Conservative Dynamical Systems 2010/2011

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The last two exercises are homework, to be handed in on Tuesday 21 September.

2.1 Kepler's 3rd law

The 3rd law of Kepler states that the ratio of the square of the period of an elliptic orbit and the cube of its semi major axis is constant. The aim of this exercise is to check the relation to the $\frac{1}{r}$–form of the gravitational potential, from which one obtaines an effective potential after reduction to one degree of freedom.

1. Check that the minimum of the effective potential leads to a circular orbit which satisfies Kepler's 3rd law.
2. Derive the $\frac{1}{r}$–form of the gravitational potential from Kepler's 3rd law.

2.2 Collision time

Two particles P_i with mass m_i, $(i = 1, 2)$ attract each other according to Newton's law with attraction constant Γ. In the initial position, they are at rest and their distance is $2a$. When do they meet?

2.3 The Legendre transformation

Let $H(x,p) = \frac{p^2}{2m} + V(x)$ be the energy function on $\mathbb{R}^2$, where $V : \mathbb{R} \longrightarrow \mathbb{R}$ is the potential energy. Compute the Legendre transformation

$$L(x,v) \; := \; \sup_{p \in \mathbb{R}} \left(v \cdot p \, - \, H(x,p) \right)$$

and clarify the situation with a figure. The function L obtained this way is the Lagrangean function of the system. Show that Hamilton's equations $\dot{x} = \frac{\partial H}{\partial p}$, $\dot{p} = -\frac{\partial H}{\partial x}$ turn into

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial v} \; = \; \frac{\partial L}{\partial x} \; .$$

2.4 Isoperimetric problem

In $\mathbb{R}^3 = \{x, y, z\}$ consider the (x, y)–plane. In the (x, y)–plane a curve is given, connecting the points $(x_1, y_1, 0)$ and $(x_2, y_2, 0)$. Revolve this curve around the x–axis. For which curve does the corresponding surface of revolution have minimal area.