Conservative Dynamical Systems 2010/2011

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The last two exercises are homework, to be handed in on Tuesday 14 September.

1.1 Phase portraits

Rewrite the second order equation $\ddot{x} = -\frac{\partial V}{\partial x}$ as a first order system with energy function $E : \mathbb{R}^2 \to \mathbb{R}$, $E(x, y) = \frac{1}{2}y^2 + V(x)$. For several shapes of the graph of V, sketch the phase portrait of the vector field $(\dot{x}, \dot{y})$. Recall that the integral curves are given by the level sets of the function E. What do they look like in the cases where V has a local maximum or minimum or a horizontal asymptote? How do the integral curves intersect the x–axis?

1.2 Gradient- and Hamiltonian vector fields

Let $E : \mathbb{R}^2 \to \mathbb{R}$ be a smooth energy function. Consider the (Hamiltonian) vector field $X = (\partial E/\partial y)\ \mathrm{e}_1 - (\partial E/\partial x)\ \mathrm{e}_2$ and the gradient vector field $\mathrm{grad}E = (\partial E/\partial x)\ \mathrm{e}_1 + (\partial E/\partial y)\ \mathrm{e}_2$. Prove that at each point of $\mathbb{R}^2$, the integral curves of X and $\mathrm{grad}E$ are orthogonal. What is the relation to the level curves of E ? Discuss the changes in the phase portraits of X and $\mathrm{grad}E$ if E is replaced by $-E$. In particular consider a neighbourhood of a minimum and a saddle point of E.

1.3 Lissajous figures

Consider the two-degree-of-freedom system consisting of a particle of unit mass moving in the plane with potential energy $U(x_1, x_2) = \frac{1}{2}(x_1^2 + \omega^2 x_2^2)$. Show that the general solution is of the form $(x_1(t), x_2(t)) = (A_1 \sin(t + \varphi_1), A_2 \sin(\omega t + \varphi_2)$. Discuss the orbits in the (x_1, x_2) plane for the cases where ω is 1 or 2 and the phase difference, $\varphi_1 - \varphi_2$, has the value 0, $\pi/2$ or π. Show that for integer ω there is a phase difference and amplitudes A_1 and A_2 such that the orbit in the (x_1, x_2) plane agrees with the graph of a Chebychev polynomial. Show that for rational ω the orbits are closed, and for irrational ω the orbits densely fill a rectangle in the (x_1, x_2) plane.

1.4 Period and area

Let $E : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a smooth function given by $E(x, y) = \frac{1}{2}y^2 + V(x)$ and assume that for $E_0 \in \mathbb{R}$, the motion in the level set $E^{-1}(E_0)$ is periodic. Show that for energies near E_0 the motion is also periodic. Let $A(E_0)$ denote the area enclosed by the level set $E^{-1}(E_0)$, and let $T(E_0)$ denote the period of the motion in this level set. Show that $T(E_0) = \left.\frac{\mathrm{d}A(z)}{\mathrm{d}z}\right|_{z=E_0}$.