant $\kappa > 0$. Consequently, $S(t) = \mu G(t)^{2/3}$, for an appropriate $\lambda > 0$. The mass lost in unit time is proportional to the surface area. Let us write this as

$$G(t + \Delta t) - G(t) = -\lambda S(t),$$

for some $\lambda > 0$. The first term can be expanded in Taylor series to give

$$G(t + \Delta t) = G(t) + \Delta t G'(t) + \frac{\Delta t^2}{2!} G''(t) + \cdots$$

Assuming Δt is small, we neglect the second and higher order terms. Inserting the second expression in the first one gives

$$\Delta t G'(t) = -\lambda S(t), \qquad G'(t) = -cG(t)^{2/3}$$

for $c = \lambda\mu/\Delta t$. To answer (b), solve the differential equation using separation of variables to obtain

$$G(t) = (-ct + G_0^{1/3})^3.$$

It follows that $G = 0$ at time $t = G_0^{1/3}/c$. Using $G_0 = 10$ and $G(10) = 1$, we find $c = (10^{1/3} - 1)/10$. Consequently, the mothball evaporates at time $t = 10^{4/3}/(10^{1/3} - 1)$, which is approximately $t = 18.66$ weeks.

Exercise 2.5

Consider a lake with a constant volume V in km^3 which contains a certain quantity of chemical waste. Assume the waste is always uniformly distributed in the lake water. At time t the concentration of waste is give by $c(t)$. A polluted river empties into the lake. The concentration of waste in this river is k, and it delivers an amount s of polluted water into the lake per time unit. Another river removes water at the same rate. Derive a differential equation for $c(t)$ and solve it.
Compute $\lim_{t\to\infty} c(t)$.

Exercise 2.6

(a) Classify the values of t_0 and x_0 for which the following ODE has a unique solution:

$$\dot{x} = \sqrt{x}, \quad x(t_0) = x_0.$$

(b) What is the first time $t > 0$ for which the solutions of the following initial value problems cease to exist?

(i) $\dot{x} = x^3$, $x(0) = 1$;

(ii) $\dot{x} = x^2 + 2x + 1$, $x(0) = -\frac{1}{2}$.

Solution. (a) We suppose real solutions are desired. Let $f(x) = \sqrt{x}$; this function is only defined for $x \geq 0$. Consider an interval $x \in [a, \infty)$, $a > 0$. On such an interval f is Lipschitz continuous with Lipschitz constant $L = f'(a)$. To see this, check that the graph of $f(x)$ is everywhere bounded above by its tangent lines. The tangent line through the point $(a, f(a))$ may be parameterized by $y(x) = f(a) + f'(a)(x - a)$. Since this line bounds the graph of f from above, we find for any $x > a$ that $f(a) + f'(a)(x - a) - f(x) \geq 0$, where all terms are positive. Rearranging gives $f(x) - f(a) \leq f'(a)(x - a)$. From this it follows that $L = f'(a)$ is a Lipschitz constant. Since f is Lipschitz on such an interval for any $a > 0$, the ODE has a unique solution for any $x_0 > 0$. If $x_0 = 0$, the derivative of f is unbounded at x_0, and there is no Lipschitz constant. Indeed there is no unique solution in this case. Since the differential equation is autonomous, the initial time t_0 is irrelevant.

Exercise 2.7

We iterate the function $f : x \mapsto \kappa x$ on the whole real line $\mathbf{R}$, (hence allowing also negative values of x, and also κ may be any real number. How does the recursively defined sequence $x_{n+1} = f(x_n)$ behave as $n \to \infty$? Answer this question for fixed κ, and as determine how many 'different' cases there are. How important is the choice of x_0?

Exercise 2.8

Consider a function f defined on an open interval of $\mathbf{R}$ and twice continuously differentiable there. The Newton process attempts to determine a zero α of f as follows. Choose a number x_0 as an approximation to α. Construct a correction to x_0: $\alpha = x_0 - h$. Hence,

$$0 = f(\alpha) = f(x_0 - h) = f(x_0) - hf'(x_0) + \frac{1}{2}h^2 f''(\xi).$$

Here we have written the Taylor series for f about x_0, where ξ is a number between x_0 and $x_0 - h$. If h is small, the h^2-term will be much smaller than the other terms, $f(x_0)$ and $hf'(x_0)$. If we neglect the h^2-term, then we can compute h: $h = f(x_0)/f'(x_0)$. Of course, this is not completely correct, but $x_1 := x_0 - f(x_0)/f'(x_0)$ will probably be a much better approximation of α than x_0. Iterating this approximation process leads to the Newton process:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad \text{voor } n = 0, 1, 2, \ldots.$$

(a) Show that the recursion in Exercise 2.7 represents the Newton process for solving $f(x) := x^2 - A = 0$.

(b) Interpret the Newton process as finding the intersection of the tangent to the graph of f at $(x_n, f(x_n))$ with the x-axis: x_{n+1} is the intersection.

(c) For $h_n := x_n - \alpha$ it follows that $\alpha = x_n - h_n$ and $0 = f(x_n) - h_n f'(x_n) + \frac{1}{2}h_n^2 f''(\xi)$. Hence

$$h_{n+1} = h_n - \frac{f(x_n)}{f'(x_n)} = \frac{1}{2}h_n^2 \frac{f''(\xi)}{f'(x_n)}.$$

Check this. Write $B := \frac{1}{2}f''(\alpha)/f'(\alpha)$. Suppose that $x_n \approx \alpha$. Check that then $\frac{1}{2}f''(\xi)/f'(x_n) \approx B$ and conclude that $Bh_{n+1} \approx (Bh_n)^2$. Explain the 'super-fast' convergence of Exercise 2.7.

(d) Consider next the recursion

$$x_{n+1} = \frac{1}{3}\left(2x_n + \frac{A}{x_n^2}\right).$$

Prove that $\lim_{n\to\infty} x_n = \sqrt[3]{A}$.

Exercise 2.9

Apply Euler's method to solve the Lotka-Volterra system

$$\frac{dq}{dt} = rq(1-p), \qquad \frac{dp}{dt} = p(q-1), \qquad q_0 = p_0 = \frac{1}{2}, \quad t \in [0, 10],$$

with $r = 2$ and step sizes $\tau = 0.1$, $\tau = 0.01$, $\tau = 0.001$. Plot the solutions in *phase space*: that is, plot the points (q_n, p_n), $n = 0, 1, 2, \ldots$ as a sequence of points in the plane $\mathbf{R}^2$. One can prove using analysis that all solutions $(q(t), p(t))$ of the Lotka-Volterra system are closed curves in $\mathbf{R}^2$. In other words, there is some T (dependent on the initial condition) for which $q(t+T) = q(t)$ and $p(t+T) = p(t)$ for all t. Consequently, any solution remains in bounded subset of $\mathbf{R}^2$. Compare this fact with what you observe for Euler's method.

Exercise 2.10

Solve the Verhulst model

$$\frac{dp}{dt} = p(1-p), \quad p(0) = \frac{1}{2}, \quad 0 \le t \le 5,$$

with the trapezoidal rule. Derive the quadratic equation for p_{n+1} as a function of p_n. Which branch (root) of the quadratic equation is appropriate for numerical integration? Repeat the computation, this time solving the quadratic equation numerically using Newton's method.

3 Exercises for Chapter 3

Exercise 3.1

Consider the function $f(x) = A(1-x)x^2$ with $0 \le A \le 27/4$ on the interval $[0, 1]$.

(a) Show that f maps the interval $[0, 1]$ to itself.

Consider the recursion

$$x_0 \in [0, 1], \quad x_{n+1} = f(x_n), \qquad n \ge 0.$$

(b) For general A determine the fixed points of the recursion and check whether they are stable or unstable.

Exercise 3.2

Consider the recursion

$$x_{n+1} = f(x_n), \qquad f(x) = r^2 x(1-x)(1-rx+rx^2), \quad r = 1+\sqrt{5}.$$

Use the graphical analysis method to identify (graphically) the steady states of this function and establish their stability. There are two stable steady states; call them α and β. Associated to these are sets A and B of initial conditions whose iterates eventually converge to α and β, respectively. That is

$$A = \{x \in [0,1] \,|\, x_0 = x \Rightarrow \lim_{n\to\infty} x_n = \alpha\},$$

and similarly for B. Consider how you might construct the sets A and B. (*Hint:* $f'(\alpha) = f'(\beta) = 0$.)

Exercise 3.3

Consider a recursion $x_{n+1} = f(x_n)$ where $f : \mathbf{R} \longrightarrow \mathbf{R}$.

(a) Construct an example (d.m.v. formule/grafiek/programma/...) of a function f that has a fixed point that is attracting, but not stable in the sense of Lyapunov. *Hint: what value will the derivative of f take at the fixed point?*

(b) Does such a function exist with the property of (a) if we demand that f be continuous?

(c) Think of a continuous function f with a stable fixed point α such that for a certain sequence (x_n) of iterates that converge to α, it holds that $|x_{2n+1} - \alpha| \geq 10\,|x_{2n} - \alpha|$ $(n = 0, 1, 2, \dots)$.

(d) Does there exist a function with the property of (c) if we demand that f be continuously differentiable?